Uniform estimates for 2D quasilinear wave

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A R T I C L E I N F O

A B S T R A C T

We consider two-dimensional quasilinear wave equations with
standard null-type quadratic nonlinearities. In 2001 Alinhac
proved that such systems possess global in time solutions
for compactly supported initial data with sufficiently small
Sobolev norm. The highest norm of the constructed solution
grows polynomially in time. In this work we develop a new
strategy and prove uniform boundedness of the highest order
norm of the solution for all time.

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1. Introduction

Let $\square = \partial_{tt} - \Delta = \partial_{tt} - \partial_{x_1 x_1} - \partial_{x_2 x_2}$ be the usual d'Alembertian operator in $(2+1)$ space-time. We consider the following quasilinear wave equation:

$$\begin{cases} \square u = g^{kij} \partial_k u \partial_{ij} u, \quad t > 2, \quad x \in \mathbb{R}^2; \\ (u, \partial_t u)\Big|_{t=2} = (f_1, f_2), \end{cases} \tag{1.1}$$

where the functions f_1, $f_2 : \mathbb{R}^2 \to \mathbb{R}$ are initial data. On the RHS of (1.1) we employ the Einstein summation convention with $\partial_0 = \partial_t$, $\partial_l = \partial_{x_l}$, $l = 1, 2$. For simplicity we

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assume g^{kij} are constant coefficients, $g^{kij} = g^{kji}$ for any i, j, and satisfy the standard null condition:

$$g^{kij}\omega_k\omega_i\omega_j = 0, \qquad \text{for any null } \omega, \text{ i.e., } \omega = (-1, \cos\theta, \sin\theta), \ \theta \in [0, 2\pi]. \quad (1.2)$$

In the seminal work [1], Alinhac showed that for compactly supported initial data which have sufficiently small Sobolev norm, the system (1.1) with the null condition (1.2) admits global in time solutions. The main ingredients of Alinhac's proof are two:

1) construction of an approximation solution;
2) time-dependent weighted energy estimates known as the ghost weight method.

The name ghost weight derives from a judiciously chosen bounded space-time weight which seems negligible by itself but after differentiation produces a remarkable stabilization term helping to balance the critical decay of the solution near the light-cone. Besides the aforementioned ghost weight, the weighted energy estimates typically involve a number of vector fields which are the infinitesimal generators of certain symmetry groups, for example:

- Spatial rotation: $\partial_\theta = x^\perp \cdot \nabla = x_1\partial_2 - x_2\partial_1$, $x^\perp = (-x_2, x_1)$.
- Lorentz boost: $L_i = x_i\partial_0 + t\partial_i$, $i = 1, 2$.
- Scaling: $L_0 = t\partial_t + r\partial_r$.

In particular, the Lorentz boost vector fields were employed together with the scaling operator in order to extract sufficient time-decay of the solution. While the Lorentz boost vector fields can lead to strong time-decay estimates, they are not suitable for general wave systems which are not Lorentz invariant. To name a few we mention non-relativistic wave systems with multiple wave speeds (cf. [17,23]), nonlinear wave equations on non-flat space-time (cf. [26]) and exterior domains (cf. [19]). From this perspective it is of fundamental importance to remove the Lorentz boost operator and develop a new strategy for the general non-Lorentz-invariant systems. In [13], Hoshiga considered a quasilinear system with multiple speeds of propagation, and proved global wellposedness under some suitable strong null conditions. In [27] (see also [20]), Zha considered (1.1)–(1.2) with the additional symmetry condition: $g^{kij} = g^{ikj} = g^{jik}$, $\forall i, j, k$. For this case Zha developed the first proof of global wellposedness without using the Lorentz boost vector fields. Note that the additional symmetry condition introduced by Zha appears to be a bit restrictive. For example, it does not include the standard nonlinearity $\partial(|\partial_t u|^2 - |\nabla u|^2)$. In recent work [7], a novel strong null form which includes several prototypical strong null forms such as $\partial(|\partial_t u|^2 - |\nabla u|^2)$ in the literature and also some null forms in [6] as special cases. Moreover for this class of new null forms, a new normal-form type Lorentz-boost-free strategy was developed in [7] to prove global wellposedness

and uniform boundedness of the highest norm of the solution. We refer to the papers [5,6,9–11,14–16,18,21,22,24,28] for other related developments and different strategies.

We now mention a few other important works on somewhat related systems. In [18], by using Alinhac's method, Lei established small data global wellposedness for 2D incompressible elastodynamics. A similar result was obtained independently by X. Wang in [25] using a normal form method. In [6], Cai, Lei and Masmoudi considered the quasilinear wave equations of the form $\Box u = A_l \partial_l (N_{ij} \partial_i u \partial_j u)$, where A_l, N_{ij} are constants, and $N_{ij} \omega_i \omega_j = 0$ for any null vector ω. A special case is the equation $\Box u = \partial_t (|\partial_t u|^2 - |\nabla u|^2)$. In [6] by using a nonlocal transformation (see Remark 1.3 therein) it was shown that the above system has a uniform bound of the highest-order energy for all time. More recently by using Alinhac's ghost weight and the null structure in the Lagrangian formulation, Cai [5] showed uniform boundedness of the highest-order energy for 2D incompressible elastodynamics. In [10], by using the hyperbolic foliation method which goes back to Hörmander and Klainerman, Dong, LeFloch and Lei showed that the top-order energy of the system (1.1) with the null condition (1.2) is uniformly bounded for all time. The main advantage of the hyperbolic change of variable is that one can gain better control of the conformal energy thanks to the extra integrability in the hyperbolic time $s = \sqrt{t^2 - r^2}$. One should note, however, that if one works with the advanced coordinate $s = t - r$, then there is certain degeneracy in the ∂_s direction which renders (even any generalized) conformal energy out of control. In this connection an interesting further issue is to explore the monotonicity of the conformal energy (and possible generalizations) with respect to different space-time foliations. Another subtle technical issue in the hyperbolic foliation method is the extensive use of Lorentz boost vector fields which appears not suitable for general non-Lorentz-invariant systems.

In Alinhac's work [1], the highest energy of the constructed solution has an upper bound which grows polynomially in time. An ensuing open question is whether this growth is genuinely true phenomena known as the "blowup-at-infinity" conjecture ([2–4]). The purpose of this work is to develop further the program initiated in [7,8,20,27], and obtain the uniform boundedness of highest norm under the generic null condition (1.2). We introduce a new approach and settle the blowup-at-infinity conjecture without employing the Lorentz boost vector fields. The main result is the following.

Theorem 1.1. *Consider* (1.1) *with* g^{kij} *satisfying the standard null condition* (1.2). *Let* $m \geq 5$ *and assume* $f_1 \in H^{m+1}(\mathbb{R}^2)$, $f_2 \in H^m(\mathbb{R}^2)$ *are compactly supported in the disk* $\{|x| \leq 1\}$. *There exists* $\varepsilon_0 > 0$ *depending on* g^{kij} *and* m *such that if* $\|f_1\|_{H^{m+1}} + \|f_2\|_{H^m} < \varepsilon_0$, *then the system* (1.1) *has a unique global solution. Furthermore, the highest norm of the solution remains uniformly bounded, namely*

$$\sup_{t \geq 2} \sum_{|\alpha| \leq m} \|(\partial \Gamma^\alpha u)(t, \cdot)\|_{L_x^2(\mathbb{R}^2)} < \infty. \tag{1.3}$$

Here $\Gamma = \{\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_\theta, t\partial_t + r\partial_r\}$ *does not include the Lorentz boost (see* (2.3) *for notation).*

Remark 1.1. The regularity constraint $m \geq 5$ can be lowered further by optimizing some technical arguments. However we shall not dwell on this issue in this work.

We now explain the key steps of the proof of Theorem 1.1 (see section 2 for the relevant notation). Fix any multi-index α with $|\alpha| \leq m$ and consider $\Gamma^\alpha u$. By Lemma 2.4, we have

$$\Box \Gamma^\alpha u = \sum_{\alpha_1 + \alpha_2 \leq \alpha} g^{kij}_{\alpha;\alpha_1,\alpha_2} \partial_k \Gamma^{\alpha_1} u \partial_{ij} \Gamma^{\alpha_2} u, \qquad (1.4)$$

where $g^{kij}_{\alpha;\alpha,0} = g^{kij}_{\alpha;0,\alpha} = g^{kij}$, and $g^{kij}_{\alpha;\alpha_1,\alpha_2}$ still satisfies the null condition (1.2) for all other values of (α_1, α_2).

Step 1. Weighted energy estimates: LHS of (1.4). We choose $p(r,t) = q(r-t)$ with $q'(s)$ nearly scales as $\langle s \rangle^{-1}$ to derive

$$\int \Box \Gamma^\alpha u \partial_t \Gamma^\alpha u e^p dx = \frac{1}{2} \frac{d}{dt} (\|e^{\frac{p}{2}} \partial \Gamma^\alpha u\|_2^2) + \frac{1}{2} \int e^p q' |T\Gamma^\alpha u|^2 dx. \qquad (1.5)$$

Summing over $|\alpha| \leq m$, we have (below $v = \Gamma^\alpha u$)

$$\sum_{|\alpha| \leq m} \|e^{\frac{p}{2}} \partial v\|_2^2 \sim E_m = \sum_{|\alpha| \leq m} \|\partial \Gamma^\alpha u(t,\cdot)\|_2^2; \qquad (1.6)$$

$$\sum_{|\alpha| \leq m} \int e^p q' |Tv|^2 dx = \sum_{|\alpha| \leq m} \int e^p q' |T\Gamma^\alpha u|^2 dx. \qquad (1.7)$$

Step 2. Refined decay estimates. One way to remedy the lack of Lorentz boost vector fields is to employ L^∞ and L^2 estimates involving the weight-factor $\langle r-t \rangle$. At the expense of certain smallness of $E_{\lfloor \frac{m}{2} \rfloor + 3}$ and using in an essential way the nonlinear null form (see Lemma 2.6), we obtain

$$\|\langle r-t \rangle (\partial^2 \Gamma^{\leq l_0} u)(t,\cdot)\|_2 \lesssim \|(\partial \Gamma^{\leq l_0+1} u)(t,\cdot)\|_2, \qquad \forall l_0 \leq m-1; \qquad (1.8)$$

$$|\langle r-t \rangle (\partial^2 \Gamma^{\leq l_0} u)(t,x)| \lesssim |(\partial \Gamma^{l_0+1})(t,x)|, \qquad \forall r \geq t/10, \, l_0 \leq m-1; \qquad (1.9)$$

$$\|(\partial \Delta \Gamma^{\leq m-3} u)(t,\cdot)\|_{L^2_x(|x| \leq \frac{2}{3} t)} \lesssim t^{-2} \|(\partial \Gamma^{\leq m-1} u)(t,\cdot)\|_2. \qquad (1.10)$$

These in turn lead to a handful of new strong decay estimates (see Lemma 2.7):

$$t^{\frac{1}{2}} \|\partial \Gamma^{\leq m-3} u\|_\infty + t^{\frac{3}{2}} \left\| \frac{T\Gamma^{\leq m-3} u}{\langle r-t \rangle} \right\|_\infty + t^{\frac{1}{2}} \left\| \frac{T\Gamma^{\leq m-2} u}{\langle r-t \rangle} \right\|_2 \lesssim E_{m-1}^{\frac{1}{2}}; \qquad (1.11)$$

$$t^{\frac{1}{2}} \|\langle r-t \rangle \partial^2 \Gamma^{\leq m-4} u\|_\infty + t^{\frac{3}{2}} \|T\partial \Gamma^{\leq m-4} u\|_\infty + t \|T\partial \Gamma^{\leq m-4} u\|_2 \lesssim E_{m-1}^{\frac{1}{2}}. \qquad (1.12)$$

These decay estimates play an important role in the nonlinear energy estimates.

Step 3. Weighted energy estimates: nonlinear terms. We discuss several cases.

Case 1: $\alpha_1 < \alpha$ and $\alpha_2 < \alpha$. Since $g^{kij}_{\alpha;\alpha_1,\alpha_2}$ still satisfies the null condition, by Lemma 2.4 we rewrite

$$
\sum_{\substack{\alpha_1 < \alpha, \alpha_2 < \alpha \\ \alpha_1 + \alpha_2 \le \alpha}} g^{kij}_{\alpha;\alpha_1,\alpha_2} \partial_k \Gamma^{\alpha_1} u \partial_{ij} \Gamma^{\alpha_2} u
$$
$$
= \sum_{\substack{\alpha_1 < \alpha, \alpha_2 < \alpha \\ \alpha_1 + \alpha_2 \le \alpha}} g^{kij}_{\alpha;\alpha_1,\alpha_2} (T_k \Gamma^{\alpha_1} u \partial_{ij} \Gamma^{\alpha_2} u - \omega_k \partial_t \Gamma^{\alpha_1} u T_i \partial_j \Gamma^{\alpha_2} u + \omega_k \omega_i \partial_t \Gamma^{\alpha_1} u T_j \partial_t \Gamma^{\alpha_2} u).
$$

$$(1.13)$$

By using the decay estimates obtained in Step 2, we show that

$$
\sup_{|\alpha| \le m} \| \sum_{\substack{\alpha_1 < \alpha, \alpha_2 < \alpha \\ \alpha_1 + \alpha_2 \le \alpha}} g^{kij}_{\alpha;\alpha_1,\alpha_2} \partial_k \Gamma^{\alpha_1} u \partial_{ij} \Gamma^{\alpha_2} u \|_2 \lesssim t^{-\frac{3}{2}} E_{\lfloor \frac{m}{2} \rfloor + 3}^{\frac{1}{2}} E_m^{\frac{1}{2}}. \qquad (1.14)
$$

Case 2: The quasilinear piece $\alpha_1 = 0$, $\alpha_2 = \alpha$. Recall that $g^{kij}_{\alpha;0,\alpha} = g^{kij}$. By using successive integration by parts, we have

$$
\int g^{kij} \partial_k u \partial_{ij} \Gamma^\alpha u \partial_t \Gamma^\alpha u e^p = \mathrm{OK}, \qquad (1.15)
$$

where OK is in the sense of (3.5). Here we exploit an important algebraic identity (see (3.16))

$$
- \partial_j \varphi \partial_i v \partial_t v + \partial_t \varphi \partial_i v \partial_j v - \partial_i \varphi \partial_t v \partial_j v
$$
$$
= - T_j \varphi \partial_i v \partial_t v + \partial_t \varphi T_i v T_j v - T_i \varphi \partial_t v \partial_j v - \omega_i \omega_j \partial_t \varphi (\partial_t v)^2, \qquad (1.16)
$$

where φ is taken to be either $\partial_k u$ or e^p, and $v = \Gamma^\alpha u$. The standard null form condition amounts to the annihilation of the term $\omega_i \omega_j \omega_k$ when $\varphi = \partial_k u$ and $\partial_t \varphi$ is replaced by $T_k \partial_t u - \omega_k \partial_{tt} u$.

Case 3: the main piece $\alpha_1 = \alpha$, $\alpha_2 = 0$. By using Lemma 2.4 with the decay estimates, we derive

$$
\int g^{kij} \partial_k \Gamma^\alpha u \partial_{ij} u \partial_t \Gamma^\alpha u e^p = \mathrm{OK} + \underbrace{\int g^{kij} \omega_i \omega_j T_k \Gamma^\alpha u \partial_{tt} u \partial_t \Gamma^\alpha u e^p}_{=:Y_1}. \qquad (1.17)
$$

We perform a further refined decomposition of the term Y_1. By using $T_1 = \omega_1 \partial_+ - \frac{\omega_2}{r} \partial_\theta$ and $T_2 = \omega_2 \partial_+ + \frac{\omega_1}{r} \partial_\theta$, we obtain (below we denote $v = \Gamma^\alpha u$)

$$
g^{kij} \omega_i \omega_j T_k v = g^{1ij} \omega_i \omega_j (\omega_1 \partial_+ v - \frac{\omega_2}{r} \partial_\theta v) + g^{2ij} \omega_i \omega_j (\omega_2 \partial_+ v + \frac{\omega_1}{r} \partial_\theta v)
$$
$$
= (g^{1ij} \omega_1 \omega_i \omega_j + g^{2ij} \omega_2 \omega_i \omega_j) \partial_+ v + \omega_i \omega_j (g^{2ij} \omega_1 - g^{1ij} \omega_2) \frac{1}{r} \partial_\theta v
$$

$$=: h_1(\theta)\partial_+ v + h_2(\theta)\frac{1}{r}\partial_\theta v.$$

We decompose Y_1 accordingly as

$$Y_1 = \int h_1(\theta)\partial_+ v \partial_t v \partial_{tt} u e^p + \int h_2(\theta)\frac{1}{r}\partial_\theta v \partial_t v \partial_{tt} u e^p$$

$$=: Y_A + Y_B. \tag{1.18}$$

Step 4. Estimate of Y_A: localization, further decomposition and normal form transformation. We use a bump function ϕ which is localized to $r \in [\frac{t}{2}, 2t]$ such that the main part of Y_A becomes

$$\int h(\theta)\partial_+ v \partial_t v \partial_{tt} u e^p \phi. \tag{1.19}$$

The contribution of the regimes $r \le \frac{t}{2}$ and $r > 2t$ corresponding to the cut-off $1 - \phi$ can be shown to be negligible. We further use the decomposition $\partial_t = \frac{\partial_+ + \partial_-}{2}$ to transform (1.19) as

$$\frac{1}{2}\int h(\theta)\partial_+ v \partial_- v \partial_{tt} u e^p \phi + \text{OK}. \tag{1.20}$$

At this point, the crucial observation is to use the fundamental identity $\partial_+ \partial_- = \Box + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}$ to transform (1.20) into an expression which contains an "inflated" nonlinearity. After this novel normal form type transformation and further technical estimates the term Y_A can be shown to be under control.

Step 5. Estimate of Y_B: localization and further transformation. By using the estimate $\|\langle r - t \rangle \partial_{tt} u\|_\infty \lesssim t^{-\frac{1}{2}}$ (see Lemma 2.7), we have

$$Y_B = \text{OK} + \int h_2(\theta)\frac{1}{r}\partial_\theta v \partial_{tt} u \partial_t v \tilde{\phi}(\frac{x}{t}) e^p, \tag{1.21}$$

where $\tilde{\phi}$ is a radial bump function localized to $|z| \sim 1$. Denote $\phi(z) = \frac{1}{|z|}\tilde{\phi}(z)$. Using integration by parts in θ, we obtain

$$Y_B = \text{OK} + \frac{1}{t}\int h_2(\theta)v \partial_{tt} u \partial_t \partial_\theta v \phi(\frac{x}{t}) e^p. \tag{1.22}$$

We then proceed to bound the second term (without the $\frac{1}{t}$ factor) as

$$\int e^p \phi(\frac{x}{t})h_2(\theta)v \partial_{tt} u \partial_t \partial_\theta v$$

$$\lesssim \left(\|e^p \phi(\frac{x}{t})h_2(\theta)v \partial_{tt} u\|_2 + \|\nabla(e^p \phi(\frac{x}{t})h_2(\theta)v \partial_{tt} u)\|_2\right) \cdot \|\langle\nabla\rangle^{-1}\partial_t \partial_\theta v\|_2$$

$$\lesssim t^{-\frac{1}{2}} E_m^{\frac{1}{2}}\|\langle\nabla\rangle^{-1}\partial_t \partial_\theta v\|_2. \tag{1.23}$$

By Lemma 2.1 the norm $\|\langle\nabla\rangle^{-1}\partial_t\partial_\theta v\|_2$ is well-defined. The employment of the nonlocal norm $\|\langle\nabla\rangle^{-1}\partial_t\partial_\theta v\|_2$ is the key to obtaining sufficient time-decay estimates of Y_B. In the next step we show $\|\langle\nabla\rangle^{-1}\partial_t\partial_\theta v\|_2 \lesssim t^\delta$ for some $\delta < \frac{1}{2}$ which suffices for time-integrability.

Step 6. Estimate of $\|\langle\nabla\rangle^{-1}\partial_t\Gamma^{\le m+1}u\|_2$. This is the most technical part of the proof. Due to nonlocality we work with a frequency localized energy which in the main order is given by

$$\tilde{E}_m = \sum_{J\ge 0}\sum_{|\beta|\le m+1} 2^{-2J}\|e^{\frac{p}{2}}\partial P_J\Gamma^\beta u\|_2^2, \tag{1.24}$$

where $(P_J)_{J\ge 0}$ are the Littlewood-Paley frequency projection operators. By using a number of delicate commutator estimates and deeply exploiting the null form structure, we show $\tilde{E}_m \lesssim t^{0+}$ which is just enough for closing the uniform estimates. Here t^{0+} means t^c for some sufficiently small exponent $c > 0$.

The rest of this paper is organized as follows. In Section 2 we collect some preliminaries and useful lemmas. In Sections 3, 4 and 5 we give the proof of Theorem 1.1.

Acknowledgment

2. Preliminaries

Notation. For any two quantities A, $B \ge 0$, we write $A \lesssim B$ if $A \le CB$ for some unimportant constant $C > 0$. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We write $A \ll B$ if $A \le cB$ and $c > 0$ is a sufficiently small constant. The needed smallness is clear from the context.

We shall use the Japanese bracket notation: $\langle x\rangle = \sqrt{1+|x|^2}$, for $x \in \mathbb{R}^2$. For $s \in \mathbb{R}$, we denote the smoothed fractional Laplacian $\langle\nabla\rangle^s = (I-\Delta)^{s/2}$ which corresponds to the Fourier multiplier $(1+|\xi|^2)^{s/2}$.

We denote $\partial_0 = \partial_t$, $\partial_i = \partial_{x_i}$, $i = 1,2$ and (below ∂_θ and ∂_r correspond to the usual polar coordinates)

$$\partial = (\partial_i)_{i=0}^2, \ \partial_\theta = x_1\partial_2 - x_2\partial_1, \ L_0 = t\partial_t + r\partial_r; \tag{2.1}$$

$$\Gamma = (\Gamma_i)_{i=1}^5, \quad \text{where } \Gamma_1 = \partial_t, \Gamma_2 = \partial_1, \Gamma_3 = \partial_2, \Gamma_4 = \partial_\theta, \Gamma_5 = L_0; \tag{2.2}$$

$$\Gamma^\alpha = \Gamma_1^{\alpha_1}\Gamma_2^{\alpha_2}\Gamma_3^{\alpha_3}\Gamma_4^{\alpha_4}\Gamma_5^{\alpha_5}, \qquad \alpha = (\alpha_1,\cdots,\alpha_5) \text{ is a multi-index}; \tag{2.3}$$

$$\partial_+ = \partial_t + \partial_r, \qquad \partial_- = \partial_t - \partial_r; \tag{2.4}$$

$$T_i = \omega_i\partial_t + \partial_i, \ \omega_0 = -1, \ \omega_i = x_i/r, \ i = 1,2. \tag{2.5}$$

Note that in (2.2) we do not include the Lorentz boosts. Note that $T_0 = 0$. For simplicity of notation, we define for any integer $k \geq 1$, $\Gamma^k = (\Gamma^\alpha)_{|\alpha|=k}$, $\Gamma^{\leq k} = (\Gamma^\alpha)_{|\alpha| \leq k}$. In particular

$$|\Gamma^{\leq k} u| = \Big(\sum_{|\alpha| \leq k} |\Gamma^\alpha u|^2 \Big)^{\frac{1}{2}}. \tag{2.6}$$

Informally speaking, it is useful to think of $\Gamma^{\leq k}$ as any one of the vector fields Γ^α with $|\alpha| \leq k$.

For integer $I \geq 3$, we shall denote

$$E_I = E_I(u(t,\cdot)) = \|(\partial \Gamma^{\leq I} u)(t,\cdot)\|_{L^2_x(\mathbb{R}^2)}^2. \tag{2.7}$$

In Section 5 of this paper we will need to use the Littlewood–Paley (LP) frequency projection operators. To fix the notation, let ϕ_0 be a radial function in $C_c^\infty(\mathbb{R}^2)$ and satisfy

$$0 \leq \phi_0 \leq 1, \quad \phi_0(\xi) = 1 \text{ for } |\xi| \leq 1, \quad \phi_0(\xi) = 0 \text{ for } |\xi| \geq 7/6.$$

Let $\phi(\xi) := \phi_0(\xi) - \phi_0(2\xi)$ which is supported in $\frac{1}{2} \leq |\xi| \leq \frac{7}{6}$. For any Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, $j \in \mathbb{Z}$, define

$$\begin{aligned}
\widehat{P_0 f}(\xi) &= \phi_0(\xi)\hat{f}(\xi); \\
\widehat{P_j f}(\xi) &= \phi(2^{-j}\xi)\hat{f}(\xi), \qquad \xi \in \mathbb{R}^2, j \geq 1; \\
\widehat{P_{\leq j} f}(\xi) &= \phi_0(2^{-j}\xi)\hat{f}(\xi), \qquad \xi \in \mathbb{R}^2, j \geq 1.
\end{aligned} \tag{2.8}$$

Note that for $j \geq 1$, $P_j f$ is supported in the annulus $\frac{1}{2} \cdot 2^j \leq |\xi| \leq \frac{7}{6} \cdot 2^j$.

More generally, one can take $\psi \in C_c^\infty(\mathbb{R}^2)$ with compact support in $\{\xi : a_1 < |\xi| < a_2\}$, and $0 < a_1 < a_2 < \infty$ are constants. To spell out the explicit dependence on ψ, one can define for $j \geq 0$

$$\widehat{P_j^\psi f}(\xi) = \psi(2^{-j}\xi)\widehat{f}(\xi). \tag{2.9}$$

In this way P_j^ψ is a smooth frequency cut-off localized to $|\xi| \sim 2^j$. In later computations we often write $\tilde{P}_j = P_j^\psi$ where ψ may vary from line to line. This notation is convenient for intermediate calculations.

For $j \geq 2$, we will denote $P_{<j} = P_{\leq j-1}$, $P_{>j} = I - P_{\leq j}$ (I is the identity operator), $P_{\geq j} = I - P_{\leq j-1}$.

We begin with the following innocuous lemma which justifies the legitimacy of the norm $\|\langle \nabla \rangle^{-1} \partial \Gamma^{\leq m+1} u\|_2$.

Lemma 2.1 *(The nonlocal norm is well-defined). Let u be the solution to* (1.1). *We have*

$$\left\| \langle \nabla \rangle^{-1} \partial \Gamma^{\leq m+1} u \Big|_{t=2} \right\|_{L^2(\mathbb{R}^2)} \leq D_1, \tag{2.10}$$

where $D_1 > 0$ is a finite constant depending on $\|f_1\|_{H^{m+1}(\mathbb{R}^2)}$ and $\|f_2\|_{H^m(\mathbb{R}^2)}$.

Proof. Clearly we only need to consider the case $\left\| \langle \nabla \rangle^{-1} \partial_t \Gamma^{\leq m+1} u \Big|_{t=2} \right\|_2$. Since $\Gamma = \{\partial_t, \partial_1, \partial_2, x^\perp \cdot \nabla, t\partial_t + x \cdot \nabla\}$ and f_1, f_2 are both compactly supported, we have

$$\begin{aligned}
&\left\| \langle \nabla \rangle^{-1} \partial_t \Gamma^{\leq m+1} u \Big|_{t=2} \right\|_{L^2(\mathbb{R}^2)} \\
&\lesssim \sum_{m_1+m_2+m_3 \leq m+2} \left\| \langle \nabla \rangle^{-1} \left(\phi_{m_1,m_2,m_3}(x) \cdot \partial_t^{m_1} \partial_1^{m_2} \partial_2^{m_3} u \Big|_{t=2} \right) \right\|_{L^2(\mathbb{R}^2)},
\end{aligned} \tag{2.11}$$

where $\phi_{m_1,m_2,m_3} \in C_c^\infty(\mathbb{R}^2)$. It is not difficult to check that for each (m_1, m_2, m_3), we have

$$\partial_t^{m_1} \partial_1^{m_2} \partial_2^{m_3} u \Big|_{t=2} = F_{m_1,m_2,m_3}^{(0)} + \sum_{j=1}^{2} \partial_j F_{m_1,m_2,m_3}^{(j)}, \tag{2.12}$$

where

$$\sum_{j=0}^{2} \| F_{m_1,m_2,m_3}^{(j)} \|_{L^2(\mathbb{R}^2)} \leq D_{m_1,m_2,m_3} < \infty, \tag{2.13}$$

and $D_{m_1,m_2,m_3} > 0$ are constants depending on $\|f_1\|_{H^{m+1}(\mathbb{R}^2)}$ and $\|f_2\|_{H^m(\mathbb{R}^2)}$. The desired result follows. $\square$

Lemma 2.2 *(Sobolev decay). For $v \in \mathcal{S}(\mathbb{R}^2)$, we have*

$$\sup_{x \in \mathbb{R}^2} \left(|x|^{\frac{1}{2}} |v(x)| \right) \lesssim \| \partial_\theta^{\leq 1} \partial_r^{\leq 1} v \|_2 = \|v\|_2 + \|\partial_r v\|_2 + \|\partial_\theta v\|_2 + \|\partial_r \partial_\theta v\|_2.$$

Proof. For a one-variable smooth function h which decays sufficiently fast at the spatial infinity, we have

$$\rho |h(\rho)|^2 \leq \int_0^\infty |h(r)|^2 r \, dr + \int_0^\infty |h'(r)|^2 r \, dr, \quad \forall \rho > 0. \tag{2.14}$$

It follows that (below we slightly abuse the notation and denote $v(\rho, \theta) = v(x)$ for $x = (\rho \cos \theta, \rho \sin \theta)$)

$$\rho \|\partial_\theta v(\rho,\theta)\|^2_{L^2_\theta} = \rho \int_0^{2\pi} |\partial_\theta v(\rho,\theta)|^2 d\theta \lesssim \|\partial_\theta v\|^2_{L^2(\mathbb{R}^2)} + \|\partial_r \partial_\theta v\|^2_{L^2(\mathbb{R}^2)}.$$

Denote $\overline{v}(\rho)$ as the average of $v(\rho,\theta)$ over θ. By (2.14), we have

$$\rho |\overline{v}(\rho)|^2 = \rho \left(\frac{1}{2\pi} \int_0^{2\pi} v(\rho,\theta) d\theta \right)^2 \lesssim \rho \int_0^{2\pi} v(\rho,\theta)^2 d\theta \lesssim \|v\|^2_{L^2(\mathbb{R}^2)} + \|\partial_r v\|^2_{L^2(\mathbb{R}^2)}.$$

Note that $|v(\rho,\theta) - \overline{v}(\rho)|^2 \lesssim |\partial_\theta v|^2_{L^2_\theta}$ by the Poincaré inequality. Thus

$$|x||v(x)|^2 \lesssim \|v\|^2_2 + \|\partial_r v\|^2_2 + \|\partial_\theta v\|^2_2 + \|\partial_r \partial_\theta v\|^2_2. \quad \square$$

Lemma 2.3 *(Refined Hardy's inequality). For any real-valued $h \in C_c^\infty([0, M+1))$ with $M > 0$, we have*

$$\int_0^{M+1} \frac{h(\rho)^2}{(2+M-\rho)^2} \rho d\rho \leq 4 \int_0^\infty (h'(\rho))^2 \rho d\rho. \tag{2.15}$$

For $u \in C^\infty([0,T] \times \mathbb{R}^2)$ with support in $\{(t,x) : |x| \leq 1+t\}$, we have

$$\|\langle |x|-t \rangle^{-1} u\|_{L^2_x(\mathbb{R}^2)} \lesssim \|\partial_r u\|_{L^2_x(\mathbb{R}^2)}, \qquad \langle |x|-t \rangle^{-1} |u(t,x)| \lesssim \langle x \rangle^{-\frac{1}{2}} \|\partial \Gamma^{\leq 1} u\|_{L^2_x(\mathbb{R}^2)}.$$

Proof. The inequality (2.15) follows from integrating by parts:

$$\text{LHS of } (2.15) = - \int_0^{M+1} \frac{h^2}{2+M-\rho} d\rho - \int_0^{M+1} \frac{2hh'}{2+M-\rho} \rho d\rho. \tag{2.16}$$

The second inequality follows from (2.15) and the fact that $\langle |x|-t \rangle^{-2} \sim (2+t-|x|)^{-2}$ for $|x| \leq 1+t$. For the third inequality, consider first the case $|x| > 1$. By Lemma 2.2, we have

$$\langle |x|-t \rangle^{-1} |u(t,x)| \lesssim \langle x \rangle^{-\frac{1}{2}} \|\partial_r^{\leq 1} \partial_\theta^{\leq 1} (\langle r-t \rangle^{-1} u)\|_2 \sim \langle x \rangle^{-\frac{1}{2}} \|\partial_r^{\leq 1} (\langle r-t \rangle^{-1} \partial_\theta^{\leq 1} u)\|_2$$

$$\lesssim \langle x \rangle^{-\frac{1}{2}} \|\partial_r \partial_\theta^{\leq 1} u\|_2 \lesssim \langle x \rangle^{-\frac{1}{2}} \|\partial \Gamma^{\leq 1} u\|_2. \tag{2.17}$$

On the other hand, for $|x| \leq 1$, we have

$$\langle |x|-t \rangle^{-1} |u(t,x)| \lesssim \langle t \rangle^{-1} (\|u\|_{L^2_x(|x| \leq 1)} + \|\partial^2 u\|_{L^2_x(|x| \leq 1)})$$

$$\lesssim \|\langle |x|-t \rangle^{-1} u\|_{L^2_x(\mathbb{R}^2)} + \|\Delta u\|_{L^2_x(\mathbb{R}^2)} \lesssim \|\nabla u\|_2 + \|\Delta u\|_2. \quad \square$$

Lemma 2.4. *If g^{kij} satisfies the null condition, then for $t > 0$ we have*

$$g^{kij}\partial_k f\partial_{ij}h = g^{kij}(T_k f\partial_{ij}h - \omega_k\partial_t fT_i\partial_j h + \omega_k\omega_i\partial_t fT_j\partial_t h), \qquad (2.18)$$

where $T = (T_1, T_2)$ is defined in (2.5). It follows that

$$|g^{kij}\partial_k f\partial_{ij}h| \lesssim |Tf||\partial^2 h| + |\partial f||T\partial h| \qquad (2.19)$$

$$\lesssim \frac{1}{\langle r+t\rangle}(|\Gamma f||\partial^2 h| + |\partial f||\Gamma\partial h| + |\partial f|\cdot|\partial^2 h|\cdot|r-t|). \qquad (2.20)$$

Suppose g^{kij} satisfies the null condition and $\Box u = g^{kij}\partial_k u\partial_{ij}u$. Then for any multi-index α, we have

$$\Box\Gamma^\alpha u = \sum_{\alpha_1+\alpha_2\leq\alpha} g^{kij}_{\alpha;\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u, \qquad (2.21)$$

where for each $(\alpha, \alpha_1, \alpha_2)$, $g^{kij}_{\alpha;\alpha_1,\alpha_2}$ also satisfies the null condition. In addition, we have $g^{kij}_{\alpha;\alpha,0} = g^{kij}_{\alpha;0,\alpha} = g^{kij}$.

Proof. The identity (2.18) follows by applying repeatedly the identity $\partial_l = T_l - \omega_l\partial_t$ and using the null condition at the last step. The inequality (2.20) is obvious if $r \leq \frac{t}{2}$ or $r \geq 2t$, or $r \sim t \lesssim 1$ since $\langle r+t\rangle \sim \langle r-t\rangle$ in these regimes. On the other hand, if $r \sim t \gtrsim 1$, then one can use the identities

$$T_1 = \omega_1\partial_+ - \frac{\omega_2}{r}\partial_\theta, \quad T_2 = \omega_2\partial_+ + \frac{\omega_1}{r}\partial_\theta; \quad \partial_+ = \frac{1}{t+r}(2L_0 - (t-r)\partial_-). \quad (2.22)$$

The identity (2.21) follows from Hörmander [12]. $\quad\Box$

Lemma 2.5. *For any $f \in \mathcal{S}(\mathbb{R}^2)$, we have*

$$\sup_{x_0\in\mathbb{R}^2} \langle|x_0|-t\rangle^{\frac{1}{2}}|f(x_0)| \lesssim \|f\|_2 + \|\langle|x|-t\rangle\nabla f\|_2 + \|\langle|x|-t\rangle\partial_1\partial_2 f\|_2, \qquad \forall\, t \geq 0; \tag{2.23}$$

$$\|\langle|x|-t\rangle\partial f\|_\infty \lesssim \|\langle|x|-t\rangle\partial f\|_2 + \|\langle|x|-t\rangle\partial^2 f\|_2 + \|\langle|x|-t\rangle\partial^3 f\|_2, \quad \forall\, t \geq 0. \tag{2.24}$$

It follows that

$$\|f\|_{L_x^\infty(\mathbb{R}^2)} \lesssim \langle t\rangle^{-\frac{1}{2}}(\|f\|_2 + \|\partial_\theta f\|_2 + \|\langle|x|-t\rangle\nabla\tilde{\Gamma}^{\leq 1}f\|_2), \qquad \forall\, t \geq 0, \quad (2.25)$$

where $\tilde{\Gamma} = (\partial_1, \partial_2, \partial_\theta)$.

Proof. The case $||x_0|-t| \leq 2$ follows from the inequality $|f(x_0)|^2 \leq \int |\partial_1\partial_2(f(x)^2)|dx_1dx_2$. For $||x_0|-t| > 2$, we note that $\langle x_0\rangle + t \sim |x_0| + t \gtrsim 1$ and $\langle|x_0|-t\rangle \sim \frac{\langle|x_0|^2-t^2\rangle}{\langle x_0\rangle+t} =: W(x_0)$. Observe that

$$\sum_{1 \le i \le 2} \|\partial_i W\|_\infty + \sum_{1 \le i,j \le 2} \|\partial_i \partial_j W\|_\infty \lesssim 1, \quad W(x) \lesssim \langle |x| - t \rangle, \, \forall\, x \in \mathbb{R}^2. \quad (2.26)$$

By using the Fundamental Theorem of Calculus we have

$$\langle |x_0| - t \rangle |f(x_0)|^2 \lesssim W(x_0)|f(x_0)|^2 \lesssim \int_{\mathbb{R}^2} \left| \partial_1 \partial_2 \Big(W(x) f(x)^2 \Big) \right| dx_1 dx_2$$

$$\lesssim \|f\|_2^2 + \|\nabla f\|_2 \|f\|_2 + \|\langle |x| - t \rangle \nabla f\|_2 \|\nabla f\|_2 + \|\langle |x| - t \rangle \partial_1 \partial_2 f\|_2 \|f\|_2$$

$$\lesssim \|f\|_2^2 + \|\langle |x| - t \rangle \nabla f\|_2^2 + \|\langle |x| - t \rangle \partial_1 \partial_2 f\|_2^2. \quad (2.27)$$

Thus (2.23) follows. The proof of (2.24) is similar by working with the expression $W(x_0)^2 |\partial f(x_0)|^2$ for the case $|x_0 - t| > 2$. For (2.25) we may assume $t \ge 2$. The case $|x_0| \le t/2$ follows from (2.23). The case $|x_0| > t/2$ follows from Lemma 2.2. □

Lemma 2.6. *Suppose* $\tilde{u} = \tilde{u}(t, x)$ *has continuous second order derivatives. Then*

$$|\langle r - t \rangle \partial_{tt} \tilde{u}(t, x)| + |\langle r - t \rangle \partial_t \nabla \tilde{u}(t, x)| + |\langle r - t \rangle \Delta \tilde{u}(t, x)|$$

$$\lesssim |(\partial \Gamma^{\le 1} \tilde{u})(t, x)| + (r + t)|(\Box \tilde{u})(t, x)|, \quad r = |x|, \, t \ge 0; \quad (2.28)$$

and

$$|\langle r - t \rangle \partial^2 \tilde{u}(t, x)| \lesssim |(\partial \Gamma^{\le 1} \tilde{u})(t, x)| + (r + t)|(\Box \tilde{u})(t, x)|, \quad \forall\, r \ge t/10, \, t \ge 1. \quad (2.29)$$

Suppose $T_0 \ge 2$ *and* $u \in C^\infty([2, T_0] \times \mathbb{R}^2)$ *solves* (1.1) *with support in* $|x| \le t + 1$, $2 \le t \le T_0$. *For any integer* $l_0 \ge 2$, *there exists* $\epsilon_1 > 0$ *depending only on* l_0, *such that if at some* $2 \le t \le T_0$,

$$\|(\partial \Gamma^{\le \lceil \frac{l_0}{2} \rceil + 2} u)(t, \cdot)\|_{L_x^2(\mathbb{R}^2)} \le \epsilon_1, \qquad (\text{here } \lceil z \rceil = \min\{n \in \mathbb{N} : n \ge z\}) \quad (2.30)$$

then for the same t, *we have the* L^2 *estimate:*

$$\|(\langle r - t \rangle \partial^2 \Gamma^{\le l_0} u)(t, \cdot)\|_{L_x^2(\mathbb{R}^2)} \lesssim \|(\partial \Gamma^{\le l_0 + 1} u)(t, \cdot)\|_{L_x^2(\mathbb{R}^2)}. \quad (2.31)$$

For any integer $l_1 \ge 2$, *there exists* $\epsilon_2 > 0$ *depending only on* l_1, *such that if at some* $2 \le t \le T_0$,

$$\|(\partial \Gamma^{\le l_1 + 1} u)(t, \cdot)\|_{L_x^2(\mathbb{R}^2)} \le \epsilon_2, \quad (2.32)$$

then for the same t, *we have the point-wise estimate:*

$$|(\langle r - t \rangle \partial^2 \Gamma^{\le l_1} u)(t, x)| \lesssim |(\partial \Gamma^{\le l_1 + 1} u)(t, x)|, \qquad \forall\, r \ge t/10. \quad (2.33)$$

Moreover, we have

$$\|\partial\Delta\Gamma^{\le l_1-1}u\|_{L^2_x(|x|\le\frac{2}{3}t)} \lesssim t^{-2}\|(\partial\Gamma^{\le l_1+1}u)(t,\cdot)\|_{L^2_x(\mathbb{R}^2)}. \tag{2.34}$$

Proof. In the 3D case, the estimate (2.28) is an elementary but deep observation of Sideris (cf. [17]). To prove the 2D case we denote $Y = |(\partial\Gamma^{\le1}\tilde{u})(t,x)| + (r+t)|(\Box\tilde{u})(t,x)|$. By using $L_0\tilde{u} = t\partial_t\tilde{u} + r\partial_r\tilde{u}$, we obtain

$$\partial_t L_0\tilde{u} = \partial_t\tilde{u} + t\partial_{tt}\tilde{u} + r\partial_t\partial_r\tilde{u} \Rightarrow |r\partial_t\partial_r\tilde{u} + t\partial_{tt}\tilde{u}| \lesssim Y; \tag{2.35}$$

$$\partial_r L_0\tilde{u} = t\partial_t\partial_r\tilde{u} + \partial_r\tilde{u} + r\partial_{rr}\tilde{u} \Rightarrow |t\partial_t\partial_r\tilde{u} + r\partial_{rr}\tilde{u}| \lesssim Y. \tag{2.36}$$

Since $\Box\tilde{u} = \partial_{tt}\tilde{u} - \partial_{rr}\tilde{u} - \frac{1}{r}\partial_r\tilde{u} - \frac{1}{r^2}\partial_{\theta\theta}\tilde{u}$ and $|\frac{1}{r}\partial_{\theta\theta}\tilde{u}| \lesssim |\nabla\partial_\theta\tilde{u}| \lesssim Y$, we have

$$|r\partial_{tt}\tilde{u} - r\partial_{rr}\tilde{u}| \lesssim Y. \tag{2.37}$$

It follows that

$$\langle r-t\rangle(|\partial_{tt}\tilde{u}| + |\partial_t\partial_r\tilde{u}|) \lesssim Y. \tag{2.38}$$

By using $\partial_\theta L_0\tilde{u} = t\partial_t\partial_\theta\tilde{u} + r\partial_r\partial_\theta\tilde{u} = (t-r)\partial_t\partial_\theta\tilde{u} + r(\partial_t + \partial_r)\partial_\theta\tilde{u}$, we obtain

$$|(t-r)\partial_t(\frac{1}{r}\partial_\theta\tilde{u})| \lesssim Y. \tag{2.39}$$

The estimates of $\partial_t\partial_r\tilde{u}$ and $\partial_t(\frac{1}{r}\partial_\theta\tilde{u})$ settle the point-wise estimate of $\partial_t\nabla\tilde{u}$. It follows that

$$\langle r-t\rangle(|\partial_{tt}\tilde{u}| + |\partial_t\nabla\tilde{u}| + |\Delta\tilde{u}|) \lesssim Y \tag{2.40}$$

which is exactly (2.28). To derive the estimate (2.29) we only need to bound $|(r-t)\partial_i\partial_j\tilde{u}|$ for $1 \le i, j \le 2$. By using the identity $\nabla = \omega\partial_r + \frac{\omega^\perp}{r}\partial_\theta$ where $\omega = (\cos\theta, \sin\theta)$, $\omega^\perp = (-\sin\theta, \cos\theta)$, it is not difficult to check that for $r \gtrsim t$,

$$\sum_{1\le i,j\le2} |(r-t)\partial_i\partial_j\tilde{u}| \lesssim |(r-t)\partial_{rr}\tilde{u}| + Y$$

$$\lesssim |(r-t)(\Delta\tilde{u} - \frac{1}{r}\partial_r\tilde{u} - \frac{1}{r^2}\partial_{\theta\theta}\tilde{u})| + Y \lesssim Y. \tag{2.41}$$

Thus (2.29) easily follows.

For (2.31), by using a simple integration-by-parts argument, one has (below $k_0 \ge 0$ is a running parameter)

$$\sum_{i,j=1}^2 \|\langle r-t\rangle\partial_i\partial_j\Gamma^{\le k_0}u\|_2 \lesssim \|\partial\Gamma^{\le k_0}u\|_2 + \|\langle r-t\rangle\Delta\Gamma^{\le k_0}u\|_2. \tag{2.42}$$

By using (2.28) and (2.20) we have

$$|(\langle r - t\rangle \partial_{tt}\Gamma^{\leq k_0}u)(t,x)| + |(\langle r - t\rangle \partial_t \nabla \Gamma^{\leq k_0}u)(t,x)| + |(\langle r - t\rangle \Delta\Gamma^{\leq k_0}u)(t,x)|$$
$$\lesssim |(\partial\Gamma^{\leq k_0+1}u)(t,x)|$$
$$+ \sum_{m+l\leq k_0}(|\Gamma^{\leq m+1}u||\partial^2\Gamma^{\leq l}u| + |\partial\Gamma^{\leq m}u||\Gamma^{\leq l+1}\partial u| + |\partial\Gamma^{\leq m}u||\partial^2\Gamma^{\leq l}u||r-t|).$$
$$(2.43)$$

By (2.42), we obtain

$$\|\langle r - t\rangle\partial^2\Gamma^{\leq k_0}u\|_2 \lesssim \|\partial\Gamma^{\leq k_0+1}u\|_2$$
$$+ \sum_{m+l\leq k_0}(\||\Gamma^{\leq m+1}u||\partial^2\Gamma^{\leq l}u|\|_2 + \||\partial\Gamma^{\leq m}u||\Gamma^{\leq l+1}\partial u|\|_2 + \||\partial\Gamma^{\leq m}u||\partial^2\Gamma^{\leq l}u||r-t|\|_2).$$
$$(2.44)$$

If $m \leq l + 1$, then we use the estimates (note that $m + 2 \leq \lfloor\frac{k_0+1}{2}\rfloor + 2 \leq \lceil\frac{k_0}{2}\rceil + 2$)

$$\langle r - t\rangle^{-1}|(\Gamma^{\leq m+1}u)(t,x)| \lesssim \|\partial\Gamma^{\leq m+2}u\|_2, \quad \|\partial\Gamma^{\leq m}u\|_\infty \lesssim \|\partial\Gamma^{\leq m+2}u\|_2. \quad (2.45)$$

If $m \geq l + 2$, then $l \leq \frac{k_0-2}{2}$ and we use the estimates (see (2.24) for the second estimate)

$$\left\|\frac{|\Gamma^{\leq m+1}u|}{\langle r - t\rangle}\right\|_2 \lesssim \|\partial\Gamma^{\leq m+1}u\|_2, \quad |\langle r - t\rangle\partial^2\Gamma^{\leq l}u(t,x)| \lesssim \|\langle r - t\rangle\partial^2\Gamma^{\leq l+2}u\|_2. (2.46)$$

Thus if $\|\partial\Gamma^{\leq k_0+1}u\|_2 \ll 1$, we obtain

$$\|\langle r - t\rangle\partial^2\Gamma^{\leq k_0}u(t,\cdot)\|_2 \lesssim \|\partial\Gamma^{\leq k_0+1}u\|_2 \ll 1. \quad (2.47)$$

To prove (2.31) under the assumption (2.30) we first take $k_0 = \lceil\frac{l_0}{2}\rceil + 1$ and show that

$$\|\langle r - t\rangle\partial^2\Gamma^{\leq\lceil\frac{l_0}{2}\rceil+1}u)(t,\cdot)\|_2 \lesssim \|(\partial\Gamma^{\leq\lceil\frac{l_0}{2}\rceil+2}u)(t,\cdot)\|_2 \ll 1. \quad (2.48)$$

We then use this smallness in (2.46) and obtain the desired result for $k_0 = l_0$ (note that $\lceil\frac{l_0-2}{2}\rceil + 2 \leq \lceil\frac{l_0}{2}\rceil + 1$). The estimate of (2.33) follows from (2.29).

We turn now to (2.34). Applying (2.28) to $\tilde{u} = \partial\Gamma^{\leq l_1-1}u$ with $r \leq \frac{2}{3}t$, we get

$$|\Delta\partial\Gamma^{\leq l_1-1}u| \lesssim \frac{1}{t}|\partial^2\Gamma^{\leq l_1}u| + |\partial\Box\Gamma^{\leq l_1-1}u|. \quad (2.49)$$

By Lemma 2.4, we have

$$|\partial\Box\Gamma^{\leq l_1-1}u| \lesssim \sum_{a+b\leq l_1-1}|\partial(\partial\Gamma^a u\partial^2\Gamma^b u)|$$
$$\lesssim |\partial^2\Gamma^{\leq l_1-1}u||\partial^2\Gamma^{\leq l_1-1}u| + |\partial\Gamma^{\leq l_1-1}u||\partial^3\Gamma^{\leq l_1-1}u|. \quad (2.50)$$

Note that

$$|\partial^3 \Gamma^{\leq l_1 - 1} u| \lesssim | \underbrace{\partial_{tt} \partial \Gamma^{\leq l_1 - 1} u}_{\partial_t \text{ appears twice or more}} | + \sum_{1 \leq i_1, i_2 \leq 2} | \underbrace{\partial \partial_{i_1} \partial_{i_2} \Gamma^{\leq l_1 - 1} u}_{\partial_t \text{ appears at most once}} |$$

$$\lesssim |\partial \Box \Gamma^{\leq l_1 - 1} u| + |\partial \tilde{\partial}^2 \Gamma^{\leq l_1 - 1} u|, \tag{2.51}$$

where we have denoted $\tilde{\partial} = (\partial_1, \partial_2)$. By using the smallness of the pre-factor $\|\partial \Gamma^{\leq l_1 - 1} u\|_\infty$ and (2.51), we then derive from (2.50)

$$|\partial \Box \Gamma^{\leq l_1 - 1} u| \lesssim |\partial^2 \Gamma^{\leq l_1 - 1} u||\partial^2 \Gamma^{\leq l_1 - 1} u| + |\partial \Gamma^{\leq l_1 - 1} u||\partial \tilde{\partial}^2 \Gamma^{\leq l_1 - 1} u|. \tag{2.52}$$

By the standard Sobolev embedding $H^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$, we have

$$\|\langle r - t \rangle \partial^2 \Gamma^{\leq l_1 - 1} u\|_4 \lesssim \|\partial^{\leq 1}(\langle r - t \rangle \partial^2 \Gamma^{\leq l_1 - 1} u)\|_2 \lesssim \|(\partial \Gamma^{\leq l_1 + 1} u)(t, \cdot)\|_2; \tag{2.53}$$

$$\|\langle r - t \rangle^{\frac{1}{2}} \partial \Gamma^{\leq l_1 - 1} u\|_\infty \lesssim \|(\partial \Gamma^{\leq l_1 + 1} u)(t, \cdot)\|_2 \qquad \text{(by Lemma 2.5)}. \tag{2.54}$$

By using a smooth cut-off function localized to $|x| \leq \frac{2}{3} t$, we then derive

$$\|\Delta \partial \Gamma^{\leq l_1 - 1} u\|_{L_x^2(|x| \leq \frac{2}{3} t)} \lesssim t^{-\frac{3}{2}} \|(\partial \Gamma^{\leq l_1 + 1} u)(t, \cdot)\|_2. \tag{2.55}$$

It follows that (recall $\tilde{\partial} = (\partial_1, \partial_2)$)

$$\|\tilde{\partial}^2 \partial \Gamma^{\leq l_1 - 1} u\|_{L_x^2(|x| \leq \frac{2}{3} t)} \lesssim t^{-\frac{3}{2}} \|(\partial \Gamma^{\leq l_1 + 1} u)(t, \cdot)\|_2. \tag{2.56}$$

Plugging this estimate into (2.52), we obtain the estimate (2.34). $\quad \Box$

Lemma 2.7 *(Decay estimates). Suppose $T_0 \geq 2$ and $u \in C^\infty([2, T_0] \times \mathbb{R}^2)$ solves (1.1) with support in $|x| \leq t + 1$, $2 \leq t \leq T_0$. Suppose $I \geq 4$ is an integer and*

$$E_I(u(t, \cdot)) = \|(\partial \Gamma^{\leq I} u)(t, \cdot)\|_2^2 \leq \tilde{\epsilon}, \tag{2.57}$$

where $\tilde{\epsilon} > 0$ is sufficiently small. Then we have the following decay estimates:

$$t^{\frac{1}{2}} \|\partial \Gamma^{\leq I-2} u\|_{L_x^\infty} + t^{\frac{1}{2}} \|\langle |x| - t \rangle \partial^2 \Gamma^{\leq I-3} u\|_{L_x^\infty(|x| > \frac{t}{10})} + \|\langle |x| - t \rangle \partial^2 \Gamma^{\leq I-3} u\|_{L_x^\infty} \lesssim E_I^{\frac{1}{2}};$$

$$\tag{2.58}$$

$$\|\partial^2 \Gamma^{\leq I-3} u\|_{L_x^\infty(|x| < \frac{t}{2})} \lesssim t^{-\frac{3}{2}} E_I^{\frac{1}{2}}; \tag{2.59}$$

$$\|\langle |x| - t \rangle \partial^2 \Gamma^{\leq I-3} u\|_{L_x^\infty} \lesssim t^{-\frac{1}{2}} E_I^{\frac{1}{2}}; \tag{2.60}$$

$$\left\| \frac{T \Gamma^{\leq I-2} u}{\langle |x| - t \rangle} \right\|_{L_x^\infty} + \|T \partial \Gamma^{\leq I-3} u\|_{L_x^\infty} \lesssim t^{-\frac{3}{2}} E_I^{\frac{1}{2}}; \tag{2.61}$$

$$\left\| \frac{T \Gamma^{\leq I-1} u}{\langle |x| - t \rangle} \right\|_{L_x^2} + \|T \partial \Gamma^{\leq I-1} u\|_{L_x^2} \lesssim t^{-1} E_I^{\frac{1}{2}}. \tag{2.62}$$

More generally, for any integer $I_1 \geq 1$, we have

$$\|\frac{T\Gamma^{\leq I_1}u}{\langle |x|-t\rangle}\|_{L_x^2} \lesssim t^{-1}\|\partial\Gamma^{\leq I_1+1}u\|_{L_x^2}. \tag{2.63}$$

Also we have

$$\|\langle |x|-t\rangle^2\partial^3 u\|_{L_x^\infty(|x|\geq\frac{t}{2})} \lesssim t^{-\frac{1}{2}}. \tag{2.64}$$

Proof. We shall take $\tilde{\epsilon}$ sufficiently small so that Lemma 2.6 can be applied. The estimate (2.58) follows from Lemma 2.5 and Lemma 2.6. To derive the estimate (2.59), we choose $\psi \in C_c^\infty(\mathbb{R}^2)$ such that $\psi(z) \equiv 1$ for $|z| \leq 0.5$ and $\psi(z) \equiv 0$ for $|z| \geq 0.52$. Applying the interpolation inequality $\|\tilde{v}\|_\infty \lesssim \|\tilde{v}\|_2^{\frac{1}{2}}\|\Delta\tilde{v}\|_2^{\frac{1}{2}}$ with $\tilde{v}(x) = \psi(\frac{x}{t})\partial^2\Gamma^{\leq J-3}u$, we obtain

$$\|\psi(\frac{x}{t})\partial^2\Gamma^{\leq J-3}u\|_\infty \lesssim \|\psi(\frac{x}{t})\partial^2\Gamma^{\leq I-3}u\|_2^{\frac{1}{2}}\|\Delta(\psi(\frac{x}{t})\partial^2\Gamma^{\leq I-3}u)\|_2^{\frac{1}{2}}. \tag{2.65}$$

By Lemma 2.6, it is not difficult to check that

$$\|\Delta(\psi(\frac{x}{t})\partial^2\Gamma^{\leq I-3}u)\|_2 \lesssim t^{-2}E_I^{\frac{1}{2}}, \quad \|\psi(\frac{x}{t})\partial^2\Gamma^{\leq I-3}u\|_2 \lesssim t^{-1}E_I^{\frac{1}{2}}. \tag{2.66}$$

The estimate (2.59) then follows. For the estimate (2.60) we only need to examine the regime $|x| \geq t/2$. But this follows from Lemma 2.6 and 2.5.

For (2.61), we note that the case $|x| \leq \frac{t}{2}$ follows from (2.58) and (2.59). On the other hand, for $|x| > \frac{t}{2}$ we denote $\tilde{u} = \Gamma^{\leq I-2}u$ and estimate $\|\frac{T_1\tilde{u}}{\langle |x|-t\rangle}\|_{L_x^\infty(|x|>\frac{t}{2})}$ (the estimate for T_2 is similar). Recall that

$$T_1\tilde{u} = \omega_1\partial_t\tilde{u} + \partial_1\tilde{u} = \omega_1(\partial_t + \partial_r)\tilde{u} - \frac{\omega_2}{r}\partial_\theta\tilde{u}$$

$$= \omega_1\frac{1}{t+r}(2L_0\tilde{u} - (t-r)\partial_-\tilde{u}) - \frac{\omega_2}{r}\partial_\theta\tilde{u}. \tag{2.67}$$

Clearly for $r = |x| \geq \frac{t}{2}$,

$$\left|\frac{T_1\tilde{u}}{\langle r-t\rangle}\right| \lesssim \frac{1}{t}\left(\left|\frac{L_0\tilde{u}}{\langle r-t\rangle}\right| + |\partial\tilde{u}|\right) + \left|\frac{\partial_\theta\tilde{u}}{r\langle r-t\rangle}\right|$$

$$\lesssim t^{-1}\cdot t^{-\frac{1}{2}}\|\partial\Gamma^{\leq 2}\tilde{u}\|_2 + t^{-\frac{3}{2}} + t^{-1}\cdot t^{-\frac{1}{2}}\|\partial\Gamma^{\leq 1}\partial_\theta\tilde{u}\|_2 \lesssim t^{-\frac{3}{2}}E_I^{\frac{1}{2}}, \tag{2.68}$$

where in the second last step we used Lemma 2.3 (for the term $|\partial\tilde{u}|$ we use (2.58)). The estimates for (2.62)–(2.63) are similar. We omit the details. We now sketch how to prove (2.64). By using (2.29) (applied to $\tilde{u} = \partial u$), we obtain

$$|\langle r-t\rangle\partial^3 u| \lesssim |\partial^2\Gamma^{\leq 1}u| + (r+t)|\Box\partial u|. \tag{2.69}$$

The contribution of the term $|\partial^2\Gamma^{\leq 1}u|$ is clearly OK for us since it can absorb a factor of $\langle r - t\rangle$. By Lemma 2.4 (with $f = \partial u$, $h = u$ or $f = u$, $h = \partial u$), we have

$$
(r+t)|\Box \partial u| \lesssim |\Gamma \partial u||\partial^2 u| + |\partial^2 u|^2|r-t| + |\Gamma u||\partial^3 u| + |\partial u||\Gamma \partial^2 u| + |\partial u||\partial^3 u||r-t|
$$

$$
\lesssim |\partial\Gamma^{\leq 1}u||\partial^2\Gamma^{\leq 1}u| + |\partial^2 u|^2|r-t| + \left|\frac{\Gamma u}{\langle r - t\rangle}\right| \cdot \langle r - t\rangle|\partial^3 u|
$$

$$
+ |\partial u||\partial^3 u||r - t|. \tag{2.70}
$$

The desired estimate clearly follows by using smallness of the pre-factors. $\square$

3. Proof of Theorem 1.1

In this section and later sections, we carry out the proof of Theorem 1.1. We fix a multi-index α with $|\alpha| \leq m$ and for simplicity denote $v = \Gamma^\alpha u$. By Lemma 2.4 we have (below for simplicity of notation we write $g^{kij}_{\alpha_1,\alpha_2} = g^{kij}_{\alpha;\alpha_1,\alpha_2}$)

$$
\Box v = \sum_{\alpha_1+\alpha_2\leq\alpha} g^{kij}_{\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u \tag{3.1}
$$

$$
= g^{kij}\partial_k v\partial_{ij}u + g^{kij}\partial_k u\partial_{ij}v + \sum_{\substack{\alpha_1<\alpha,\alpha_2<\alpha;\\ \alpha_1+\alpha_2\leq\alpha}} g^{kij}_{\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u. \tag{3.2}
$$

Choose $p(t, r) = q(r - t)$, where

$$
q(s) = \int_0^s \langle\tau\rangle^{-1}\left(\log(2+\tau^2)\right)^{-2}d\tau, \quad s \in \mathbb{R}. \tag{3.3}
$$

Clearly

$$
-\partial_t p = \partial_r p = q'(r - t) = \langle r - t\rangle^{-1}\left(\log(2+(r-t)^2)\right)^{-2}. \tag{3.4}
$$

Multiplying both sides of (3.1) by $e^p\partial_t v$, we obtain

$$
\text{LHS} = \int e^p\partial_{tt}v\partial_t v - \int e^p\Delta v\partial_t v = \int e^p\partial_{tt}v\partial_t v + \int e^p\nabla v\cdot\nabla\partial_t v + \int e^p\nabla v\cdot\nabla p\partial_t v
$$

$$
= \frac{1}{2}\frac{d}{dt}\int e^p(\partial v)^2 - \frac{1}{2}\int e^p|\partial v|^2 p_t + \int e^p\nabla v\cdot\nabla p\partial_t v
$$

$$
= \frac{1}{2}\frac{d}{dt}\|e^{\frac{p}{2}}\partial v\|_{L^2}^2 + \frac{1}{2}\int e^p q'\cdot\left(|\partial_+v|^2 + \frac{|\partial_\theta v|^2}{r^2}\right) = \frac{1}{2}\frac{d}{dt}\|e^{\frac{p}{2}}\partial v\|_{L^2}^2 + \frac{1}{2}\int e^p q'|Tv|^2.
$$

To simplify the notation in the subsequent nonlinear estimates, we introduce the following terminology.

Notation. For a quantity $X(t)$, we shall write $X(t) = \text{OK}$ if $X(t)$ can be written as

$$X(t) = \frac{d}{dt}X_1(t) + X_2(t) + X_3(t), \tag{3.5}$$

where (below $\alpha_0 > 0$ is some constant)

$$|X_1(t)| \ll \|(\partial\Gamma^{\leq m}u)(t,\cdot)\|^2_{L^2_x(\mathbb{R}^2)}, \quad |X_2(t)| \ll \sum_{|\alpha|\leq m}\int e^p q' |(T\Gamma^\alpha u)(t,x)|^2 dx,$$

$$|X_3(t)| \lesssim \langle t\rangle^{-1-\alpha_0}. \tag{3.6}$$

In yet other words, the quantity X will be controllable if either it can be absorbed into the energy, or can be controlled by the weighted L^2-norm of the good unknowns from the Alinhac weight, or it is integrable in time.

We now proceed with the nonlinear estimates. We shall discuss several cases.

3.1. The case $\alpha_1 < \alpha$ and $\alpha_2 < \alpha$

Since $g^{kij}_{\alpha_1,\alpha_2}$ still satisfies the null condition, by (2.18) we have

$$\sum_{\substack{\alpha_1<\alpha,\alpha_2<\alpha \\ \alpha_1+\alpha_2\leq\alpha}} g^{kij}_{\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u$$

$$= \sum_{\substack{\alpha_1<\alpha,\alpha_2<\alpha \\ \alpha_1+\alpha_2\leq\alpha}} g^{kij}_{\alpha_1,\alpha_2}(T_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u - \omega_k\partial_t\Gamma^{\alpha_1}uT_i\partial_j\Gamma^{\alpha_2}u + \omega_k\omega_i\partial_t\Gamma^{\alpha_1}uT_j\partial_t\Gamma^{\alpha_2}u). \tag{3.7}$$

`Estimate of` $\|T_k\Gamma^{\alpha_1}u\partial^2\Gamma^{\alpha_2}u\|_2$. If $|\alpha_1| \leq |\alpha_2|$, then by Lemma 2.7 we have

$$\|T_k\Gamma^{\alpha_1}u\partial^2\Gamma^{\alpha_2}u\|_2 \lesssim \|\frac{T_k\Gamma^{\alpha_1}u}{\langle r-t\rangle}\|_\infty \cdot \|\langle r-t\rangle\partial^2\Gamma^{\alpha_2}u\|_2 \lesssim t^{-\frac{3}{2}}. \tag{3.8}$$

If $|\alpha_1| > |\alpha_2|$, then we have

$$\|T_k\Gamma^{\alpha_1}u\partial^2\Gamma^{\alpha_2}u\|_2 \lesssim \|\frac{T_k\Gamma^{\alpha_1}u}{\langle r-t\rangle}\|_2 \cdot \|\langle r-t\rangle\partial^2\Gamma^{\alpha_2}u\|_\infty \lesssim t^{-1}\cdot t^{-\frac{1}{2}} \lesssim t^{-\frac{3}{2}}. \tag{3.9}$$

`Estimate of` $\|\partial\Gamma^{\alpha_1}uT\partial\Gamma^{\alpha_2}u\|_2$. If $|\alpha_1| \leq |\alpha_2|$ we have

$$\|\partial\Gamma^{\alpha_1}uT\partial\Gamma^{\alpha_2}u\|_2 \lesssim \|\partial\Gamma^{\alpha_1}u\|_\infty \cdot \|T\partial\Gamma^{\alpha_2}u\|_2 \lesssim t^{-\frac{3}{2}}. \tag{3.10}$$

If $|\alpha_1| > |\alpha_2|$ we have

$$\|\partial\Gamma^{\alpha_1}uT\partial\Gamma^{\alpha_2}u\|_2 \lesssim \|\partial\Gamma^{\alpha_1}u\|_2 \cdot \|T\partial\Gamma^{\alpha_2}u\|_\infty \lesssim t^{-\frac{3}{2}}. \tag{3.11}$$

Collecting the estimates, we have proved

$$\|\sum_{\substack{\alpha_1<\alpha,\alpha_2<\alpha \\ \alpha_1+\alpha_2\le\alpha}} g_{\alpha_1,\alpha_2}^{kij}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u\|_2 \lesssim t^{-\frac{3}{2}}. \tag{3.12}$$

3.2. The case $\alpha_2 = \alpha$

Noting that $g_{0,\alpha}^{kij} = g^{kij}$, we have

$$\int g^{kij}\partial_k u\partial_{ij}v\partial_t v e^p = \text{OK} - \underbrace{\int g^{kij}\partial_{jk}u\partial_i v\partial_t v e^p}_{I_1} - \underbrace{\int g^{kij}\partial_k u\partial_i v\partial_t v\partial_j(e^p)}_{I_2}$$
$$- \int g^{kij}\partial_k u\partial_i v\partial_{tj}v e^p. \tag{3.13}$$

Here in the above, the term "OK" is zero if $\partial_j = \partial_1$ or ∂_2. This term is nonzero when $\partial_j = \partial_t$, i.e. we should absorb it into the energy when integrating by parts in the time variable.

Further integration by parts gives

$$- \int g^{kij}\partial_k u\partial_i v\partial_{tj}v e^p = \text{OK} + \underbrace{\int g^{kij}\partial_{tk}u\partial_i v\partial_j v e^p}_{I_3} + \underbrace{\int g^{kij}\partial_k u\partial_i v\partial_j v\partial_t(e^p)}_{I_4}$$
$$+ \int g^{kij}\partial_k u\partial_{it}v\partial_j v e^p. \tag{3.14}$$

$$\int g^{kij}\partial_k u\partial_{it}v\partial_j v e^p = \text{OK} - \underbrace{\int g^{kij}\partial_{ik}u\partial_t v\partial_j v e^p}_{I_5} - \underbrace{\int g^{kij}\partial_k u\partial_t v\partial_j v\partial_i(e^p)}_{I_6}$$
$$- \int g^{kij}\partial_k u\partial_t v\partial_{ij}v e^p. \tag{3.15}$$

It follows that

$$2\int g^{kij}\partial_k u\partial_{ij}v\partial_t v e^p = (I_1 + I_3 + I_5) + (I_2 + I_4 + I_6) + \text{OK}.$$

Observe that if $\varphi = \partial_k u$ or $\varphi = e^p$, then

$$- \partial_j\varphi\partial_i v\partial_t v + \partial_t\varphi\partial_i v\partial_j v - \partial_i\varphi\partial_t v\partial_j v$$
$$= - T_j\varphi\partial_i v\partial_t v + \omega_j\partial_t\varphi\partial_i v\partial_t v + \partial_t\varphi\partial_i v\partial_j v - T_i\varphi\partial_t v\partial_j v + \omega_i\partial_t\varphi\partial_t v T_j v - \omega_i\omega_j\partial_t\varphi(\partial_t v)^2$$
$$= - T_j\varphi\partial_i v\partial_t v + \partial_t\varphi\partial_i v T_j v - T_i\varphi\partial_t v\partial_j v + \omega_i\partial_t\varphi\partial_t v T_j v - \omega_i\omega_j\partial_t\varphi(\partial_t v)^2$$
$$= - T_j\varphi\partial_i v\partial_t v + \partial_t\varphi T_i v T_j v - T_i\varphi\partial_t v\partial_j v - \omega_i\omega_j\partial_t\varphi(\partial_t v)^2. \tag{3.16}$$

By (3.16) and rewriting $\partial_t \varphi = \partial_k \partial_t u = T_k \partial_t u - \omega_k \partial_{tt} u$, we have

$$I_1 + I_3 + I_5 = \int g^{kij}(-T_j \partial_k u \partial_i v \partial_t v + \partial_t \partial_k u T_i v T_j v - T_i \partial_k u \partial_t v \partial_j v$$
$$- \omega_i \omega_j T_k \partial_t u (\partial_t v)^2) e^p dx. \tag{3.17}$$

By Lemma 2.7, we have $\|T \partial u\|_\infty \lesssim t^{-\frac{3}{2}}$ and $\|\langle r - t\rangle \partial^2 u\|_\infty \lesssim t^{-\frac{1}{2}}$. Clearly then

$$\int\limits_{r < \frac{t}{2} \text{ or } r > 2t} |\partial^2 u||Tv|^2 dx \lesssim t^{-\frac{3}{2}}, \quad \int\limits_{r \sim t} |\partial^2 u||Tv|^2 dx \ll \int e^p q'|Tv|^2 dx. \tag{3.18}$$

It follows that

$$I_1 + I_3 + I_5 = \text{OK}. \tag{3.19}$$

Plugging $\varphi = e^p$ in (3.16) and noting that $T_j(e^p) = 0$, we have

$$I_2 + I_4 + I_6 = \int g^{kij} \partial_k u \Big(-T_j(e^p)\partial_i v \partial_t v - T_i(e^p)\partial_t v \partial_j v$$
$$- \omega_i \omega_j (\partial_t v)^2 \partial_t(e^p) + \partial_t(e^p) T_i v T_j v \Big)$$
$$= \int g^{kij} \left(-T_k u \cdot \omega_i \omega_j (\partial_t v)^2 \partial_t(e^p) + \partial_k u \partial_t(e^p) T_i v T_j v \right).$$

By Lemma 2.7 we have $\left| |Tu||\partial_t(e^p)| \right| \lesssim t^{-\frac{3}{2}}$. Clearly

$$\|\partial u \partial_t(e^p)\|_{L_x^\infty (r < \frac{t}{2}, \text{ or } r > 2t)} \lesssim t^{-\frac{3}{2}}, \quad \int\limits_{r \sim t} |\partial u \partial_t(e^p)||Tv|^2 dx \ll \int e^p q'|Tv|^2 dx. \tag{3.20}$$

Thus

$$I_2 + I_4 + I_6 = \text{OK}.$$

This concludes the case $\alpha_2 = \alpha$. In the next section we deal with the main piece $\alpha_1 = \alpha$.

4. Estimate of the main piece $\alpha_1 = \alpha$, $\alpha_2 = 0$

In this section we estimate the main piece $\alpha_1 = \alpha$. By (2.18), we have

$$\int g^{kij} \partial_k v \partial_{ij} u \partial_t v e^p = \int g^{kij}(T_k v \partial_{ij} u - \omega_k \partial_t v T_i \partial_j u + \omega_k \omega_i \partial_t v T_j \partial_t u)\partial_t v e^p$$
$$= \int g^{kij}(T_k v T_i \partial_j u - \omega_i T_k v T_j \partial_t u + \omega_i \omega_j T_k v \partial_{tt} u$$
$$- \omega_k \partial_t v T_i \partial_j u + \omega_k \omega_i \partial_t v T_j \partial_t u)\partial_t v e^p.$$

By Lemma 2.7, all terms containing $T\partial u$ decay as $O(t^{-\frac{3}{2}})$. Thus

$$\int g^{kij}\partial_k v\partial_{ij}u\partial_t v e^p = \text{OK} + \int g^{kij}\omega_i\omega_j T_k v\partial_{tt}u\partial_t v e^p. \tag{4.1}$$

Recall $T_0 = 0$, $T_1 = \omega_1\partial_+ - \frac{\omega_2}{r}\partial_\theta$, $T_2 = \omega_2\partial_+ + \frac{\omega_1}{r}\partial_\theta$. We have

$$g^{kij}\omega_i\omega_j T_k v = g^{1ij}\omega_i\omega_j(\omega_1\partial_+ v - \frac{\omega_2}{r}\partial_\theta v) + g^{2ij}\omega_i\omega_j(\omega_2\partial_+ v + \frac{\omega_1}{r}\partial_\theta v)$$

$$= (g^{1ij}\omega_1\omega_i\omega_j + g^{2ij}\omega_2\omega_i\omega_j)\partial_+ v + \omega_i\omega_j(g^{2ij}\omega_1 - g^{1ij}\omega_2)\frac{1}{r}\partial_\theta v$$

$$=: h_1(\theta)\partial_+ v + h_2(\theta)\frac{1}{r}\partial_\theta v.$$

We first estimate the piece

$$\int h_1(\theta)\partial_+ v\partial_{tt}u\partial_t v e^p. \tag{4.2}$$

The other piece will be estimated in the next section.

Choose nonnegative radial $\tilde{\phi}_1 \in C_c^\infty(\mathbb{R}^2)$ such that $\tilde{\phi}_1(z) = 1$ for $\frac{2}{3} \le |z| \le \frac{3}{2}$ and $\tilde{\phi}_1(z) = 0$ for $|z| \le \frac{1}{3}$ or $|z| \ge 2$. Denote $\phi(x) = \tilde{\phi}_1(\frac{x}{t})$. Then

$$\int h_1(\theta)\partial_+ v\partial_{tt}u\partial_t v e^p = \int h_1(\theta)\partial_+ v\partial_{tt}u\partial_t v e^p \cdot (1 - \phi) + \int h_1(\theta)\partial_+ v\partial_{tt}u\partial_t v e^p \phi. \tag{4.3}$$

By Lemma 2.7, we have

$$\int h(\theta)\partial_+ v\partial_{tt}u\partial_t v e^p \cdot (1 - \phi) \lesssim t^{-1}\int |\partial v|^2|\langle r - t\rangle\partial_{tt}u| \lesssim t^{-\frac{3}{2}} = \text{OK}. \tag{4.4}$$

By using the identity $\partial_t = \frac{\partial_+ + \partial_-}{2}$ and the fact that $\|\langle r - t\rangle\partial^2 u\|_\infty \lesssim t^{-\frac{1}{2}}$, we get

$$2\int h_1(\theta)\partial_+ v\partial_{tt}u\partial_t v e^p\phi = \int h_1(\theta)\partial_+ v\partial_{tt}u\partial_+ v e^p\phi + \int h_1(\theta)\partial_+ v\partial_{tt}u\partial_- v e^p\phi$$

$$= \text{OK} + \int h_1(\theta)\partial_+ v\partial_{tt}u\partial_- v e^p\phi. \tag{4.5}$$

Integrating by parts, we have

$$\int h_1(\theta)\partial_+ v\partial_{tt}u\partial_- v e^p\phi \cdot r dr d\theta$$

$$= \frac{d}{dt}\int h_1(\theta)v\partial_{tt}u\partial_- v e^p\phi dx - \int h_1(\theta)v\partial_- v\partial_+\left(\partial_{tt}u e^p\phi\right)dx$$

$$- \int h_1(\theta)v\partial_{tt}u\partial_+\partial_- v e^p\phi dx - \int h_1(\theta)v\partial_{tt}u\partial_- v e^p\phi\frac{1}{r}dx. \tag{4.6}$$

In the above computation, one should note that when integrating by parts in r we should take into consideration the factor r in the metric rdr. The fourth term exactly corresponds to the derivative of the metric factor. The first and fourth terms are clearly acceptable by using Hardy and the decay of $\langle r - t \rangle \partial_{tt} u$. For the second term we have

$$
\begin{aligned}
|\langle r - t \rangle \partial_+ (\partial_{tt} u e^p \phi)| &\lesssim |\langle r - t \rangle \partial_+ \partial_{tt} u \phi| + |\langle r - t \rangle \partial_{tt} u \partial_+ \phi| \\
&\lesssim t^{-1} \| 2 \langle r - t \rangle L_0 \partial_{tt} u - \langle r - t \rangle (t - r) \partial_- \partial_{tt} u \|_{L_x^\infty (|x| > \frac{t}{10})} + t^{-\frac{3}{2}} \\
&\lesssim t^{-\frac{3}{2}}.
\end{aligned} \tag{4.7}
$$

Here in the derivation of (4.7), we used Lemma 2.7 and the inequalities

$$
|\langle r - t \rangle L_0 \partial_{tt} u| \lesssim |\langle r - t \rangle \partial_{tt} \Gamma^{\leq 1} u| \lesssim t^{-\frac{1}{2}}, \qquad \text{for } r \geq t/10. \tag{4.8}
$$

For the third term we use the identity $\partial_+ \partial_- v = \Box v + \frac{\partial_r v}{r} + \frac{\partial_{\theta\theta} v}{r^2}$ and compute it as

$$
\begin{aligned}
&\int h_1(\theta) v \partial_{tt} u \partial_+ \partial_- v e^p \phi \\
&= \int h_1(\theta) v \partial_{tt} u \left(\frac{\partial_r v}{r} + \frac{\partial_{\theta\theta} v}{r^2} \right) e^p \phi + \sum_{\beta_1 + \beta_2 \leq \alpha} \int h(\theta) v \partial_{tt} u \cdot g^{kij}_{\beta_1, \beta_2} \partial_k \Gamma^{\beta_1} u \partial_{ij} \Gamma^{\beta_2} u e^p \phi.
\end{aligned} \tag{4.9}
$$

Integrating by parts (for the term $\partial_{\theta\theta} v$), we have

$$
\begin{aligned}
&\int h_1(\theta) v \partial_{tt} u \left(\frac{\partial_r v}{r} + \frac{\partial_{\theta\theta} v}{r^2} \right) e^p \phi \\
&= \int h_1(\theta) \frac{v}{\langle r - t \rangle} \langle r - t \rangle \partial_{tt} u \partial_r v \cdot \frac{1}{r} e^p \phi - \int h_1(\theta) \partial_{tt} u \left(\frac{\partial_\theta v}{r} \right)^2 e^p \phi \\
&\quad - \int \partial_\theta (h_1(\theta) \partial_{tt} u) v \frac{\partial_\theta v}{r^2} e^p \phi \\
&= \mathrm{OK}.
\end{aligned}
$$

By (3.12), we have

$$
\sum_{\substack{\beta_1 < \alpha, \beta_2 < \alpha, \\ \beta_1 + \beta_2 \leq \alpha}} \int h_1(\theta) v \partial_{tt} u \cdot g^{kij}_{\beta_1, \beta_2} \partial_k \Gamma^{\beta_1} u \partial_{ij} \Gamma^{\beta_2} u e^p \phi \lesssim t^{-2} = \mathrm{OK}.
$$

For the term $\beta_1 = \alpha$, $\beta_2 = 0$ in (4.9), it follows from (2.19) that

$$
\int g^{kij} h_1(\theta) v \partial_{tt} u \partial_k v \partial_{ij} u e^p \phi \lesssim \int |v \partial_{tt} u| |T v \partial^2 u| e^p \phi + \int |v \partial_{tt} u| |\partial v| |T \partial u| e^p \phi
$$

$$\lesssim \int |Tv|^2 |\partial^2 u| e^p \phi + t^{-\frac{3}{2}} \left\| \langle r - t \rangle^{-1} v \right\|^2_{L^2_x(\mathbb{R}^2)} + t^{-2}$$

$$= \mathrm{OK}.$$

For the term $\beta_1 = 0$, $\beta_2 = \alpha$ in (4.9), we apply (2.18) to obtain

$$\int g^{kij} h_1(\theta) v \partial_{tt} u \partial_k u \partial_{ij} v e^p \phi$$

$$= \int g^{kij} h_1(\theta) v \partial_{tt} u \cdot (T_k u \partial_{ij} v - \omega_k \partial_t u T_i \partial_j v + \omega_k \omega_i \partial_t u T_j \partial_t v) e^p \phi.$$

We rewrite it as

$$\int g^{kij} h_1(\theta) v \partial_{tt} u T_k u \partial_{ij} v e^p \phi$$

$$= \int g^{kij} \partial_i (h_1(\theta) v \partial_{tt} u T_k u \partial_j v e^p \phi) - \int g^{kij} \partial_i (h_1(\theta) T_k u e^p \phi) v \partial_{tt} u \partial_j v$$

$$- \int g^{kij} h_1(\theta) v \partial_i \partial_{tt} u T_k u \partial_j v e^p \phi - \int g^{kij} h_1(\theta) \partial_i v \partial_{tt} u T_k u \partial_j v e^p \phi.$$

The term $\int g^{kij} \partial_i (h_1(\theta) v \partial_{tt} u T_k u \partial_j v e^p \phi)$ is zero for $i \neq 0$. For $i = 0$ it is clearly acceptable since it can be absorbed into the time derivative of the energy due to its smallness. By Lemma 2.6 and 2.7, we have

$$|\partial_i (h_1(\theta) T_k u e^p \phi)|$$

$$\lesssim |\partial_i h_1(\theta) T_k u e^p \phi| + |h_1(\theta) \partial_i T_k u e^p \phi| + |h_1(\theta) T_k u \partial_i e^p \phi| + |h_1(\theta) T_k u e^p \partial_i \phi|$$

$$\lesssim t^{-\frac{3}{2}} + |h_1(\theta) \partial_i \omega_k \partial_t u e^p \phi| + |h_1(\theta) T_k \partial_i u e^p \phi| + \left| h_1(\theta) \frac{T_k u}{\langle r - t \rangle} \phi \right| \lesssim t^{-\frac{3}{2}}.$$

The term containing $v \partial_i \partial_{tt} u$ can be handled by (2.63). Thus

$$\int g^{kij} h_1(\theta) v \partial_{tt} u T_k u \partial_{ij} v e^p \phi = \mathrm{OK}.$$

Similarly, we have

$$\int g^{kij} \omega_k h_1(\theta) v \partial_{tt} u \partial_t u T_i \partial_j v e^p \phi = \int g^{kij} \omega_k h_1(\theta) v \partial_{tt} u \partial_t u \left(\partial_j T_i v - \partial_j \omega_i \partial_t v \right) e^p \phi = \mathrm{OK},$$

$$\int g^{kij} \omega_k \omega_i h_1(\theta) v \partial_{tt} u \partial_t u T_j \partial_t v e^p \phi = \int g^{kij} \omega_k \omega_i h_1(\theta) v \partial_{tt} u \partial_t u \partial_t T_j v e^p \phi = \mathrm{OK}.$$

This concludes the estimate of the first part of the main piece.

5. Further estimates

We now denote $h(\theta) = h_2(\theta)$ and consider the second part of the main piece

$$\int h(\theta) \frac{1}{r} \partial_\theta v \partial_{tt} u \partial_t v e^p dx. \tag{5.1}$$

Since $\|\langle r - t \rangle \partial_{tt} u\|_\infty \lesssim t^{-\frac{1}{2}}$, it follows that

$$\int h(\theta) \frac{1}{r} \partial_\theta v \partial_{tt} u \partial_t v e^p dx = \mathrm{OK} + \int h(\theta) \frac{1}{r} \partial_\theta v \partial_{tt} u \partial_t v \tilde{\phi}(\frac{x}{t}) e^p dx, \tag{5.2}$$

where $\tilde{\phi}$ is a radial bump function localized to $|z| \sim 1$. Denote $\phi(z) = \frac{1}{|z|} \tilde{\phi}(z)$. Then

$$\int h(\theta) \frac{1}{r} \partial_\theta v \partial_{tt} u \partial_t v \tilde{\phi}(\frac{x}{t}) e^p dx$$

$$= \frac{1}{t} \int h(\theta) \partial_\theta v \partial_{tt} u \partial_t v \phi(\frac{x}{t}) e^p dx$$

$$= \mathrm{OK} + \frac{1}{t} \int h(\theta) v \partial_{tt} u \partial_t \partial_\theta v \phi(\frac{x}{t}) e^p dx$$

$$= \mathrm{OK} + \frac{1}{t} \int \underbrace{h(\theta) \langle r - t \rangle \partial_{tt} u \phi(\frac{x}{t}) e^p}_{=:F} \underbrace{\frac{v}{\langle r - t \rangle}}_{=:\tilde{v}} \partial_t \partial_\theta v dx. \tag{5.3}$$

Note that

$$\|\partial F\|_\infty + \|F\|_\infty \lesssim t^{-\frac{1}{2}} E_4^{\frac{1}{2}}, \quad \|\tilde{v}\|_2 + \|\nabla \tilde{v}\|_2 \lesssim E_m^{\frac{1}{2}}; \tag{5.4}$$

$$\|\langle \nabla \rangle (F\tilde{v})\|_2 \lesssim \|F\tilde{v}\|_2 + \|\nabla(F\tilde{v})\|_2 \lesssim t^{-\frac{1}{2}} E_4^{\frac{1}{2}} E_m^{\frac{1}{2}}. \tag{5.5}$$

It follows that

$$\left| \int F\tilde{v} \partial_t \partial_\theta v dx \right| \lesssim t^{-\frac{1}{2}} E_4^{\frac{1}{2}} E_m^{\frac{1}{2}} \|\langle \nabla \rangle^{-1} \partial_t \partial_\theta v\|_2. \tag{5.6}$$

Recall that $v = \Gamma^\alpha u$ with $|\alpha| \leq m$. Thus we only need to show (below we take $0 < \delta < 1/4$)

$$\|\langle \nabla \rangle^{-1} \partial_t \Gamma^{\leq m+1} u\|_2 \leq D_1 t^\delta, \tag{5.7}$$

where D_1 is a small constant whose smallness can be ensured by the smallness of E_m. The legitimacy of the nonlocal norm $\|\langle \nabla \rangle^{-1} \partial \Gamma^{\leq m+1} u\|_2$ is ensured by Lemma 2.1.

5.1. Estimate of $\|\langle\nabla\rangle^{-1}\partial\Gamma^{\leq m+1}u\|_2$

For each multi-index β with $|\beta| \leq m+1$, we have

$$\Box\Gamma^\beta u = \sum_{\alpha_1+\alpha_2\leq\beta} g^{kij}_{\beta;\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u, \tag{5.8}$$

where $g^{kij}_{\beta,\alpha_1,\alpha_2}$ still satisfies the null conditions for each $(\beta,\alpha_1,\alpha_2)$. Moreover $g^{kij}_{\beta,\beta,0} = g^{kij}_{\beta,0,\beta} = g^{kij}$.

We first compute the left hand side. By using the Littlewood-Paley decomposition (see (2.8)), we have

$$\sum_{J\geq 0}\sum_{|\beta|\leq m+1} 2^{-2J}\int \Box P_J\Gamma^\beta u\partial_t P_J\Gamma^\beta u e^p dx$$

$$=\sum_{J\geq 0}\sum_{|\beta|\leq m+1} 2^{-2J}\Big(\frac{1}{2}\frac{d}{dt}\|e^{\frac{p}{2}}\partial P_J\Gamma^\beta u\|_2^2 + \frac{1}{2}\int e^p q'|TP_J\Gamma^\beta u|^2 dx\Big). \tag{5.9}$$

It is not difficult to check that

$$\sum_{J\geq 0}\sum_{|\beta|\leq m+1} 2^{-2J}\|e^{\frac{p}{2}}\partial P_J\Gamma^\beta u\|_2^2 \sim \sum_{|\beta|\leq m+1}\|\langle\nabla\rangle^{-1}\partial\Gamma^\beta u\|_2^2. \tag{5.10}$$

To simplify the notation in the subsequent nonlinear estimates, we introduce the following terminology.

Notation. For a quantity $X(t)$, we shall write $X(t) = \text{NICE}$ if $X(t)$ can be written as

$$X(t) = \frac{d}{dt}X_1(t) + X_2(t) + X_3(t), \tag{5.11}$$

where (below $\alpha_0 > 0$ is some constant)

$$|X_1(t)| \ll \sum_{|\beta|\leq m+1}\|(\langle\nabla\rangle^{-1}\partial\Gamma^\beta u)(t,\cdot)\|^2_{L_x^2(\mathbb{R}^2)},$$

$$|X_2(t)| \ll \sum_{J\geq 0}\sum_{|\beta|\leq m+1} 2^{-2J}\int e^p q'|(TP_J\Gamma^\beta u)(t,x)|^2 dx;$$

$$|X_3(t)| \lesssim \langle t\rangle^{-1-\alpha_0}. \tag{5.12}$$

Next we shall deal with the RHS, namely

$$\sum_{|\beta|\leq m+1}\sum_{\alpha_1+\alpha_2\leq\beta}\sum_{J\geq 0} 2^{-2J}\Big(g^{kij}_{\beta;\alpha_1,\alpha_2}\int P_J(\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u)\partial_t P_J(\Gamma^\beta u)e^p dx\Big). \tag{5.13}$$

We shall discuss several cases. To simplify the notation, we fix β and denote $w = \Gamma^\beta u$. The most difficult case is the quasilinear piece which will be discussed in detail below.

Case 1: <u>the quasilinear piece $\alpha_1 = 0, \alpha_2 = \beta$</u>. In this case we need to estimate

$$\sum_{J \geq 0} 2^{-2J} g^{kij} \int P_J(\partial_k u \partial_{ij} w) \partial_t P_J w e^p dx. \tag{5.14}$$

We discuss several further subcases.

Case 1a: the piece

$$\sum_{J \geq 8} 2^{-2J} g^{kij} \int P_J(\partial_k u \partial_{ij} P_{[J-3,J+3]} w) \partial_t P_J w e^p dx$$

$$= \sum_{J \geq 8} 2^{-2J} g^{kij} \int \partial_k u \partial_{ij} P_J w \partial_t P_J w e^p dx \tag{5.15}$$

$$+ \sum_{J \geq 8} 2^{-2J} g^{kij} \int \left([P_J, \partial_k u] \partial_{ij} P_{[J-3,J+3]} w \right) \partial_t P_J w e^p dx. \tag{5.16}$$

It is not difficult to check that the contribution of (5.15) is acceptable for us. We now focus on the estimate of (5.16). For simplicity of notation, we denote

$$\boxed{w_J = P_{[J-3,J+3]} w.} \tag{5.17}$$

Clearly

$$\sum_{J \geq 8} 2^{-2J} g^{kij} \int \left([P_J, \partial_k u] \partial_{ij} w_J \right) \partial_t P_J w e^p dx$$

$$= \sum_{J \geq 8} 2^{-2J} g^{kij} \iint 2^{2J} \varphi(2^J y)((\partial_k u)(x-y) - (\partial_k u)(x))(\partial_{ij} w_J)(x-y) dy \partial_t P_J w e^p dx$$

$$= \sum_{J \geq 8} 2^{-2J} g^{kij} \sum_{m=1}^{2} \iint \int_0^1 2^J \varphi_m(2^J y)(\partial_m \partial_k u)(x - \theta y)(\partial_{ij} w_J)(x-y) \partial_t P_J w e^p d\theta dy dx, \tag{5.18}$$

where φ and ϕ_m are Schwartz functions. Here $2^{2J} \varphi(2^J \cdot)$ is the kernel function corresponding to P_J. For $J = 0$ and $J \geq 1$ we have slightly different expressions for φ. But we shall ignore this difference for simplicity of notation.

We first need an auxiliary estimate.

Lemma 5.1. *We have*

$$\sum_{i=1}^{2} \| \partial \partial_i P_{\leq J+3} w \|_2 \lesssim 2^J \| \partial P_{\leq J+3} w \|_2;$$

$$\|\Box P_{\leq J+3}w\|_2 \lesssim 2^J t^{-\frac{1}{2}}\|\partial P_{\leq J+5}w\|_2 + t^{-\frac{1}{2}}\|\partial P_{\leq J+5}\Gamma^{\leq m+1}u\|_2 + t^{-\frac{1}{2}}\|\langle\nabla\rangle^{-1}\partial\Gamma^{\leq m+1}u\|_2;$$

$$\|\partial_{tt}P_{\leq J+3}w\|_2 \lesssim 2^J\|\partial P_{\leq J+3}w\|_2 + 2^J t^{-\frac{1}{2}}\|\partial P_{\leq J+5}w\|_2$$
$$+ t^{-\frac{1}{2}}\|\partial P_{\leq J+5}\Gamma^{\leq m+1}u\|_2 + t^{-\frac{1}{2}}\|\langle\nabla\rangle^{-1}\partial\Gamma^{\leq m+1}u\|_2.$$

The same estimates hold when $P_{\leq J+3}w$ on the LHS above is replaced by

$$\boxed{w_J = P_{[J-3,J+3]}w}.$$

Proof. The first estimate is obvious. We only need to show the second estimate since the third estimate follows from the identity $\partial_{tt} = \Box + \Delta$. Observe that (for simplicity denote $g^{kij}_{\alpha_1,\alpha_2} = g^{kij}_{\beta;\alpha_1,\alpha_2}$)

$$\Box w = \sum_{\alpha_1+\alpha_2\leq\beta} g^{kij}_{\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u. \tag{5.19}$$

The main difficult term on the RHS is the case $\partial_{ij} = \partial_{tt}$, $\alpha_2 = \beta$. We rewrite the above as

$$\Box w = g^{k00}\partial_k u(\Box w + \Delta w) + \sum_{\alpha_1+\alpha_2\leq\beta,\alpha_2<\beta} g^{kij}_{\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u + \sum_{(i,j)\neq(0,0)} g^{kij}\partial_k u\partial_{ij}w. \tag{5.20}$$

Thus (below the Einstein summation convention is still in force, e.g. $g^{k00}\partial_k u = \sum_{k=0}^2 g^{k00}\partial_k u$)

$$\Box w = \frac{1}{1-g^{k00}\partial_k u}\Big(g^{k00}\partial_k u\Delta w + \sum_{\alpha_1+\alpha_2\leq\beta,\alpha_2<\beta} g^{kij}_{\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u$$
$$+ \sum_{(i,j)\neq(0,0)} g^{kij}\partial_k u\partial_{ij}w\Big). \tag{5.21}$$

Denote $\tilde{f} = \frac{g^{k00}\partial_k u}{1-g^{k00}\partial_k u}$. Since $\|\partial\Gamma^{\leq 3}u\|_\infty \lesssim t^{-\frac{1}{2}}E_5^{\frac{1}{2}}$, we have $\|\partial^{\leq 3}\tilde{f}\|_\infty \lesssim t^{-\frac{1}{2}}$. Clearly

$$\|P_{\leq J+3}(\tilde{f}\Delta w)\|_2 \leq \|P_{\leq J+3}(\tilde{f}P_{\leq J+5}\Delta w)\|_2 + \|P_{\leq J+3}(\tilde{f}P_{\geq J+6}\Delta w)\|_2$$
$$\lesssim 2^J t^{-\frac{1}{2}}\|\partial P_{\leq J+5}w\|_2 + t^{-\frac{1}{2}}\|\langle\nabla\rangle^{-1}\partial w\|_2. \tag{5.22}$$

By a similar estimate, we have

$$\Big\|P_{\leq J+3}\Big(\frac{1}{1-g^{k00}\partial_k u}\sum_{(i,j)\neq(0,0)} g^{kij}\partial_k u\partial_{ij}w\Big)\Big\|_2 \lesssim 2^J t^{-\frac{1}{2}}\|\partial P_{\leq J+5}w\|_2 + t^{-\frac{1}{2}}\|\langle\nabla\rangle^{-1}\partial w\|_2. \tag{5.23}$$

To estimate $\|P_{\leq J+3}\Big(\frac{1}{1-g^{k00}\partial_k u}\sum_{\alpha_1+\alpha_2\leq\beta,\,\alpha_2<\beta}g^{kij}_{\alpha_1,\alpha_2}\partial_k\Gamma^{\alpha_1}u\partial_{ij}\Gamma^{\alpha_2}u\Big)\|_2$, we denote $\tilde f_2 = \frac{1}{1-g^{k00}\partial_k u}$ and consider the general expression

$$\|P_{\leq J+3}(\tilde f_2\partial\Gamma^{\alpha_1}u\partial^2\Gamma^{\alpha_2}u)\|_2,\quad \alpha_1+\alpha_2\leq\beta,\ \alpha_2<\beta.\tag{5.24}$$

We discuss a few cases. Recall $|\beta|\leq m+1$, $\|\partial\Gamma^{\leq m-2}u\|_\infty\lesssim t^{-\frac12}E_m^{\frac12}$, and $\|\partial^2\Gamma^{\leq m-3}u\|_\infty\lesssim t^{-\frac12}E_m^{\frac12}$.

Case 1: $|\alpha_2|=m$ or $|\alpha_2|=m-1$. Clearly $|\alpha_1|\leq 2$ and we have

$$\begin{aligned}
&\|P_{\leq J+3}(\tilde f_2\partial\Gamma^{\leq 2}u\partial^2\Gamma^{\leq m}u)\|_2\\
&\lesssim \|\tilde f_2\partial\Gamma^{\leq 2}u\|_\infty\|\partial^2 P_{\leq J+5}\Gamma^{\leq m}u\|_2 + \sum_{l\geq J+6}\|P_l(\tilde f_2\partial\Gamma^{\leq 2}u)\|_\infty\|\tilde P_l(\partial^2\Gamma^{\leq m}u)\|_2\\
&\lesssim t^{-\frac12}\Big(\|\partial P_{\leq J+5}\Gamma^{\leq m+1}u\|_2 + \|\partial\langle\nabla\rangle^{-1}\Gamma^{\leq m+1}u\|_2\Big).
\end{aligned}\tag{5.25}$$

Case 2: $|\alpha_1|\leq|\alpha_2|\leq m-2$. We have

$$\begin{aligned}
&\|P_{\leq J+3}(\tilde f_2\partial\Gamma^{\alpha_1}u\partial^2\Gamma^{\alpha_2}u)\|_2\\
&\lesssim \|\tilde f_2\partial\Gamma^{\alpha_1}u\|_\infty\|P_{\leq J+5}(\partial^2\Gamma^{\alpha_2}u)\|_2 + \sum_{l\geq J+6}\|P_l(\tilde f_2\partial\Gamma^{\alpha_1}u)\|_\infty\|\tilde P_l(\partial^2\Gamma^{\alpha_2}u)\|_2\\
&\lesssim t^{-\frac12}\Big(\|\partial P_{\leq J+5}\Gamma^{\leq m+1}u\|_2 + \|\partial\langle\nabla\rangle^{-1}\Gamma^{\leq m+1}u\|_2\Big).
\end{aligned}\tag{5.26}$$

Case 3: $|\alpha_2|<|\alpha_1|\leq m-2$. We have

$$\begin{aligned}
&\|P_{\leq J+3}(\tilde f_2\partial\Gamma^{\alpha_1}u\partial^2\Gamma^{\alpha_2}u)\|_2\\
&\lesssim \|P_{\leq J+5}(\partial\Gamma^{\alpha_1}u)\|_2\|\tilde f_2\partial^2\Gamma^{\alpha_2}u\|_\infty + \sum_{l\geq J+6}\|P_l(\tilde f_2\partial^2\Gamma^{\alpha_2}u)\|_\infty\|\tilde P_l(\partial\Gamma^{\alpha_1}u)\|_2\\
&\lesssim t^{-\frac12}\Big(\|\partial P_{\leq J+5}\Gamma^{\leq m+1}u\|_2 + \|\partial\langle\nabla\rangle^{-1}\Gamma^{\leq m+1}u\|_2\Big).
\end{aligned}$$

Case 4: $|\alpha_1|=m-1$, $|\alpha_2|\leq 2$, or $|\alpha_1|=m$, $|\alpha_2|\leq 1$, or $|\alpha_1|=m+1$, $|\alpha_2|=0$. Easy to check that we also have

$$\begin{aligned}
&\|P_{\leq J+3}(\tilde f_2\partial\Gamma^{\alpha_1}u\partial^2\Gamma^{\alpha_2}u)\|_2\\
&\lesssim \|P_{\leq J+5}(\partial\Gamma^{\alpha_1}u)\|_2\|\tilde f_2\partial^2\Gamma^{\alpha_2}u\|_\infty + \sum_{l\geq J+6}\|P_l(\tilde f_2\partial^2\Gamma^{\alpha_2}u)\|_\infty\|\tilde P_l(\partial\Gamma^{\alpha_1}u)\|_2\\
&\lesssim t^{-\frac12}\Big(\|\partial P_{\leq J+5}\Gamma^{\leq m+1}u\|_2 + \|\partial\langle\nabla\rangle^{-1}\Gamma^{\leq m+1}u\|_2\Big).
\end{aligned}$$

The desired estimate then easily follows. $\quad\square$

We now continue the estimate of (5.18). In (5.18), it suffices for us to treat the case $m = 1$ since the estimate for $m = 2$ is similar. We write

$$\sum_{J \geq 8} 2^{-2J} g^{kij} \iint \int_0^1 2^J \varphi_1(2^J y)(\partial_1 \partial_k u)(x - \theta y)(\partial_{ij} w_J)(x - y) \partial_t P_J w e^p \, d\theta dy dx$$

$$= \sum_{J \geq 8} 2^{-2J} g^{kij} \iint \int_0^1 2^J \varphi_1(2^J y) \chi(t^{-\frac{2}{3}} y)(\partial_1 \partial_k u)(x - \theta y)(\partial_{ij} w_J)(x - y) \partial_t P_J w e^p \, d\theta dy dx$$

$$\tag{5.27}$$

$$+ \sum_{J \geq 8} 2^{-2J} g^{kij} \iint \int_0^1 2^J \varphi_1(2^J y) \cdot (1 - \chi(t^{-\frac{2}{3}} y))(\partial_1 \partial_k u)(x - \theta y)(\partial_{ij} w_J)(x - y)$$

$$\times \partial_t P_J w e^p \, d\theta dy dx, \tag{5.28}$$

where $\chi \in C_c^\infty(\mathbb{R}^2)$ satisfies $\chi(z) \equiv 1$ for $|z| \leq 0.01$ and $\chi(z) \equiv 0$ for $|z| \geq 0.02$. In yet other words the cut-off function $\chi(t^{-\frac{2}{3}} y)$ is to localize y to the regime $|y| \ll t^{\frac{2}{3}}$. In (5.28), since $|y| \gtrsim t^{\frac{2}{3}}$, we clearly have (by using Lemma 5.1)

$$|(5.28)| \lesssim \sum_{J \geq 0} 2^{-10J} t^{-10} \|\partial \langle \nabla \rangle^{-1} \Gamma^{\leq m+1} u\|_2^2. \tag{5.29}$$

The contribution of this term is clearly acceptable for us.

To estimate (5.27), we choose $\phi_1(t, x) = a(x/t)$ where $a \in C_c^\infty(\mathbb{R}^2)$ is such that $a(x) = 1$ for $0.9 \leq |x| \leq 1.1$, and $a(x) = 0$ for $|x| \leq 0.8$ or $|x| \geq 1.2$. We decompose (5.27) as

$$(5.27)$$

$$= \sum_{J \geq 8} 2^{-2J} g^{kij} \iint \int_0^1 2^J \varphi_1(2^J y) \chi(t^{-\frac{2}{3}} y)(1 - \phi_1(t, x))(\partial_1 \partial_k u)(x - \theta y)(\partial_{ij} w_J)(x - y)$$

$$\times \partial_t P_J w e^p \, d\theta dy dx \tag{5.30}$$

$$+ \sum_{J \geq 8} 2^{-2J} g^{kij} \iint \int_0^1 2^J \varphi_1(2^J y) \chi(t^{-\frac{2}{3}} y) \phi_1(t, x)(\partial_k \partial_1 u)(x - \theta y)(\partial_{ij} w_J)(x - y)$$

$$\times \partial_t P_J w e^p \, d\theta dy dx. \tag{5.31}$$

Observe that in (5.30), since $|y| \ll t$ and $|x|$ is away from the light cone, the variable $x - \theta y$ is also away from the light cone. We have

$$\sup_{0 \leq \theta \leq 1} \|\chi(t^{-\frac{2}{3}} y)(1 - \phi_1(t, x))(\partial_1 \partial_k u)(x - \theta y)\|_{L_x^\infty L_y^\infty} \lesssim t^{-\frac{3}{2}} E_5^{\frac{1}{2}}. \tag{5.32}$$

By using this estimate together with Lemma 5.1, it is not difficult to check that the contribution of (5.30) is acceptable for us. It remains for us to estimate (5.31). In this case observe that $|x| \sim t$, $|y| \ll t$, $|y| \ll |x|$.

We shall use the identity:

$$g^{kij}\partial_k a \partial_{ij} b = g^{kij}(T_k a \partial_{ij} b - \omega_k \partial_t a T_i \partial_j b + \omega_k \omega_i \partial_t a T_j \partial_t b). \tag{5.33}$$

One has to be extremely careful here due to the shifts in x induced by convolution! In particular

$$T_k(a(x+h)) \neq (T_k a)(x+h). \tag{5.34}$$

In (5.31), we shall apply the above identity with

$$a(x) = (\partial_1 u)(x - \theta y), \quad b(x) = w_J(x - y). \tag{5.35}$$

Subcase 1: the piece

$$\sum_{J \geq 8} 2^{-2J} g^{kij} \iint \int_0^1 2^J \varphi_1(2^J y) \chi(t^{-\frac{2}{3}} y)\phi_1(t,x) T_k a \partial_{ij} b \partial_t P_J w e^p \, d\theta dy dx. \tag{5.36}$$

Observe that

$$T_k a = \left(\omega_k(x)\partial_t + \partial_{x_k} \right)\left((\partial_1 u)(x - \theta y) \right).$$

Since $|x| \sim t$ and $|y| \ll |x|$, we have

$$|\omega_k(x) - \omega_k(x - \theta y)| \lesssim \frac{1}{t} \cdot |y|. \tag{5.37}$$

Thus we only need to work with the piece

$$\sum_{J \geq 8} 2^{-2J} g^{kij} \iint \int_0^1 2^J \varphi_1(2^J y)\chi(t^{-\frac{2}{3}} y)\phi_1(t,x)(T_k \partial_1 u)(x - \theta y)(\partial_{ij} w_J)(x - y)$$

$$\times \partial_t P_J w e^p \, d\theta dy dx. \tag{5.38}$$

Since $\|T\partial u\|_\infty \lesssim t^{-\frac{3}{2}}$, the contribution of the term (5.38) is clearly acceptable for us with the help of Lemma 5.1.

Subcase 2: the piece

$$\sum_{J \geq 8} 2^{-2J} \sum_{1 \leq j \leq 2} g^{kij} \iint \int_0^1 2^J \varphi_1(2^J y)\chi(t^{-\frac{2}{3}} y)\omega_k(x)\phi_1(t,x)\partial_t a T_i \partial_j b \partial_t P_J w e^p \, d\theta dy dx.$$

$$\tag{5.39}$$

Here we only treat the case $j \neq 0$, i.e. we deal with $T_i \nabla b$. Note that

$$\partial_t a = (\partial_t \partial_1 u)(x - \theta y); \tag{5.40}$$

$$T_i \partial_j b = (\omega_i(x)\partial_t + \partial_{x_i})\Big((\partial_j w_J)(x - y)\Big). \tag{5.41}$$

Since $|x| \sim t$ and $|y| \ll |x|$, the contribution of the difference $\omega_i(x) - \omega_i(x - y)$ is acceptable for us. Thus we only need to estimate (for $j = 1$ or $j = 2$)

$$\sum_{J \geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y)\chi(t^{-\frac{2}{3}}y)\phi_1(t, x)\omega_k(x)(\partial_t \partial_1 u)(x - \theta y)(T_i \partial_j w_J)(x - y)$$

$$\times \partial_t P_J w e^p \, dy dx. \tag{5.42}$$

Here and below we shall neglect the integral in θ since the estimates will be uniform in $\theta \in [0, 1]$. In (5.42), note that $|x - y| \sim t$ and the contribution of the commutator (below $z = x - y$)

$$([T_i, \partial_j]w_J)(z) = -\Big(\partial_{z_j}(\omega_i(z))\Big) \cdot (\partial_t w_J)(z) \tag{5.43}$$

is clearly acceptable for us (since $\|\partial_{z_j}(\omega_i(z))\|_{L^\infty(|z| \sim t)} \lesssim \frac{1}{t}$). Thus we only need to estimate

$$\sum_{J \geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y)\chi(t^{-\frac{2}{3}}y)\phi_1(t, x)\omega_k(x)(\partial_t \partial_1 u)(x - \theta y)(\partial_j T_i w_J)(x - y)$$

$$\times \partial_t P_J w e^p \, dy dx. \tag{5.44}$$

We now write $(\partial_j T_i w_J)(x - y) = -\partial_{y_j}\Big((T_i w_J)(x - y)\Big)$. Integrating by parts in ∂_{y_j}, we obtain

(5.44)

$$= \sum_{J \geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y)\chi(t^{-\frac{2}{3}}y)\phi_1(t, x)\omega_k(x)(\partial_t \partial_j \partial_1 u)(x - \theta y) \cdot (-\theta)(T_i w_J)(x - y)$$

$$\times \partial_t P_J w e^p \, dy dx \tag{5.45}$$

$$+ \sum_{J \geq 8} 2^{-2J} \iint 2^{2J}(\partial_j \varphi_1)(2^J y)\chi(t^{-\frac{2}{3}}y)\phi_1(t, x)\omega_k(x)(\partial_t \partial_1 u)(x - \theta y)(T_i w_J)(x - y)$$

$$\times \partial_t P_J w e^p \, dy dx \tag{5.46}$$

$$+ \sum_{J \geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y)t^{-\frac{2}{3}}(\partial_j \chi)(t^{-\frac{2}{3}}y)\phi_1(t, x)\omega_k(x)(\partial_t \partial_1 u)(x - \theta y)(T_i w_J)(x - y)$$

$$\times \partial_t P_J w e^p \, dy dx. \tag{5.47}$$

For (5.46), we have (below $\delta_1 > 0$ is a small constant)

$$|(5.46)|$$

$$\leq \sum_{J \geq 8} 2^{-2J} \iint 2^{2J} |(\nabla \varphi_1)(2^J y)| |\phi_1(t, x)| \Big(\epsilon |(T w_J)(x - y)|^2 \langle |x - y| - t \rangle^{-1 - \delta_1}$$

$$+ C_\epsilon \langle |x - y| - t \rangle^{1 + \delta_1} |(\partial^2 u)(x - \theta y)|^2 |\chi(t^{-\frac{2}{3}} y)|^2 |(\partial_t P_J w)(x)|^2 \Big) dx dy, \qquad (5.48)$$

where $\epsilon > 0$ can be taken sufficiently small, and $C_\epsilon > 0$ depends on ϵ. Note that

$$\langle |x - y| - t \rangle^{1 + \delta_1} |(\partial^2 u)(x - \theta y)|^2 \lesssim \frac{E_5}{t}(1 + |y|). \qquad (5.49)$$

Due to the cut-off function $|(\nabla \varphi_1)(2^J y)|$, the factor $(1 + |y|)$ is certainly harmless for us. It is then not difficult to check that the contribution of (5.48) is acceptable for us.

It is not difficult to check that the contribution of the term (5.47) is acceptable for us.

The estimate of (5.45) follows along similar lines. We omit the details.

Subcase 3: the piece

$$\sum_{J \geq 8} 2^{-2J} \iint \int_0^1 2^J \varphi_1(2^J y) \chi(t^{-\frac{2}{3}} y) \phi_2(t, x) \partial_t a \, T_j \partial_t b \, \partial_t P_J w e^p \, d\theta \, dy \, dx, \qquad (5.50)$$

where $j = 1$ or $j = 2$, and $\phi_2(t, x)$ is localized to $|x| \sim t$. Here $\phi_2(t, x)$ corresponds to $\phi_1(t, x)\omega_k(x)$ or $\phi_1(t, x)\omega_k(x)\omega_i(x)$. Recall $b(x) = w_J(x - y)$ and note that

$$(T_j \partial_t b)(x) - (T_j \partial_t w_J)(x - y) = (\omega_j(x) - \omega_j(x - y))(\partial_{tt} w_J)(x - y). \qquad (5.51)$$

Since $|x| \sim t$ and $|y| \ll t$, the contribution of (5.51) is acceptable by using Lemma 5.1. Thus we only need to estimate

$$\sum_{J \geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y) \chi(t^{-\frac{2}{3}} y) \phi_2(t, x)(\partial_t \partial_1 u)(x - \theta y)(T_j \partial_t w_J)(x - y) \partial_t P_J w e^p \, dy \, dx. \qquad (5.52)$$

We rewrite (5.52) as

$$(5.52)$$

$$= \frac{d}{dt} \Big(\sum_{J \geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y) \chi(t^{-\frac{2}{3}} y) \phi_2(t, x)(\partial_t \partial_1 u)(x - \theta y)(T_j w_J)(x - y)$$

$$\times \partial_t P_J w e^p \, dy \, dx \Big) \qquad (5.53)$$

$$-\sum_{J\geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y)\partial_t(\chi(t^{-\frac{2}{3}}y)\phi_2(t,x))(\partial_t\partial_1 u)(x-\theta y)(T_j w_J)(x-y)$$

$$\times \partial_t P_J w e^p\, dy dx \tag{5.54}$$

$$-\sum_{J\geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y)\chi(t^{-\frac{2}{3}}y)\phi_2(t,x)(\partial_{tt}\partial_1 u)(x-\theta y)(T_j w_J)(x-y)$$

$$\times \partial_t P_J w e^p\, dy dx \tag{5.55}$$

$$-\sum_{J\geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y)\chi(t^{-\frac{2}{3}}y)\phi_2(t,x)(\partial_t\partial_1 u)(x-\theta y)(T_j w_J)(x-y)$$

$$\times \partial_{tt} P_J w e^p\, dy dx \tag{5.56}$$

$$-\sum_{J\geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y)\chi(t^{-\frac{2}{3}}y)\phi_2(t,x)(\partial_t\partial_1 u)(x-\theta y)(T_j w_J)(x-y)$$

$$\times \partial_t P_J w e^p \partial_t p\, dy dx. \tag{5.57}$$

It is not difficult to check that

$$|(5.54)| + |(5.55)| = \text{NICE}. \tag{5.58}$$

For (5.56) we can choose $\tilde{\phi}_1 \in C_c^\infty$ such that $\tilde{\phi}_1 \varphi_1 \equiv \varphi_1$. Then

$$|(5.56)|$$

$$= \left| \sum_{J\geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y)\chi(t^{-\frac{2}{3}}y)\phi_2(t,x)(\partial_t\partial_1 u)(x-\theta y)(T_j w_J)(x-y)\tilde{\phi}_1(2^J y) \right.$$

$$\left. \times \partial_{tt} P_J w e^p\, dy dx \right|$$

$$\leq \sum_{J\geq 8} 2^{-2J} \iint \left(2^{2J}|\varphi_1(2^J y)|^2 \epsilon \frac{|(T w_J)(x-y)|^2}{\langle |x-y|-t\rangle^{1+\delta_1}} \right.$$

$$+ C_\epsilon |\tilde{\phi}_1(2^J y)|^2 \langle |x-y|-t\rangle^{1+\delta_1}|(\partial^2 u)(x-\theta y)|^2|\chi(t^{-\frac{2}{3}}y)|^2|\phi_1(t,x)|^2$$

$$\left. \times |(\partial_{tt} P_J w)(x)|^2 \right) dx dy$$

$$\leq \sum_{J\geq 8} 2^{-2J}\text{const}\cdot \epsilon \cdot \int |T w_J|^2(x)q'(|x|-t)dx$$

$$+ \sum_{J\geq 8} 2^{-4J} \cdot C_\epsilon^{(1)} \cdot \frac{E_5}{t}\|\partial_{tt} P_J w\|_2^2. \tag{5.59}$$

In the above $\epsilon > 0$ can be taken sufficiently small, and $C_\epsilon > 0$, $C_\epsilon^{(1)} > 0$ depend on ϵ. The term $\|\partial_{tt} P_J w\|_2^2$ can be controlled with the help of Lemma 5.1. Thus

$$|(5.56)| \leq \text{NICE} + \frac{\text{const} \cdot E_5}{t} \| \langle \nabla \rangle^{-1} \partial \Gamma^{\leq m+1} u \|_2^2. \tag{5.60}$$

The term (5.57) is easier and can be estimated along similar lines. We omit the details.

Now observe

$$\left| \sum_{J \geq 8} 2^{-2J} \iint 2^J \varphi_1(2^J y) \chi(t^{-\frac{2}{3}} y) \phi_2(t,x) (\partial_t \partial_1 u)(x - \theta y)(T_j w_J)(x - y) \partial_t P_J w e^p \, dy \, dx \right|$$

$$\lesssim E_5^{\frac{1}{2}} t^{-\frac{1}{2}} \| \langle \nabla \rangle^{-1} \partial w \|_2^2. \tag{5.61}$$

Thus the contribution of the term (5.53) is acceptable for us.

This concludes the estimate of Subcase 3 and Case 1a.

Case 1b: the piece

$$\sum_{J \geq 8} 2^{-2J} g^{kij} \int P_J(\partial_k u \partial_{ij} P_{\leq J-4} w) \partial_t P_J w e^p \, dx$$

$$= \sum_{J \geq 8} 2^{-2J} g^{kij} \int P_J(\partial_k \tilde{P}_J u \partial_{ij} P_{\leq J-4} w) \partial_t P_J w e^p \, dx \tag{5.62}$$

$$= \sum_{J \geq 8} 2^{-2J} g^{kij} \int \left([P_J, \partial_k \tilde{P}_J u] \partial_{ij} P_{\leq J-4} w \right) \partial_t P_J w e^p \, dx. \tag{5.63}$$

This case can be similarly treated along the lines in Case 1a. To overcome the issue of summability due to $P_{\leq J-4} w$, one can make use of Lemma 5.2 and Lemma 5.3. For example, the analogue of (5.46) is

$$\sum_{J \geq 8} 2^{-2J} \iint 2^{2J} (\partial_j \varphi_1)(2^J y) \chi(t^{-\frac{2}{3}} y) \phi_1(t,x) \omega_k(x) (\partial_t \partial_1 u_J)(x - \theta y)(T_i w_{\leq J-4})(x - y)$$

$$\times \partial_t P_J w e^p \, dy \, dx, \tag{5.64}$$

where $w_{\leq J-4} = P_{\leq J-4} w$ and $u_J = \tilde{P}_J u$. In lieu of (5.48), we bound it as

$$|(5.64)|$$

$$\leq \sum_{J \geq 8} 2^{-2J} \iint 2^{2J} |(\nabla \varphi_1)(2^J y)| |\phi_1(t,x)| \left(\epsilon |(T w_{\leq J-4})(x - y)|^2 \langle |x - y| - t \rangle^{-1 - \delta_1} \cdot 2^{-J \delta_2} \right.$$

$$\left. + C_\epsilon \langle |x - y| - t \rangle^{1 + \delta_1} 2^{J \delta_2} |(\partial^2 u_J)(x - \theta y)|^2 |\chi(t^{-\frac{2}{3}} y)|^2 |(\partial_t P_J w)(x)|^2 \right) dx \, dy, \tag{5.65}$$

where $\delta_2 > 0$ is a small exponent. The term containing $|T w_{\leq J-4}(x - y)|^2$ is clearly manageable due to the decay factor $2^{-J \delta_2}$. For the second term, by using Lemma 5.3, we have (for $|x| \sim t$, $|y| \ll |x|$)

$$|\langle |x - \theta y| - t \rangle \partial^2 u_J(x - \theta y)|^2 \lesssim t^{-1} 2^{-4J}. \tag{5.66}$$

Since $\langle |x-y|-t \rangle \lesssim \langle |x-\theta y|-t \rangle + |y|$, this term is under control. Thus both terms are easily estimated. We omit further details.

Case 1c: the piece

$$\sum_{J \geq 0} 2^{-2J} g^{kij} \int P_J (\partial_k u \partial_{ij} P_{\geq J+4} w) \partial_t P_J w e^p dx$$

$$= \sum_{J \geq 0} 2^{-2J} g^{kij} \sum_{l \geq J+4} \int P_J (\partial_k \tilde{P}_l u \partial_{ij} P_l w) \partial_t P_J w e^p dx$$

$$= \sum_{J \geq 0} 2^{-2J} g^{kij} \sum_{l \geq J+4} \int P_J ((1-\phi_1) \partial_k \tilde{P}_l u \partial_{ij} P_l w) \partial_t P_J w e^p dx \tag{5.67}$$

$$+ \sum_{J \geq 0} 2^{-2J} g^{kij} \sum_{l \geq J+4} \int P_J (\phi_1 \partial_k \tilde{P}_l u \partial_{ij} P_l w) \partial_t P_J w e^p dx, \tag{5.68}$$

where $\phi_1(t,x) = a(x/t)$ and $a \in C_c^\infty(\mathbb{R}^2)$ is a radial bump function such that $a(x) = 1$ for $0.9 \leq |x| \leq 1.1$, and $a(x) = 0$ for $|x| \leq 0.8$ or $|x| \geq 1.2$.

Lemma 5.2. *We have for $l \geq 1$,*

$$\|(1-\phi_1)\partial P_l u\|_\infty \lesssim t^{-\frac{3}{2}} 2^{-3l}; \tag{5.69}$$

$$\|\phi_1 T P_l u\|_\infty \lesssim t^{-\frac{3}{2}} 2^{-3l}. \tag{5.70}$$

Proof. Note that away from the light cone $\partial^2 \Gamma^{\leq m-3} u$ has $O(t^{-\frac{3}{2}})$ decay. The estimate (5.69) then follows from a mismatch estimate. For (5.70), we can take $T_1 = \omega_1 \partial_t + \partial_1$ (the estimate for T_2 is similar) and observe that

$$\|\phi_1 T_1 P_l u\|_\infty \lesssim \sum_{i,j=1}^{2} \|\phi_1 T_1 \Delta^{-2} \partial_{ii} \partial_{jj} P_l u\|_\infty$$

$$\lesssim 2^{-3l} \sum_{i=1}^{2} \|\phi_1 T_1 Q_l^{(i)} \tilde{\partial}^3 u\|_\infty, \tag{5.71}$$

where $Q_l^{(i)}$ is modified frequency projection still localized to $|\xi| \sim 2^l$, and $\tilde{\partial} = \partial_1$ or ∂_2. Note that

$$\phi_1 T_1 Q_l^{(i)} \tilde{\partial}^3 u = \phi_1 (\omega_1 \partial_t + \partial_1) Q_l^{(i)} \tilde{\partial}^3 u$$

$$= [\phi_1 \omega_1, Q_l^{(i)}] \partial_t \tilde{\partial}^3 u + [\phi_1, Q_l^{(i)}] \tilde{\partial}^3 u + Q_l^{(i)} (\phi_1 T_1 \tilde{\partial}^3 u). \tag{5.72}$$

Since

$$\|\nabla(\phi_1 \omega_1)\|_\infty + \|\nabla \phi_1\|_\infty \lesssim \frac{1}{t}, \tag{5.73}$$

the commutators $[\phi_1\omega_1, Q_l^{(i)}]$, $[\phi_1, Q_l^{(i)}]$ are under control. The desired result follows easily. $\quad\square$

By using (5.69), it is not difficult to check that the contribution of (5.67) is acceptable for us. For (5.68), we note that

$$g^{kij}\partial_k \tilde{P}_l u \partial_{ij} P_l w = g^{kij}(T_k \tilde{P}_l u \partial_{ij} P_l w - \omega_k \partial_t \tilde{P}_l u T_i \partial_j P_l w + \omega_k \omega_i \partial_t \tilde{P}_l u T_j \partial_t P_l w). \tag{5.74}$$

By (5.70) and Lemma 5.1, we have

$$\left| \sum_{J \geq 0} 2^{-2J} g^{kij} \sum_{l \geq J+4} \int P_J(\phi_1 T_k \tilde{P}_l u \partial_{ij} P_l w)\partial_t P_J w e^p dx \right|$$

$$\lesssim \sum_{J \geq 0} 2^{-2J} \sum_{l \geq J+4} 2^{-3l} t^{-\frac{3}{2}} \cdot \left(2^{2l} \|\partial\langle\nabla\rangle^{-1}\Gamma^{\leq m+1} u\|_2 \right) \cdot 2^J \|\partial\langle\nabla\rangle^{-1}w\|_2. \tag{5.75}$$

Clearly the contribution of this term is acceptable for us.

Next we estimate the piece

$$\left| \sum_{J \geq 0} 2^{-2J} \sum_{1 \leq i,j \leq 2, 0 \leq k \leq 2} g^{kij} \sum_{l \geq J+4} \int P_J(\phi_1 \omega_k \partial_t \tilde{P}_l u T_i \partial_j P_l w)\partial_t P_J w e^p dx \right|$$

$$\lesssim \sum_{1 \leq i,j \leq 2, 0 \leq k \leq 2} \left| \sum_{J \geq 0} 2^{-2J} \sum_{l \geq J+4} \int P_J(\phi_1 \omega_k \partial_t \tilde{P}_l u T_i \partial_j P_l w)\partial_t P_J w e^p dx \right|. \tag{5.76}$$

In yet other words, we first treat the terms containing $T\nabla w$.

Estimate of (5.76). With no loss we take $k = 1$, $i = 1$, $j = 1$. Note that

$$\left| \sum_{J \geq 0} 2^{-2J} \sum_{l \geq J+4} \int P_J(\phi_1 \omega_1 \partial_t \tilde{P}_l u[T_1, \partial_1] P_l w)\partial_t P_J w e^p dx \right|$$

$$\lesssim \sum_{J \geq 0} 2^{-2J} \sum_{l \geq J+4} \frac{1}{t}\|\partial\tilde{P}_l u\|_\infty \|\partial P_l w\|_2 \|\partial P_J w\|_2 \lesssim t^{-\frac{3}{2}} E_5^{\frac{1}{2}} \|\partial\langle\nabla\rangle^{-1}w\|_2^2. \tag{5.77}$$

Thus the commutator piece is under control. We now consider

$$\left| \sum_{J \geq 0} 2^{-2J} \sum_{l \geq J+4} \int P_J(\phi_1 \omega_1 \partial_t \tilde{P}_l u \partial_1(T_1 P_l w))\partial_t P_J w e^p dx \right|$$

$$\leq \text{NICE} + \left| \sum_{J \geq 0} 2^{-2J} \sum_{l \geq J+4} \int P_J(\phi_1 \omega_1 \partial_1 \partial_t \tilde{P}_l u T_1 P_l w)\partial_t P_J w e^p dx \right| \tag{5.78}$$

$$+ \left| \sum_{J \geq 0} 2^{-2J} \sum_{l \geq J+4} \int \partial_1 P_J(\phi_1 \omega_1 \partial_t \tilde{P}_l u T_1 P_l w)\partial_t P_J w e^p dx \right|. \tag{5.79}$$

For (5.78), we have

$$
\begin{aligned}
&(5.78) \\
&\le \sum_{J\ge 0} 2^{-2J} \Big| \sum_{l\ge J+4} \int \phi_1\omega_1\partial_1\partial_t\tilde{P}_l u T_1 P_l w P_J(\partial_t P_J w e^p) dx \Big| \\
&\lesssim \sum_{J\ge 0} 2^{-2J} \sum_{l\ge J+4} \Big(\epsilon 2^{-2l}\int |T P_l w|^2 q' dx + C_\epsilon 2^{2l}\int \frac{1}{q'}|\phi_1\omega_1\partial^2\tilde{P}_l u|^2 |P_J(\partial_t P_J w e^p)|^2 dx \Big),
\end{aligned}
\tag{5.80}
$$

where $\epsilon > 0$ can be taken sufficiently small, and $C_\epsilon > 0$ depends on ϵ.

Lemma 5.3. *We have*

$$
\|\phi_1\langle r-t\rangle\partial^2\tilde{P}_l u\|_\infty \lesssim t^{-\frac{1}{2}} 2^{-2l}.
\tag{5.81}
$$

Proof. By (2.29) and noting that $r \sim t$ (thanks to the cut-off ϕ_1), we have

$$
\begin{aligned}
\|\phi_1\langle r-t\rangle\partial^2\tilde{P}_l u\|_\infty &\lesssim \|\partial\Gamma^{\le 1}\tilde{P}_l u\|_\infty + t\|\Box\tilde{P}_l u\|_\infty \\
&\lesssim 2^{-2l} t^{-\frac{1}{2}}. \quad \Box
\end{aligned}
\tag{5.82}
$$

By using Lemma 5.3, it is not difficult to check that (5.80) is under control. Thus (5.78) is acceptable for us. The estimate of (5.79) is similar. We omit the details. This concludes the estimate of (5.76).

Next we estimate the piece

$$
\sum_{J\ge 0} 2^{-2J} \sum_{1\le i\le 2, 0\le k\le 2} g^{ki0} \sum_{l\ge J+4} \int P_J(\phi_1\omega_k\partial_t\tilde{P}_l u T_i\partial_t P_l w)\partial_t P_J w e^p dx.
\tag{5.83}
$$

The idea is to rewrite

$$
\begin{aligned}
&\int P_J(\phi_1\omega_k\partial_t\tilde{P}_l u T_i\partial_t P_l w)\partial_t P_J w e^p dx \\
&= \frac{d}{dt}\Big(\int P_J(\phi_1\omega_k\partial_t\tilde{P}_l u T_i P_l w)\partial_t P_J w e^p dx\Big) - \int P_J(\phi_1\omega_k\partial_t\tilde{P}_l u T_i P_l w)\partial_{tt} P_J w e^p dx \\
&\quad - \int P_J(\phi_1\omega_k\partial_t\tilde{P}_l u T_i P_l w)\partial_t P_J w e^p \partial_t p dx - \int P_J(\partial_t(\phi_1\omega_k\partial_t\tilde{P}_l u) T_i P_l w)\partial_t P_J w e^p dx.
\end{aligned}
\tag{5.84}
$$

It is not difficult to check that all terms are under control.

Finally we note that the piece

$$
\sum_{J\ge 0} 2^{-2J} g^{kij} \sum_{l\ge J+4} \int P_J(\phi_1\omega_k\omega_i\partial_t\tilde{P}_l u T_j\partial_t P_l w)\partial_t P_J w e^p dx
\tag{5.85}
$$

can be estimated similarly. We omit the details. This concludes the estimate of (5.68) and Case 1c.

Case 1d: the piece

$$\sum_{0 \leq J \leq 7} 2^{-2J} g^{kij} \int P_J(\partial_k u \partial_{ij} P_{\leq J+3} w) \partial_t P_J w e^p dx. \tag{5.86}$$

Since $0 \leq J \leq 7$, it is not difficult to check that this case is under control.

Case 2: $|\alpha_1| \leq \frac{m}{2}$, $|\alpha_2| \leq m$ with $\alpha_1 + \alpha_2 \leq \beta$, i.e. the piece

$$\sum_{J \geq 0} 2^{-2J} g^{kij} \int P_J(\partial_k \Gamma^{\alpha_1} u \partial_{ij} \Gamma^{\alpha_2} u) \partial_t P_J w e^p dx. \tag{5.87}$$

This case can again be treated by using the decomposition (with no loss consider the main case $J \geq 8$)

$$\sum_{J \geq 0} 2^{-2J} g^{kij} \int P_J(\partial_k \Gamma^{\alpha_1} u \partial_{ij} \Gamma^{\alpha_2} u) \partial_t P_J w e^p dx$$

$$= \sum_{J \geq 0} 2^{-2J} g^{kij} \int P_J(\partial_k \tilde{P}_J \Gamma^{\alpha_1} u \partial_{ij} P_{\leq J-3} \Gamma^{\alpha_2} u) \partial_t P_J w e^p dx$$

$$+ \sum_{J \geq 0} 2^{-2J} g^{kij} \int P_J(\partial_k \Gamma^{\alpha_1} u \partial_{ij} P_{[J-3,J+3]} \Gamma^{\alpha_2} u) \partial_t P_J w e^p dx$$

$$+ \sum_{J \geq 0} 2^{-2J} g^{kij} \int P_J(\partial_k \Gamma^{\alpha_1} u \partial_{ij} P_{\geq J+4} \Gamma^{\alpha_2} u) \partial_t P_J w e^p dx. \tag{5.88}$$

The estimates are similar to the quasilinear piece $\alpha_1 = 0$, $\alpha_2 = \beta$. We omit the details.

Case 3: $|\alpha_2| \leq \frac{m}{2}$, $|\alpha_1| \leq m$ with $\alpha_1 + \alpha_2 \leq \beta$, i.e. the piece

$$\sum_{J \geq 0} 2^{-2J} g^{kij} \int P_J(\partial_k \Gamma^{\alpha_1} u \partial_{ij} \Gamma^{\alpha_2} u) \partial_t P_J w e^p dx. \tag{5.89}$$

The situation is similar to the case $\alpha_1 = \beta$, $\alpha_2 = 0$ which is discussed below. We omit the details.

Case 4: $\alpha_1 = \beta$, $\alpha_2 = 0$. In this case we need to estimate

$$\sum_{J \geq 0} 2^{-2J} g^{kij} \int P_J(\partial_k w \partial_{ij} u) \partial_t P_J w e^p dx. \tag{5.90}$$

Case 4a: $J \geq 8$. We write

$$P_J(\partial_k w \partial_{ij} u) = P_J(\partial_k P_{\leq J-3} w \partial_{ij} \tilde{P}_J u) + P_J(\partial_k P_{[J-2,J+2]} w \partial_{ij} u)$$

$$+ \sum_{l \geq J+3} P_J(\partial_k P_l w \partial_{ij} \tilde{P}_l u), \tag{5.91}$$

where $\tilde{P}_l$ denotes the fattened Littlewood-Paley projection localized to $|\xi| \sim 2^l$.

We shall sketch the details for the second term $P_J(\partial_k P_{[J-2,J+2]} w \partial_{ij} u)$. The first and the third term can be treated along similar lines with the help of Lemma 5.3. Thus we only need to consider

$$\sum_{J \geq 8} 2^{-2J} g^{kij} \int P_J(\partial_k \tilde{w}_J \partial_{ij} u) \partial_t P_J w e^p \, dx, \tag{5.92}$$

where $\tilde{w}_J = P_{[J-2,J+2]} w$.

Subcase 4a1: the regime $|r - t| \geq \frac{1}{2} t$. Choose a radial bump function $a \in C_c^\infty(\mathbb{R}^2)$ such that $a(x) = 1$ for $0.9 \leq |x| \leq 1.1$, and $a(x) = 0$ for $|x| \leq 0.8$ or $|x| \geq 1.2$. Define $\phi_1(t, x) = a(x/t)$. We estimate the piece

$$\sum_{J \geq 8} 2^{-2J} \int P_J(\partial \tilde{w}_J \underbrace{(1 - \phi_1) \partial^2 u}_{=:F_1}) \partial_t P_J w e^p \, dx. \tag{5.93}$$

Observe that

$$\|F_1\|_\infty + \|\partial^2 F_1\|_\infty \lesssim t^{-\frac{3}{2}} E_5^{\frac{1}{2}}. \tag{5.94}$$

The contribution of this case is clearly acceptable.

Subcase 4a2: the regime $|r - t| < \frac{1}{2} t$. We estimate the piece

$$\sum_{J \geq 8} 2^{-2J} g^{kij} \int P_J(\partial_k \tilde{w}_J \phi_1 \partial_{ij} u) \partial_t P_J w e^p \, dx. \tag{5.95}$$

By using the null condition, we rewrite

$$g^{kij} \partial_k \tilde{w}_J \partial_{ij} u = g^{kij}(T_k \tilde{w}_J T_i \partial_j u - \omega_i T_k \tilde{w}_J T_j \partial_t u - \omega_k \partial_t \tilde{w}_J T_i \partial_j u + \omega_k \omega_i \partial_t \tilde{w}_J T_j \partial_t u$$
$$+ \omega_i \omega_j T_k \tilde{w}_J \partial_{tt} u). \tag{5.96}$$

The first four terms all contain $T \partial u$. To estimate them, it suffices for us to consider the general expression (below $h \in C^\infty$ corresponds to various expressions involving ω_k, ω_i which are functions of the polar angle θ)

$$\sum_{J \geq 8} 2^{-2J} \int P_J(\partial \tilde{w}_J \underbrace{h(\theta) \phi_1 T \partial u}_{=:F_2}) \partial_t P_J w e^p \, dx. \tag{5.97}$$

Observe that

$$\|F_2\|_\infty + \|\partial^2 F_2\|_\infty \lesssim t^{-\frac{3}{2}}. \tag{5.98}$$

The contribution of this piece is clearly acceptable.

We then consider the main piece

$$\sum_{J\geq 8} 2^{-2J} \int P_J(\underbrace{g^{kij}\omega_i\omega_j\phi_1}_{=:\phi_2}\partial_{tt}uT_k\tilde{w}_J)\partial_t P_J we^p dx. \tag{5.99}$$

We estimate it as follows:

$$|(5.99)| \leq \epsilon \sum_{J\geq 0} 2^{-2J} \int |T\tilde{w}_J|^2 q' e^p dx + C_\epsilon \cdot \sum_{J\geq 0} 2^{-2J} \int \frac{1}{q'}|\partial_{tt}u|^2|P_J(e^p\partial_t P_J w)|^2 dx, \tag{5.100}$$

where $\epsilon > 0$ can be taken sufficiently small and $C_\epsilon > 0$ depends on ϵ. Summing over $|\beta| = m+1$ and taking $\epsilon > 0$ sufficiently small, the first term above can be absorbed by the positive Alinhac term in (5.9). The second term can be bounded as

$$\text{const} \cdot \frac{1}{t} E_5(u) \cdot \|\langle\nabla\rangle^{-1}\partial w\|_2^2$$

which is clearly acceptable for us.

Case 4b: $0 \leq J \leq 7$. This is similar to the case $J \geq 8$ which some minor changes in numerology. We omit the details.

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