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The linearized cubic NLS has no embedded eigenvalue

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Abstract

The linearized operator $\mathcal{L}$ associated with the three-dimensional cubic nonlinear Schrödinger equation is shown to possess no embedded eigenvalues in the essential spectrum. This result rigorously confirms the key spectral hypothesis underpinning the construction of center-stable manifolds by Schlag (Ann. Math. (2) 169(1):139–227, 2009), and thereby provides the final analytical link for an unconditional stability theory near the ground state soliton. Unlike the one-dimensional case, the 3D model lacks integrability; the non-self-adjointness of $\mathcal{L}$ and the non-explicit profile of the ground state render classical techniques insufficient to exclude eigenvalues in the continuum. The proof rests on the introduction of a weight-modulated positivity trap to exploit the hyperbolic instability of the fundamental mode, combined with a novel constrained shooting method and delicate comparison arguments for higher angular momenta. Beyond the cubic model, the machinery introduced here provides a robust analytical framework for addressing the spectral coercivity conjectures central to the Merle–Raphaël's log-log blow-up on mass-critical NLS.

Contents

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1 Introduction

The three-dimensional cubic nonlinear Schrödinger equation (NLS)

$$i\partial_t \psi + \Delta\psi + |\psi|^2\psi = 0, \tag{1.1}$$

occupies a singular position in the landscape of mathematical physics. It serves as the canonical mean-field description of dilute Bose–Einstein condensates [25, 45], providing a rigorous framework for the study of macroscopic quantum phenomena. In this context, (1.1) is derived as the rigorous limit of N-boson quantum dynamics as $N \to \infty$, effectively describing the dynamics of a single macroscopic wave function where the cubic nonlinearity models the collective s-wave scattering of particles.

Beyond quantum gases, (1.1) is established as the universal envelope equation for weakly nonlinear wave packets in dispersive media [68]. As such, it governs a remarkably diverse range of classical phenomena: from the modulation of electromagnetic beams in nonlinear optics [3, 16, 39, 69] and Langmuir waves in plasmas [76], to the self-induced motion of vortex filaments in incompressible fluids via the Hasimoto transformation [36].

Concerning the general equation for $u = u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $d \geq 1$:

$$i\partial_t u + \Delta u = \mu|u|^{p-1}u, \qquad \mu = \pm 1,\ p > 1, \tag{1.2}$$

one has (for smooth solutions) mass conservation $\int_{\mathbb{R}^d} |u(t, x)|^2 dx \equiv \int_{\mathbb{R}^d} |u(0, x)|^2 \times dx$, and energy conservation:

$$\mathcal{E}(t) = \int_{\mathbb{R}^d} \left(\tfrac{1}{2}|\nabla u(t, x)|^2 + \tfrac{\mu}{p+1}|u(t, x)|^{p+1}\right)dx, \quad \mathcal{E}(t) = \mathcal{E}(0), \qquad \forall t \geq 0. \tag{1.3}$$

The cases $\mu = 1$ and $\mu = -1$ are usually called defocusing and focusing respectively. For $\lambda > 0$, the scaling transformation $u(t, x) \to \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x)$ can be invoked to determine the critical functional spaces. In particular, $p = 1 + \frac{4}{d}$ is called mass-critical for which the scaling transformation leaves the L^2 mass invariant and likewise $p = 1 + \frac{4}{d-2}$ $(d \geq 3)$ is called energy-critical. In this context the governing equation (1.1) is $\dot{H}^{\frac{1}{2}}$ critical (mass-supercritical, energy-subcritical) with focusing nonlinearity. As a result the equation (1.1) is orbitally unstable [7] whereas for mass-subcritical equations one expects orbital stability [14, 73, 74].

There are a number of natural symmetries associated with the solutions to (1.1). For example, if $\psi = \psi(t, x)$ is a smooth solution, then the reflection $\psi(t, -x)$ is also a solution. More generally for $\lambda > 0$, $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^3$, $O \in SO(3)$, $c \in \mathbb{R}^3$, $\theta_0 \in \mathbb{R}$, the twelve-parameter family of transformations (translation, rotation, Galilean boost, dilation)

$$\psi_{\lambda,t_0,x_0,O,c,\theta_0}(t, x) = \lambda\psi(\lambda^2(t - t_0), \lambda(Ox - x_0 - ct)) \cdot e^{i\theta_0} e^{i(\frac{1}{2}Ox\cdot c - \frac{1}{4}t|c|^2)} \quad (1.4)$$

generate invariant solutions to (1.1). As it later transpired, the infinitesimal generators of these "continuous" transformations contribute to certain secular directions of the linearized Schrödinger operator.

The equation (1.1) possesses soliton solutions of the form $\psi = e^{it}R(x)$, where R solves

$$\Delta R - R + |R|^2 R = 0.$$

By [8, 67] (see also [9, 10]), the above equation has infinitely many H^1 solutions. Among those we denote by Q the positive radial ground state which minimizes the functional $J(u) = \frac{\|\nabla u\|_2^3 \|u\|_2}{\|u\|_4^4}$. In the physics literature this is known as the Townes soliton [16]: a stationary, self-trapped wave profile where nonlinear self-focusing precisely balances diffractive spreading. In the radial coordinate one can write $Q(x) = y(r)$ $(r = |x|)$, where the one-variable function y solves the ODE

$$-y''(r) - \tfrac{2}{r}y'(r) + y(r) - y^3(r) = 0. \quad (1.5)$$

The uniqueness of Q can be stated in a somewhat more general setting. By Coffman [17] (see also Kwong [43] and [50, 51]), there exists a unique solution to (1.5) satisfying $y(r) > 0$ for any $r \in (0, \infty)$, $y'(0) = 0$, $y(r) \to 0$ as $r \to \infty$. By (1.4), the standing waves (here we regard $\theta_1 = \theta_0 - \lambda^2 t_0$ as a single parameter) $\lambda Q(\lambda(Ox - x_0 - ct))e^{i\theta_1} e^{i(\frac{1}{2}Ox\cdot c - \frac{1}{4}t|c|^2 + t\lambda^2)}$ form an 11-parameter family of solutions to (1.1). Since Q is radial, the infinitesimal generators for SO(3) do not contribute any nontrivial modes. Thus 8 out of 11 parameters correspond to 8 secular modes for the corresponding linearized NLS operator around the ground state soliton.

Since (1.1) is a mass-supercritical equation, its action functional exhibits a mountain-pass geometry where Q is no longer a minimizer but a saddle point. This instability [7] was formalized through the general theory of Grillakis–Shatah–Strauss [33, 34] and the spectral analysis of Weinstein [73, 74], which rigorously established that the linearized operator L_+ possesses a Morse index of one. This unstable direction implies that generic perturbations trigger either finite-time blow-up or dispersive scattering, rendering Q orbitally unstable in $H^1(\mathbb{R}^3)$.

Writing in (1.1) $\psi = e^{it}(Q + \eta)$ with $\eta = \eta_1 + i\eta_2$, we have

$$\partial_t \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}}_{=:\mathcal{L}} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \mathcal{O}(|\eta|^2),$$

$$L_+ := -\Delta + 1 - 3Q^2, L_- := -\Delta + 1 - Q^2. \tag{1.6}$$

Alternatively one can recast the above dynamics using $(\eta, \overline{\eta})$ where $\overline{\eta}$ is the complex conjugate of η:

$$\partial_t \begin{pmatrix} \eta \\ \overline{\eta} \end{pmatrix} = -i \underbrace{\begin{pmatrix} -\Delta + 1 - 2Q^2 & -Q^2 \\ Q^2 & \Delta - 1 + 2Q^2 \end{pmatrix}}_{=:\mathcal{H}} \begin{pmatrix} \eta \\ \overline{\eta} \end{pmatrix} + \mathcal{O}(|\eta|^2). \tag{1.7}$$

The operator $\mathcal{H}$ is sometimes called the matrix Hamiltonian. Denote $P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, $P_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Clearly

$$\mathcal{H} = P_1^{-1} \underbrace{\begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix}}_{=:\widetilde{A}} P_1, \qquad \mathcal{L} = iP\widetilde{A}P^{-1}. \tag{1.8}$$

As operators on $L^2(\mathbb{R}^3, \ \mathbb{C}^2)$, the spectra of $\mathcal{L}$, $\mathcal{H}$ and $\widetilde{A}$ are related by $\sigma(\mathcal{L}) = i \times \sigma(\widetilde{A}) = i\sigma(\mathcal{H})$.

The matrix operator $\mathcal{H}$ is closed with domain $D(\mathcal{H}) = H^2(\mathbb{R}^3, \mathbb{C}^2)$. The spectrum $\sigma(\mathcal{H})$ lies entirely within the real and imaginary axes, i.e., $\sigma(\mathcal{H}) \subset \mathbb{R} \cup i\mathbb{R}$. The essential spectrum[1] of $\mathcal{H}$ is $\sigma_{\text{ess}}(\mathcal{H}) = (-\infty, -1] \cup [1, \infty)$. In [73, 74], Weinstein computed the root space of $\mathcal{L}$ at zero and showed that $\ker(\mathcal{L}) \subsetneq \ker(\mathcal{L}^2) = \ker(\mathcal{L}^3)$ with $\dim(\ker(\mathcal{L})) = 4$, $\dim(\ker(\mathcal{L}^2)) = 8$. As a matter of fact, one has

$$\ker(\mathcal{L}) = \operatorname{span}\left\{ \begin{pmatrix} \partial_1 Q \\ 0 \end{pmatrix}, \begin{pmatrix} \partial_2 Q \\ 0 \end{pmatrix}, \begin{pmatrix} \partial_3 Q \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Q \end{pmatrix} \right\}; \tag{1.9}$$

$$\ker(\mathcal{L}^2) = \ker(\mathcal{L}) \oplus \operatorname{span}\left\{ \begin{pmatrix} Q + x \cdot \nabla Q \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 Q \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 Q \end{pmatrix}, \begin{pmatrix} 0 \\ x_3 Q \end{pmatrix} \right\}. \tag{1.10}$$

Note that (1.10) is connected with the identities $L_+(Q_1) = -2Q$, $L_-(x_j Q) = -2\partial_j Q$, where $Q_1 = Q + x \cdot \nabla Q$. Since $\mathcal{L} = iPP_1\mathcal{H}P_1^{-1}P^{-1}$, the operator $\mathcal{H}$ also possesses a root space of dimension eight at zero.

Assuming the technical condition that L_+ does not have any eigenvalues in (0, 1), Schlag (see Lemma 15 of [64] and Proposition 3.12 of this paper) showed that the only eigenvalue of $\mathcal{H}$ in $[-1, 1]$ is zero. Moreover, assuming L_- has neither an eigenvalue nor a resonance at 1, Schlag also showed (see Lemma 16 of [64] and Proposition 3.14 of this paper) that ± 1 are not resonances of $\mathcal{H}$. Here we recall the definition of resonance: a "resonance" at λ means the existence of some solution $\overrightarrow{f}$ to $\mathcal{H}\overrightarrow{f} = \lambda\overrightarrow{f}$ such that $\overrightarrow{f} \notin L^2(\mathbb{R}^3, \mathbb{C}^2)$ but $\int_{\mathbb{R}^3} \frac{|\overrightarrow{f}(x)|^2}{1+|x|^s} dx < \infty, \forall s > 1$.

On the other hand, by [33] and [64] (see also Proposition 3.9 of this paper), for some $\lambda_0 > 0$ one has $\sigma(\mathcal{H}) \cap i\mathbb{R} = \{0, i\lambda_0, -i\lambda_0\}$, where both $i\lambda_0$ and $-i\lambda_0$

[1] There exist at least five different definitions of the essential spectrum for non-selfadjoint operators. However they are all equivalent in the present setting (cf. Hundertmark and Lee [37]).

are simple eigenvalues with C^∞ exponentially decaying eigenfunctions. Thus assuming L_+ has no eigenvalues in (0, 1) (no "gap eigenvalues"), one has $\sigma_{\text{dis}}(\mathcal{H}) = \{0,\ i\lambda_0,\ -i\lambda_0\}$, where σ_{dis} denotes the discrete spectrum.

In the L^2-supercritical regime, the above spectral analysis of $\mathcal{H}$ reveals a structured threshold for the nonlinear flow. A systematic description of this threshold was established by the Schlag program [64], which sought to extend the modulation methods of Soffer–Weinstein [65, 66] to cases where the linearized operator exhibits exponential instabilities. This lineage began with the work of Perelman [58] on the 1D mass-critical NLS and was extended to the full mass-supercritical setting by Krieger and Schlag [42].

In this framework, the geometry of the flow near Q is dictated by three primary spectral obstructions:

(1) The Unstable Mode: The presence of a unique pair of imaginary eigenvalues $\pm\lambda_0 i$ for $\mathcal{H}$, which necessitates the construction of a codimension-one center-stable manifold $\mathcal{M}$ to suppress exponential growth.

(2) The Symmetry-Induced Null Space: As a consequence of the fundamental symmetries of the NLS (scaling, phase, translations, and Galilean boosts), the operator $\mathcal{H}$ possesses a high-dimensional generalized kernel at the origin. In the 3D case, this results in an eight-dimensional spectral pile-up of generalized eigenvectors at $0 \in \sigma(\mathcal{L})$, which must be suppressed via modulation.

(3) The Absence of Embedded Eigenvalues: The requirement that the remaining spectrum, i.e., the essential spectrum $\sigma_{\text{ess}}(\mathcal{H}) = (-\infty, -1] \cup [1, \infty)$, be "clean", meaning it is devoid of embedded eigenvalues and threshold resonances.

Whilst the first two obstructions were handled in Schlag [64], the third one was formulated as a key spectral hypothesis therein:

(i) The operators L_+ and L_- do not admit any eigenvalues in (0, 1), and the operator L_- has neither an eigenvalue nor a resonance at 1.

(ii) The matrix operator $\mathcal{H}$ does not have any embedded eigenvalues in the essential spectrum.

Note that in [64] and later works, the first condition is sometimes called the spectral gap property as it is only concerned with the elliptic operators $L_\pm$. In [64], by imposing the above spectral conditions, Schlag showed that there exists a Lipschitz manifold in $H^1(\mathbb{R}^3) \cap W^{1,1}(\mathbb{R}^3)$ of initial data that generate global $H^1 \cap W^{1,\infty}$ solutions to (1.1) around a moving ground state soliton. Due to the use of $W^{1,1}$-norm, the manifold constructed in [64] was not time invariant. The technical pinnacle of the Schlag program was reached in Beceanu [5, 6]. Building on the weighted Sobolev space construction in [5], Beceanu achieved the definitive refinement in [6] by establishing $\mathcal{M}$ as a real-analytic hypersurface in the scaling-critical space $\dot{H}^{\frac{1}{2}}$. Note that the above spectral conditions were also assumed in Beceanu's work. The spectral gap condition also played an important role in the work of Nakanishi and Schlag (cf. [55, 57] and the references therein) on the center-stable manifold for cubic-nonlinear Klein-Gordon equations. In [56], Nakanishi and Schlag investigated[2] classification (à la Kenig-Merle [40]) of the global dynamics of (1.1) for radial initial data of energy

[2]The notation of [56] differs from our equation (1.1) by a harmless minus sign, namely they considered $i\partial_t u = \Delta u + |u|^2 u$. This corresponds to changing $i \to -i$ or $t \to -t$.

slightly above the ground state. In the radial case, a very nice summary of the spectral information for $\mathcal{H}$ can be found in Proposition B.1 of [56].

However, despite the above landmark advances, the theory has long harbored an Achilles' heel: the dispersive estimates required to close the nonlinear modulation equations are fundamentally conditional upon the rigorous verification of the absence of embedded spectrum. While the center-stable manifold provides the geometric vessel for stability, any embedded eigenvalue would act as a localized resonator, trapping energy and preventing the radiation damping required for the solution to relax to the ground state. Consequently, the field has largely proceeded under the assumption that such spectral obstructions do not exist, often relying on the heuristic of the Fermi Golden Rule or numerical observations in non-integrable settings.

In [24], Erdoğan and Schlag considered the matrix Schrödinger operator of the form

$$\widetilde{\mathcal{H}} = \begin{pmatrix} -\Delta+\mu & 0 \\ 0 & \Delta-\mu \end{pmatrix} + \underbrace{\begin{pmatrix} -V_1 & -V_2 \\ V_2 & V_1 \end{pmatrix}}_{V}, \tag{1.11}$$

where $\mu > 0$ and V_1, V_2 are real-valued decaying potentials. Under the assumptions:

A1) (Here $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a Pauli matrix.) $-\sigma_3 V = \begin{pmatrix} V_1 & V_2 \\ V_2 & V_1 \end{pmatrix}$ is a positive matrix;

A2) $L_- := -\Delta + \mu - V_1 + V_2 \geq 0$;

A3) For some $\beta > 10$, $|V_1(x)| + |V_2(x)| \lesssim (1+|x|)^{-\beta}$;

A4) There are no embedded eigenvalues in $(-\infty, -\mu) \bigcup (\mu, \infty)$;

Erdoğan and Schlag developed an L^2 bound of the form $\sup_{t\in\mathbb{R}} \|e^{it\widetilde{\mathcal{H}}} P_s\|_{2\to 2} \leq C$ (this in fact only requires $\beta > 5$) and the dispersive estimate $\|e^{it\widetilde{\mathcal{H}}} P_c - t^{-\frac{1}{2}} F_t\|_{1\to\infty} \leq C t^{-\frac{3}{2}}$, where $\sup_t \|F_t\|_{L^1\to L^\infty} < \infty$.

In the numerical work [23], Demanet and Schlag employed the Birman-Schwinger method and checked the spectral gap property for the linearization of NLS with monomial nonlinearities $|\psi|^{2\beta}\psi$, $\beta_* < \beta \leq 1$, $\beta_* \approx 0.913958905$. In a subsequent numerical work [49], Marzuola and Simpson carried out extensive index computations of various quadratic forms (see also [29] for related virial arguments) to check the spectral condition (i.e., absence of embedded eigenvalues in the essential spectrum). It should be noted that the numerical work of Marzuola and Simpson also supports the spectral gap property for the 3D cubic NLS (1.1) as well as a class of one-dimensional NLS with monomial nonlinearities. These provide solid evidence that the spectral condition and the spectral gap property can be proved by completely rigorous arguments.

This is first achieved in [20], where Costin, Huang and Schlag devised a computational (but fully rigorous!) strategy to give a rigorous proof of the spectral gap property under radial assumptions. The main achievements in [20] are two:

- A remarkably accurate approximate (and explicit) ground state $\widetilde{Q}$ which differs from the true ground state by $\mathcal{O}(10^{-4})$. More precisely, the pointwise error is at most $\frac{7}{10^5} \cdot \frac{1}{1+r} e^{-r}$;
- A robust Wronskian strategy connecting two Jost quasi-solutions: one emanating from $r = 0$, and the other (decaying) solution from $r = \infty$.

The decisive step is to check $\inf_{\lambda\in[0,1]}|W(\lambda)|>0$ for L_+ and $\inf_{\lambda\in[0,1]}|W(\lambda)/\lambda|>0$ for L_-, where λ is the spectral parameter. This very involved computation was executed in [20] to prove the spectral gap property for the radial case.

In [44], by using the Costin-Huang-Schlag approximate solution and a new comparison approach, we settled the fully nonradial case and proved the spectral gap property.

Theorem ([44]) *Let Q be the ground state soliton to* (1.1) *and recall $L_+=-\Delta+1-3Q^2$, $L_-=-\Delta+1-Q^2$. The operators L_+ and L_- do not have any eigenvalues in* $(0,1]$; *also they do not have any resonances at* 1.

The theorem above completely describes the spectrum of the elliptic operators $L_\pm$. It follows that the matrix operator $\mathcal{H}$ has zero as its only eigenvalue in $[-1,1]$, and that ±1 are not resonances of $\mathcal{H}$. The method in [44], however, is confined to elliptic operators. Consequently, it does not extend to the non-selfadjoint operator $\mathcal{H}$, and leaves completely open the question of whether embedded eigenvalues exist within its essential spectrum.

In one dimension, the meromorphic structure of the linearized operators allows for an explicit exclusion of trapping modes [42, 58]. In three dimensions, however, the loss of integrability and the presence of the centrifugal barrier render classical Evans function techniques and Wronskian-based root-tracking methods ineffective. For the 3D cubic NLS, as was already mentioned earlier, numerical studies by Demanet–Schlag [23] and Marzuola–Simpson [49] have provided compelling evidence that the "absence of embedded spectrum" holds. Yet, a fully rigorous proof has remained elusive for over two decades. The absence of an explicit form for the ground state soliton Q, as well as the non-integrability in 3D, combined with the non-self-adjointness of $\mathcal{L}$, renders standard exclusion techniques insufficient.

In this work, we provide this missing analytical component, establishing the spectral stability of the ground state without further assumptions.

Theorem 1.1 (No embedded eigenvalues in the essential spectrum) *Let Q be the ground state soliton to* (1.1) *and recall the linearized matrix operator on* $L^2(\mathbb{R}^3,\mathbb{C}^2)$:

$$\mathcal{H}=\begin{pmatrix}-\Delta+1-2Q^2 & -Q^2\\ Q^2 & \Delta-1+2Q^2\end{pmatrix}.$$

The operator $\mathcal{H}$ does not have any embedded eigenvalues in the essential spectrum $\sigma_{\text{ess}}(\mathcal{H})=(-\infty,-1]\bigcup[1,\infty)$.

Remark 1.2 Same results hold for $\widetilde{A}=\begin{pmatrix}0 & L_-\\ L_+ & 0\end{pmatrix}$ or $\widetilde{\mathcal{L}}=\begin{pmatrix}0 & iL_-\\ -iL_+ & 0\end{pmatrix}$. That $\widetilde{A}$ and $\widetilde{\mathcal{L}}$ are similar (i.e., $\widetilde{\mathcal{L}}=P_2^{-1}\widetilde{A}P_2$ for some invertible constant matrix P_2) can be easily seen as follows:

$$\begin{cases} iL_- b = \tau a \\ -iL_+ a = \tau b \end{cases} \quad \Leftrightarrow \quad \begin{cases} L_-(ib) = \tau a \\ L_+ a = \tau (ib). \end{cases}$$

As a matter of fact, taking $P_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ easily leads to $\widetilde{\mathcal{L}} = P_2^{-1} \widetilde{A} P_2$.

Theorem 1.1 together with earlier results leads to the following full description of the spectrum of $\mathcal{H}$.

Corollary 1.3 (Complete characterization of the spectrum of $\mathcal{H}$) *Let Q be the ground state soliton to* (1.1). *The following hold.*

(1) *For some $\lambda_0 > 0$, we have $\sigma_{\text{dis}}(\mathcal{H}) = \{0,\ i\lambda_0,\ -i\lambda_0\}$. Both $i\lambda_0$ and $-i\lambda_0$ are simple eigenvalues with C^∞ exponentially decaying eigenfunctions. The root space of $\mathcal{H}$ has dimension* 8, *with*

$$\ker(\widetilde{A}) = \text{span}\left\{ \begin{pmatrix} \partial_1 Q \\ 0 \end{pmatrix}, \begin{pmatrix} \partial_2 Q \\ 0 \end{pmatrix}, \begin{pmatrix} \partial_3 Q \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Q \end{pmatrix} \right\}; \tag{1.12}$$

$$\ker(\widetilde{A}^2) = \ker(\widetilde{A}) \oplus \text{span}\left\{ \begin{pmatrix} Q_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 Q \end{pmatrix}, \begin{pmatrix} 0 \\ x_2 Q \end{pmatrix}, \begin{pmatrix} 0 \\ x_3 Q \end{pmatrix} \right\};$$

$$(Q_1 := Q + x \cdot \nabla Q)$$

$$\ker(\mathcal{H}) = \text{span}\left\{ \begin{pmatrix} \partial_1 Q \\ \partial_1 Q \end{pmatrix}, \begin{pmatrix} \partial_2 Q \\ \partial_2 Q \end{pmatrix}, \begin{pmatrix} \partial_3 Q \\ \partial_3 Q \end{pmatrix}, \begin{pmatrix} Q \\ -Q \end{pmatrix} \right\}; \tag{1.13}$$

$$\ker(\mathcal{H}^2) = \ker(\mathcal{H}) \oplus \text{span}\left\{ \begin{pmatrix} Q_1 \\ Q_1 \end{pmatrix}, \begin{pmatrix} x_1 Q \\ -x_1 Q \end{pmatrix}, \begin{pmatrix} x_2 Q \\ -x_2 Q \end{pmatrix}, \begin{pmatrix} x_3 Q \\ -x_3 Q \end{pmatrix} \right\}; \tag{1.14}$$

$$\ker(\mathcal{H}^n) = \ker(\mathcal{H}^2), \quad \forall n \geq 2. \tag{1.15}$$

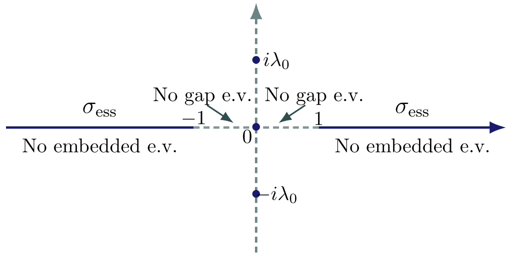
(2) $\sigma_{\text{ess}}(\mathcal{H}) = (-\infty, -1] \bigcup [1, \infty)$, *and $\mathcal{H}$ does not have any resonances at the edge* $\lambda = \pm 1$.

(3) (*No other exceptional eigenvalues*) *If $\mathcal{H}\vec{f} = \lambda \vec{f}$ for some $\lambda \in \mathbb{C}$ and some non-trivial $\vec{f} \in L^2(\mathbb{R}^3, \mathbb{C}^2)$, then $\lambda \in \sigma_{\text{dis}}(\mathcal{H})$. In particular, there are no embedded eigenvalues in the essential spectrum, and there are no "gap eigenvalues" in* $[-1, 0) \bigcup (0, 1]$. (*In short, there are no eigenvalues in $\mathbb{C} \setminus \{0, i\lambda_0, -i\lambda_0\}$.*)

Thanks to Corollary 1.3, we have a complete description of the spectrum of $\mathcal{H}$ (see Fig. 1). The proof of Corollary 1.3 is given in Sect. 3, where we also give a detailed summary of properties of L_+, L_-, $\widetilde{A}$ along with several other associated auxiliary operators.

We stress that this result supplies the final analytical link required for an unconditional stability theory near the 3D ground state soliton. It rigorously validates the

Fig. 1 The spectrum of $\mathcal{H}$



spectral hypothesis underpinning the manifold constructions in [5, 6, 64]. Furthermore, as we will discuss more fully later (see Technical Remark (iii) in Sect. 1.1), the machinery introduced here provides a robust framework for addressing the Merle–Raphaël spectral property and related coercivity conjectures [29, 52, 75] central to the log-log blow-up dynamics for mass-critical NLS.

1.1 Overview of the proof of Theorem 1.1

We assume[3] $\tau \geq 1$ is a putative embedded eigenvalue and proceed by contradiction. A spherical harmonic decomposition reduces the problem to radial ODE systems parametrized by the angular momentum index $\ell \geq 0$. The proof treats separately the regimes $\ell = 0$ and $\ell \geq 1$, each requiring original and distinct analytical approaches.

Case 1: The Monopole Mode $\ell = 0$. This regime is the most delicate due to the absence of a centrifugal barrier. We develop a weight-modulated positivity trap argument that replaces local tracking of Jost solutions with a global coercion principle.

- Large eigenvalues $\tau \geq \tau_{\max}$. We construct a weight-modulated energy functional that is fully coercive, forcing all solutions to be trivial. This relies solely on the explicit quantitative decay of the ground state.
- Intermediate regime $\tau \in [1, \tau_{\max}]$. We adopt a "divide and conquer" strategy, partitioning the spectral parameter space. On each subinterval, we introduce tailored weights that reduce the spectral problem to two algebraic criteria: one involving the ground state profile and the other (the H-criterion) detecting the onset of hyperbolic instability. We rigorously verify these criteria by constructing explicit and low-degree τ-dependent polynomial ansatz, thereby transforming the continuous spectral problem into a finite set of certified algebraic conditions.

Case 2: The Dipole Mode $\ell = 1$. The difficulty here stems from an orthogonality constraint that is not aligned with the negative direction of the Schrödinger operator. In the spirit of the Mourre commutator [54], we reduce the system to a bilinear form with a linear constraint. To establish non-negativity, a decomposition argument reduces the problem to showing the existence of a decaying solution to a certain inhomogeneous problem. The analytic challenge then lies in proving that this decaying solution also satisfies a definite integral sign condition, which demands precise quantitative control. For this we introduce a novel constrained shooting method and proceed in three stages:

[3] The case $\tau \leq -1$ follows by symmetry.

1. Via quantitative comparison, we characterize the precise sign structure and far-field asymptotics of the underlying kernel.

2. We construct an explicit comparison function that inverts the associated inhomogeneous equation, using a shooting parameter to satisfy the orthogonality constraint.

3. We develop further delicate quantitative comparison techniques to obtain sharp control of upper and lower solutions in both the near- and far-field regimes. Together with a continuity argument, this yields the desired decaying solution, which automatically satisfies the required sign condition.

Case 3: Higher Modes $\ell \geq 2$. the strategy for $\ell \geq 2$ parallels the $\ell = 1$ case but is substantially simpler. The same multiplier method produces quadratic forms whose positivity follows directly from the underlying ground state profile and the stronger centrifugal barrier ($\ell(\ell+1) \geq 6$), yielding an immediate contradiction.

We refer to Sect. 2 for a detailed structural flowchart of the proof.

Technical remarks

(i) Prior to this work, rigorous results for the full matrix Hamiltonian operator are largely restricted to one dimension, where the operators $L_\pm$ have the explicit form $-\partial_{xx} + 1 - c_1 \operatorname{sech}^2(x/c_2)$. Their essential spectrum is $[1, \infty)$ with finitely many eigenvalues below 1; the corresponding Jost solutions (generalized eigenfunctions) are explicit hypergeometric functions. The spectral gap property holds for all NLS exponents $2\sigma + 1$ with $\sigma > 1$ ([31]). For the matrix operator $\mathcal{H}$, there are no embedded eigenvalues in its essential spectrum in the full mass-supercritical regime ([42, 58]), as shown via an explicit transformation $z = \tanh(\sigma x)$, which reduces the problem to a system of ODEs with meromorphic coefficients.

(ii) For the 3D cubic NLS, the analysis is far more subtle than the one-dimensional case. As noted earlier, the spectral gap property for the scalar elliptic operators $L_\pm$ was first rigorously proved in the radial case by Costin, Huang, and Schlag in [20], and later resolved in the nonradial case in [44]. In three dimensions, this gap property turns out to depend in a delicate and pronounced way on the nonlinearity. Surprisingly, numerical calculations by Demanet and Schlag in [23], which employed the Birman–Schwinger method, indicated that the gap property can fail for nonlinearities slightly below 3. Specifically, for the NLS exponent $2\sigma + 1$ with $\sigma < \sigma_* \approx 0.91395$, the operator L_+ acquires eigenvalues in the gap $(0, 1]$. Thus, the spectral gap does not hold over the full $L^2(\mathbb{R}^3)$-mass-supercritical range $\frac{2}{3} < \sigma < 2$. This is in stark contrast to the one-dimensional results mentioned previously, where the gap property holds throughout the entire mass-supercritical regime. Our approach in this work is the first to provide the quantitative and completely rigorous precision necessary to distinguish between the stable cubic regime and the unstable regimes slightly below it, where L_+ acquires gap eigenvalues.

(iii) A very closely related question is the spectral property for mass-critical NLS and in particular the numerical work [29, 75]. Denote[4] $L_1 = -\Delta + \frac{2}{d}(\frac{4}{d} +$

[4] Recall $L_\pm = -\Delta + 1 - a_\pm Q^{\frac{4}{d}}$ with $a_+ = 1 + \frac{4}{d}$, $a_- = 1$. Let $\Lambda f = \frac{d}{2} f + x \cdot \nabla f = \partial_\lambda|_{\lambda=1}\big(\lambda^{d/2} f(\lambda x)\big)$. Then $L_{1,2} = \frac{1}{2}[L_\pm, \Lambda]$. For general $L_\pm = -\Delta + V_\pm$, this yields $L_{1,2} = -\Delta - \frac{1}{2} x \cdot \nabla V_\pm$.

$1)Q^{\frac{4}{d}-1}rQ'$, $L_2 = -\Delta + \frac{2}{d}Q^{\frac{4}{d}-1}rQ'$. Consider a complex-valued perturbation $\eta = \eta_1 + i\eta_2 \in H^1(\mathbb{R}^d, \mathbb{C})$ whose components satisfy the orthogonality conditions $\eta_1 \perp \text{span}\{Q, Q_1, x_1Q, \ldots, x_dQ\}$ and $\eta_2 \perp \text{span}\{Q_1, Q_2, \partial_{x_1}Q, \ldots, \partial_{x_d}Q\}$, where $Q_1 = \frac{d}{2}Q + x \cdot \nabla Q$, $Q_2 = \frac{d}{2}Q_1 + x \cdot \nabla Q_1$. Then there exists a constant $c_0 > 0$ such that

$$\langle L_1\eta_1, \eta_1\rangle + \langle L_2\eta_2, \eta_2\rangle \geq c_0 \int_{\mathbb{R}^d} (|\nabla\eta|^2 + |\eta|^2 e^{-|x|})dx.$$

This spectral property, introduced in Merle-Raphaël [52], is rigorously established in one spatial dimension owing to the explicit ground state soliton (see Appendix A of [52]). Its validity in dimensions $d = 2, 3, 4$ was confirmed only numerically (not rigorously) in [29]. Moreover, the formulation given above breaks down for $d = 5, 6$; specifically, for dimension $d = 5$, the orthogonality condition $\eta_1 \perp Q_1$ must be replaced by $\eta_1 \perp |x|^2 Q$. Subsequent numerical work [75] explored the modified spectral property in higher dimensions, ranging from dimensions 6 up to 11. The framework developed in this work is not idiosyncratic to 3D cubic NLS and can be adapted to address these problems completely rigorously.

(iv) In their numerical study of the 3D cubic NLS problem [49], Marzuola and Simpson adopted the framework of [29] and performed comprehensive index calculations for a range of quadratic forms to test for embedded eigenvalues within the essential spectrum. A notable technical difficulty, which is already pronounced even in the simplest case of angular momentum $\ell = 0$, is that this approach requires precise knowledge of ϕ_2, a function defined as the solution to a fourth-order eigenvalue problem given by

$$L_+L_-\phi_2 = -\lambda_0^2\phi_2.$$

Furthermore, the numerically assisted index computations, together with the inner products derived from approximate solutions (using adaptive nonlinear collocation algorithms, artificial far-field boundary conditions of Robin type, floating-point numbers, asymptotic expansion and a posteriori consistency checks), remain challenging to fully justify with complete rigor. In fact, the prevailing situation before this work is perhaps best captured by Schlag in [20] when discussing [23], who remarked that all such numerical findings "appear to be accurate on all empirical accounts, the numerical method implemented there is not a proof since it seems very difficult–if not impossible–to give rigorous error bounds for all numerical approximations and calculations....". These fundamental limitations constitute a key reason behind our development of a new framework, in which we demonstrate that the aforementioned "numerical barrier" can be overcome through a synthesis of hard analysis and certified algebraic reduction.

We also note that while the Evans function is a robust tool in 1D, extending it to probe embedded eigenvalues in the 3D essential spectrum presents significant obstacles. Techniques such as the Gap Lemma [32] or analytic continuation [38] require complex Riemann surface constructions that become prohibitively difficult in higher dimensions. Our strategy circumvents these issues entirely, replacing local tracking arguments with global coercion principles. For the monopole mode, our strategy of transforming a continuous spectral problem into finitely many certified algebraic conditions offers a robust alternative to perturbative methods. Similarly, for higher modes, the technique of establishing coercivity under linear constraints via constrained shooting is remarkably efficient, requiring minimal quantitative input regarding the linearized operator. The reduction of the spectral problem to certified algebraic conditions, combined with the novel constrained shooting method, creates a unified framework that does not rely on integrability or perturbative proximity to a known limit. Consequently, we expect these techniques to be widely applicable to other constrained, non-self-adjoint spectral problems.

1.2 Connections with earlier works and comments

- One dimensional NLS.

  For one dimensional NLS, by using ODE analysis (cf. Theorem 8.1.6., p. 259 of [13]) one can show that if $v \in H^1(\mathbb{R}, \mathbb{C})$ solves (below $0 < \sigma < \infty$)

$$v - v_{xx} - |v|^{2\sigma} v = 0,$$

  then for some $x_0 \in \mathbb{R}$, $\theta_0 \in \mathbb{R}$, it holds that $v(x) = e^{i\theta_0} Q_{1\mathrm{D}}(x - x_0)$, $\forall x \in \mathbb{R}$, where

$$Q_{1\mathrm{D}}(x) = (1+\sigma)^{\frac{1}{2\sigma}} \operatorname{sech}^{\frac{1}{\sigma}}(\sigma x).$$

  The corresponding linearized operators $L_\pm$ take the form

$$\begin{aligned} L_+ &= 1 - \partial_{xx} - (1+2\sigma) Q_{1\mathrm{D}}^{2\sigma} = 1 - \partial_{xx} - (1+\sigma)(1+2\sigma)\operatorname{sech}^2(\sigma x); \\ L_- &= 1 - \partial_{xx} - Q_{1\mathrm{D}}^{2\sigma} = 1 - \partial_{xx} - (1+\sigma)\operatorname{sech}^2(\sigma x). \end{aligned} \tag{1.16}$$

  Note that the potential term is of modified Pöschl-Teller type.[5] Thanks to this explicit form, one can work out[6] the eigenfunctions and generalized eigenfunctions of these operators in terms of certain hypergeometric functions (cf. Problem 39 on p. 94 of [31] or p. 103 of the classical book [71]). In particular for $\sigma > 1$, the operators $L_\pm$ have no eigenvalues in the interval $(0, 1]$ and no resonances at the edge $\lambda = 1$ (i.e., no nontrivial solutions to $L_\pm f = f$ with $f \in L^\infty \setminus L^2$; for a simple proof, see Lemma 9.1 of [42] which is based on the calculation in [31]). Interestingly, except for the first negative eigenvalue of L_+, all the remaining eigenvalues of L_+ and L_- exactly coincide and can be explicitly computed (cf. Theorem 3.1 of [15]). This seems to be a phenomenon exclusively occurring in one dimension. On the other hand, to investigate the discrete spectrum of $\widetilde{A} = \begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix}$

[5] The usual Pöschl-Teller potential takes the form $V(x) = C(\frac{\kappa(\kappa-1)}{\sin^2 \alpha x} + \frac{\lambda(\lambda-1)}{\cos^2 \alpha x})$ with $C > 0$, $\kappa > 1$, $\lambda > 1$ for $x \in (0, \frac{\pi\alpha}{2})$.

[6] All eigenvalues are simple.

(equivalently $\mathcal{H} = \begin{pmatrix} 1-\partial_{xx}-(\sigma+1)^2\mathrm{sech}^2(\sigma x) & -\sigma(\sigma+1)\mathrm{sech}^2(\sigma x) \\ \sigma(\sigma+1)\mathrm{sech}^2(\sigma x) & -1+\partial_{xx}+(\sigma+1)^2\mathrm{sech}^2(\sigma x) \end{pmatrix}$) or $\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}$ (note that $\sigma(\widetilde{A}) = i\sigma(\mathcal{L})$, more precisely $P_2^{-1}\widetilde{A}P_2 = i\mathcal{L}$ with $P_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$), it suffices to examine the eigenvalues of the self-adjoint operator $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ using the minmax principle.
Interestingly by using a Darboux-type factorization $L_- = S^*S$ with $S = \partial_x - Q^{-1}Q_x$ ($S^* = -\partial_x - Q^{-1}Q_x$), one can show that (see Theorem 3.6 of [15]) the eigenvalues of $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ and $H = SL_+S^*$ coincide:

$$\inf_{g\perp Q_{1\mathrm{D}}, g_k, k<j} \frac{\langle g, L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} g\rangle}{\langle g, g\rangle} = \mu_j = \inf_{f\perp f_k, k<j} \frac{\langle f, Hf\rangle}{\langle f, f\rangle},$$
$$j = 1, 2, 3, \ldots. \tag{1.17}$$

As a result, one has

$$0 < \sigma < 2 \text{ (mass-subcritical)}: \ \mu_1 = 0, \quad \mu_2 > 0; \tag{1.18}$$
$$\sigma = 2 \text{ (mass-critical)}: \ \mu_1 = \mu_2 = 0, \quad \mu_3 = 1; \tag{1.19}$$
$$2 < \sigma < \infty \text{ (mass-supercritical)}: \ \mu_1 < 0, \ \mu_2 = 0, \ \mu_3 = 1. \tag{1.20}$$

In addition it holds that ([15])

$$\tfrac{1}{2} < \sigma < 2: \ \mu_3 = 1; \tag{1.21}$$
$$\sigma \neq 1: \ \mu_2 \le C_\sigma < 1. \tag{1.22}$$

Much finer information on μ_j such as interlacing of eigenvalues can be extracted from the variational characterization (cf. Theorems 3.7 and 3.8 of [15]). On the other hand, for the mass-supercritical case, the absence of embedded eigenvalues in the essential spectrum of the operator $\widetilde{A}$ is established in Proposition 9.2 of Krieger-Schlag [42] by adapting Perelman's argument [58] from $\sigma = 2$ to $\sigma > 2$.

- Orbital stability, asymptotic stability and conditional asymptotic stability.
There is an extensive literature on the stability and instability of standing waves for nonlinear evolutionary equations. For the mass-subcritical case, by using concentration-compactness, Cazenave and Lions [14] proved orbital stability for subcritical equations including NLS with pure monomial nonlinearities, Hartree-type equations and Pekar-Choquard equations. For a class of nonlinear dispersive equations including mass-subcritical NLS with monomial power type nonlinearity, Weinstein [73, 74] gave a new proof of orbital stability by carefully exploiting the spectral properties of the linearized operator together with a natural metric taking into account the modulation parameters. For the mass-supercritical and energy-subcritical case the ground state is unstable, since minuscule H^1-perturbation of the ground state Q can lead to finite-time blowup (cf. [7, 13, 68]).

There is a natural spectral interpretation of these stability/instability results. Consider the monomial case and the ground state Q with $Q - \Delta Q = Q^{1+2\sigma}$, $x \in \mathbb{R}^d$. The corresponding linearized operator takes the form

$$\mathcal{H} = \begin{pmatrix} 1-\Delta - V_1 & -V_2 \\ V_2 & \Delta - 1 + V_1 \end{pmatrix}, \qquad V_1 = (1+2\sigma)Q^{2\sigma}, \qquad V_2 = \sigma Q^{2\sigma}.$$

For $0 < \sigma \le \frac{2}{d}$ (i.e., mass-subcritical and mass-critical) all eigenvalues of $\mathcal{H}$ lie on the real axis, whilst for $\frac{2}{d} < \sigma < \sigma_d$ ($\sigma_d = \infty$ for $d = 1, 2$ and $\sigma_d = \frac{2}{d-2}$ for $d \ge 3$) the operator $\mathcal{H}$ possesses a symmetric pair of purely imaginary eigenvalues on the imaginary axis which is the springboard of exponential instability (for a more detailed discussion see Appendix D of this paper).

It is possible to recast some of the orbital stability/instability issues into a more abstract functional framework, cf. [19, 33–35]. On the other hand, the aforementioned orbital stability results do not reveal the long time behavior of the perturbations such as diminishing to zero or perpetual oscillation around the ground state. In this regard the more difficult asymptotic stability analysis was addressed by Buslaev-Perelman [11, 12] for mass-subcritical 1D NLS, Soffer-Weinstein [65, 66] for a class of mass-subcritical NLS with certain potentials, and Cuccagna [21] which transposed the Buslaev-Perelman result to a class of NLS in dimension $n \ge 3$ (note that Cuccagna assumed the absence of embedded eigenvalues in the essential spectrum in [21]). For the multi-soliton case one can see [59, 62, 63]. Note that these results are mostly obtained via Lyapunov-Schmidt reduction which splits the evolution into a set of finite-dimensional modulation equations (corresponding to the symmetries) and a dispersive part.

For the mass-supercritical case which is orbitally unstable, in light of exponential instability one has to work with conditional asymptotic stability. This line of results was first obtained by Krieger-Schlag [42] for 1D supercritical NLS, and Schlag [64] for the 3D case under the spectral hypothesis (for more complete references on nonlinear Klein-Gordon and other equations, one can see the monograph [55] and the references therein). We also refer to the review [41] and the references therein for more discussions on related equations such as gKdV, ϕ^4 models and the like. See Morrison-Tsai [53] and Martel [48] for some recent progress on the standing waves of 1D NLS with combined power type nonlinearities.

- Future directions.

  This work orchestrates a range of techniques, which include analytic and variational ODE analysis, comparison principles, stability analysis, monotonicity formula, continuity argument, shooting methods, and so on, some of which are supported by the steady rhythm of exact computer-assisted computation. This is a flexible and evolving toolkit for a wide range of related problems. It is our hope that this perspective may serve as a point of departure for a collective endeavor, one that stimulates further progress and opens new avenues of research in the field. In this regard, several promising directions are outlined below.[7] Needless to say, the list is not exhaustive.

[7] We would like to thank one of the anonymous referees for very valuable suggestions regarding these future directions.

– General NLS. The approach developed here, though presented for the 3D cubic nonlinearity, is applicable to a broader class of energy-subcritical focusing monomial power-type nonlinear Schrödinger equations (see Appendix D for a discussion in the general setting). It relies quantitatively on the existence and decay properties of ground states, and extensions along these lines seem readily achievable. Another promising direction is the extension to more general nonlinearities or even Hartree-type equations with convolution-type nonlinearity.
– Multi-solitons and excited states. A further important direction lies in extending the spectral analysis of linearized operators to multi-soliton configurations (cf. [59, 63]) and general excited states. A key question is whether the present approach can be adapted to exclude embedded eigenvalues in superpositions of well-separated solitons, or to preclude specific internal modes arising from their interactions. At present a complete spectral analysis for excited states remains largely open, though substantial progress on the uniqueness problem has been made in [18, 70].
– Nonlinear dynamics and asymptotic stability of other PDEs. Finally, the broader applicability of the current approach to other dispersive PDEs, such as Schrödinger operators with potentials [24], the nonlinear Klein–Gordon equations [56], nonlinear wave equations and so on, remains an open and promising area of investigation.

1.3 Comments about the "computational" part of our proof

A pivotal role in our proof is played by the Costin-Huang-Schlag approximate ground state $\widetilde{Q}$ whose properties are summarized in Lemma A.1. Specifically $1/\widetilde{Q}$ is a piecewise polynomial for $t \le 5/2$, and an explicit transcendental function for $t > 5/2$. To carry out the computational analysis within the domain of rational numbers, we introduce in Lemma A.2 the following:

(1) A degree-14 polynomial Q_*^2 which accurately approximates Q^2 on the interval $[0,\ 8/5]$. Thanks to (Markov-Brothers type) Lemma 1.4, the error $|Q_*^2 - \widetilde{Q}^2|$ can be rigorously quantified (which yields the estimate of $|Q_*^2 - Q^2|$). The polynomial Q_*^2 is needed in Proposition 5.2, where we treat the $\ell = 0$ subsystem for $\tau \in [1,\ 19.2]$ using explicit polynomials in τ and t. A rewarding feature of Q_*^2 is that the corresponding residual errors for the $\ell = 0$ subsystem can be explicitly expressed as polynomial functions in τ and t which lead to explicit error bounds in terms of rational numbers.
(2) A degree-8 polynomial W_2 approximating Q on the interval $[\frac{5}{2},\ \frac{7}{2}]$. Thanks to the maximum principle, only the values of $Q(5/2)$ and $Q(7/2)$ (via $\widetilde{Q}(5/2)$ and $\widetilde{Q}(7/2)$) enter in the estimate of $W_2 - Q$. In yet other words we avoid any "transcendental" treatment.
(3) Another degree-22 polynomial W_1 approximating Q on the interval $[0,\ \frac{5}{2}]$. Both W_1 and W_2 are needed in the analysis for $\ell = 0$, $\tau \ge 19.2$ (Theorem 6.1), the rigorous verification of condition (2) in Theorem 4.5 (see Appendix B) and the analysis of the $\ell = 1$ subsystem (cf. Lemma 8.1).

To tame the $\ell = 0$ subsystem we use a divide-and-conquer approach and reduce matters to checking that the corresponding solutions satisfy the "H-criterion". The verification of the H-criterion is by no means a trivial undertaking. For this we construct a family of explicit polynomials in τ and t and rigorously prove the error bounds (see Sect. 5). We emphasize that the bulk of our analysis adopts polynomials with coefficients in $\mathbb{Q}$, avoiding numerical approximations like floating-point integration.

To demonstrate the feasibility and advantages of this purely symbolic approach, let us examine two concrete, yet manageable, problems. These "toy examples" are designed to mirror the type of computations encountered in our main analysis, but in a slightly simplified setting.

Toy problem 1: Consider the ground state soliton Q of (1.1). Show that

$$3tQ'Q + 3Q^2 - 1 \geq 0, \qquad \text{if } 0 \leq t \leq 0.5. \tag{1.23}$$

This toy problem illustrates the effectiveness and efficiency of a recurring class of polynomial inequalities in this work (cf. Appendix A), as compared to more elaborate and soft approaches that depend on monotonicity, for example by analyzing the derivatives, or on various identities of the ground state Q. To prove it, we apply the W_1-polynomial approximation of Q from Appendix A, which reduces the problem to verifying a polynomial inequality. For the latter we present a solution both novel and classical: a judiciously chosen Markov–Brothers type lemma (Lemma 1.4) bounds the polynomial's minimum via rational sampling, handling moderately high[8] degrees effectively.

Solution to Toy problem 1: By using Lemma A.2, we have

$$3tQ'Q + 3Q^2 - 1 \geq \underbrace{3t \cdot (W_1 + \tfrac{75}{10^6})(W_1' - \tfrac{42}{10^5}) + 3(W_1 - \tfrac{75}{10^6})^2 - 1}_{R}.$$

By Lemma 1.4 and exact computation (e.g., using MATHEMATICA), we find (the notation $\mathcal{M}$ is from (1.25))

$$\min_{0 \leq t \leq 0.5} R \geq \mathcal{M}(R, [0, \tfrac{1}{2}], \tfrac{5}{10^5}) > 0.$$

Toy problem 2: Let Q be the ground state soliton of (1.1). Consider the ODE

$$\begin{cases} \partial_{tt} U - (1 - \tau - 2Q^2)U = 0, \\ U(0) = 0, \ U'(0) = 1. \end{cases}$$

Show that for $T = \frac{3}{10}$ and any $\tau \in [14.4, 19.2]$, it holds that

$$|\partial_t U(\tau, T) + \tfrac{92+19\tau}{10^3}| < \tfrac{1}{10^2}.$$

[8] In general we require $N > d^2/\sqrt{6}$, where N is the number of (uniform)-sampling points, and d is the degree of the polynomial.

This toy problem is quite illustrative in that it can be regarded as a scalar trimmed-down version of Proposition 5.2, which treats an ODE system for a pair of functions U and V. Roughly speaking, the aim of Proposition 5.2 is to show that the exact ODE solution at time T must satisfy a collection of inequalities called the H-criterion. To check those inequalities we need sharp control of the terminal vector $Y(\tau, T) = (U(\tau, T), V(\tau, T), \partial_t U(\tau, T), \partial_t V(\tau, T))^\top$. Remarkably, in Proposition 5.2 it is possible to pin down the main order of Y as a linear function of τ well within the needed error margins. To convey how this is achieved without the technical overhead of the full proof therein, we examine key elements of the proof on this stripped-down model. As in Proposition 5.2, we shall employ the explicit polynomial approximation Q_*^2 given in Lemma A.2, which satisfies the estimate

$$|Q_*^2 - Q^2| \le \epsilon_Q := \tfrac{75}{10^5}, \qquad \text{on } [0, T].$$

The decisive step is to adopt a polynomial ansatz of degree six in t and linear in τ whose residual error remains negligible and can be quantified explicitly via Lemma 1.4. A direct energy estimate then yields the desired bound at time T. Notably, the low-degree ansatz in this toy model renders the computations amenable to classical pen-and-pencil verification. In this way, the proof of Proposition 5.2 effectively represents the very edge of what is achievable within the classical pen-and-pencil framework.

Solution to Toy problem 2: By a simple energy estimate (note $Q' \le 0$), we have

$$\max_{0 \le t \le T} |U'(t)| \le 1, \qquad \max_{0 \le t \le T} |U(t)| \le T.$$

We use the ansatz

$$\widetilde{U}(\tau, t) = t - \tfrac{31}{5}t^3 + \tfrac{1}{3}t^4 + \tfrac{472}{21}t^5 - 25t^6 + \left(-\tfrac{1}{6}t^3 + \tfrac{21}{20}t^5 - \tfrac{8}{7}t^6\right)\tau.$$

Denote $\eta = \widetilde{U} - U$ and $R = \partial_{tt}\widetilde{U} + (\tau + 2Q_*^2 - 1)\widetilde{U}$. Clearly

$$\partial_{tt}\eta + (\tau + 2Q_*^2 - 1)\eta = 2(Q^2 - Q_*^2)U + R.$$

A simple energy estimate together with Lemma 1.4 yields that

$$\max_{0 \le t \le T} |\partial_t \eta| \le 2\epsilon_Q T + T^{\frac{1}{2}}\Big(\int_0^T R^2 dt\Big)^{\frac{1}{2}} \le \tfrac{7}{10^3}.$$

Note that $\partial_t \widetilde{U}(\tau, T) = -\frac{26{,}793\tau}{14\cdot 10^5} - \frac{1291}{14{,}000}$. The desired result follows.

What is the point? As we demonstrated above, using a symbolic algebra system like MATHEMATICA whose arbitrary-precision arithmetic is built upon the rigorously tested GNU Multiple Precision Arithmetic Library (GMP), these problems can be solved exactly and near-instantly. The entire process relies on a finite sequence of elementary, error-free operations on rational numbers: addition, subtraction, multiplication, division, and comparison. In essence, this approach upholds the purity and certainty of traditional pencil-and-paper proof. It may be best understood as operating like a super-calculator or a modern abacus, executing only exact integer and rational arithmetic with no symbolic heuristics, approximations, or complex code.

It is in this sense that all our computations involved in this work are more or less akin to the above toy examples. This echoes the philosophy advocated in [20] (see also the recent [30]) where only exact computations are performed. As a preliminary remark, we note that the "hands-on" approach adopted in this work occupies a different point on the spectrum of formal verification methods compared to, for example, interval arithmetic (cf. [26] for stability of matter, [27, 28] for the proof of the Dirac–Schwinger conjecture and [18] for the uniqueness of excited states). Our work explores the end of this spectrum that favors a classical style of analysis, seeking to keep the proof structure directly human-intelligible and utilizing the computer as a super-calculator for explicit symbolic expressions.

At this point, we should also point it out that a fundamental mechanism for the "H-criterion" to work is hyperbolicity, i.e., the existence of certain exponentially unstable directions. Moreover the Lyapunov exponent increases as τ increases. Whilst hyperbolicity is beneficial in terms of "philosophical thinking", it inevitably leads to some pronounced difficulty in the whole analysis. In particular, the dynamics of the systems involving Q^2 and Q_*^2 may differ significantly after some $T = \mathcal{O}(1)$ especially for τ large. For this reason, we meticulously carry out the whole analysis on relatively short time intervals for which "hyperbolicity" just starts to kick in. The essence of the ballgame is to harness hyperbolicity within the stability window of the explicit approximate solution.

Organization & notation See Sect. 2 (and in particular the flowchart at the end of Sect. 2) for the outline of the proof of Theorem 1.1. The following notation will be used in this work.

- Fourier transform: $(\mathcal{F}f)(\xi) = \widehat{f}(\xi) := \int_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi}dx$, $(\mathcal{F}^{-1}g)(x) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} g(\xi)e^{ix\cdot\xi}d\xi$.
- For $p \in [1,\infty]$, $m \in \mathbb{N}$ and Lebesgue measurable $f : \mathbb{R}^m \to \mathbb{C}$, we denote

$$\|f\|_p = \|f\|_{L^p(\mathbb{R}^m,\mathbb{C})} := \left(\int_{\mathbb{R}^m} |f|^p dx\right)^{\frac{1}{p}}, \qquad p \in [1, \infty).$$

  For $p = \infty$, we denote $\|f\|_\infty := \operatorname{ess\,sup}|f|$.
- For $s \ge 0$, $\|f\|_{H^s(\mathbb{R}^3,\mathbb{C})} := \left(\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$.
- For a nonnegative quantity Y, we write $X = \mathcal{O}(Y)$ if $|X| \le CY$ for some positive constant C. For any two positive quantities X and Y, we write $X \lesssim Y$ if $X \le CY$ for some positive constant C.
- For $g_1, g_2 \in L^2(\mathbb{R}^3, \mathbb{C})$, we denote

$$\langle g_1, g_2\rangle := \int_{\mathbb{R}^3} \overline{g_1(x)} g_2(x)dx.$$

  We write $g_1 \perp g_2$ if $\langle g_1, g_2\rangle = 0$. For $h = (h_1, h_2, h_3)^\top$, $g = (g_1, g_2, g_3)^\top \in L^2(\mathbb{R}^3, \mathbb{C}^3)$, we denote

$$\langle h, g\rangle := \sum_{j=1}^{3}\int_{\mathbb{R}^3} \overline{h_j(x)} g_j(x)dx.$$

- Recall that the ground state Q is radial and can be written as $Q(x) = y(r)$ (see (1.5)). Throughout the paper we shall slightly abuse the notation and regard $Q = Q(r)$ when there is no obvious confusion. For example, we write $Q_1 = Q + x \cdot \nabla Q = Q + rQ'$, where $r = |x|$.
- We use 3D spherical harmonics (cf. [4]) $Y_\ell^m(\theta, \phi)$ ($\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$) defined by

$$Y_\ell^m(\theta, \phi) = \sqrt{\tfrac{2\ell+1}{4\pi} \cdot \tfrac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta)e^{im\phi}, \quad |m| \le \ell,\ \ell \ge 0,$$

  where $P_\ell^m(\cdot)$ is an associated Legendre polynomial. The normalization is chosen such that

$$\int_0^{2\pi} \int_0^{\pi} Y_\ell^m(\theta, \phi)\overline{Y_{\ell'}^{m'}} d\sigma = \delta_{mm'}\delta_{\ell\ell'}, \qquad d\sigma = \sin\theta d\theta d\phi.$$

  For example, $Y_0^0 = \frac{1}{2\sqrt{\pi}}$, $Y_1^0 = \frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta$, $Y_1^{-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi}$, $Y_2^0 = \frac{1}{4}\sqrt{\frac{5}{\pi}}(\cos^2\theta - 1)$ and so on.
- Notation for Decimals. For simplicity of presentation, decimals (e.g., 0.49) denote exact rational numbers (e.g., 49/100). When implementing algorithms in symbolic systems (e.g., MATHEMATICA), it is good practice to input the **exact fraction** to avoid floating-point rounding errors.

In this work, we need to bound polynomials on a compact interval by using their values at finitely many rational abscissas. The following lemma secures a modest yet nontrivial saving of $\mathsf{h}/2$ through a simple pigeonhole argument; this translates directly into a twofold speedup in practical computations.

Lemma 1.4 *Let p be a polynomial of degree $d \ge 1$ on an interval $[a, b]$. We have (below $6N^2 > d^2(d^2-1)$)*

$$\|p\|_{L^\infty([a,b])} \le \frac{1}{\lambda} \cdot \max_{0\le k\le N} |p(x_k)|, \qquad \lambda = 1 - \frac{d^2(d^2-1)\mathsf{h}^2}{6(b-a)^2} = 1 - \frac{d^2(d^2-1)}{6N^2}, \quad (1.24)$$

where $\mathsf{h} = (b-a)/N$ and $x_k = a + k\mathsf{h}\,(0 \le k \le N)$. Similarly for the minimum we have

$$\min_{x\in[a,b]} p(x) \ge \mathcal{M}(p, [a, b], \mathsf{h})$$
$$:= \min_{0\le k\le N} p(x_k) - \frac{d^2(d^2-1)}{6N^2 - d^2(d^2-1)} \max_{0\le k\le N} |p(x_k)|. \qquad (1.25)$$

Proof Denote $t = \frac{2(x-a)}{b-a} - 1$. Then $x = \frac{(1-t)a+(1+t)b}{2}$ and $p(\frac{(1-t)a+(1+t)b}{2}) =: f(t)$. From the Markov Bothers inequality (cf. [46, 47]) on $[-1, 1]$, we have

$$\|f''\|_{L^\infty([-1,1])} \le \frac{d^2(d^2-1)}{3}\|f\|_{L^\infty([-1,1])}$$
$$\Rightarrow \|p''\|_{L^\infty([a,b])} \le \frac{4d^2(d^2-1)}{3(b-a)^2}\|p\|_{L^\infty([a,b])}.$$

Now assume $\|p\|_{L^\infty([a,b])}$ is achieved at some x_*. If $x_* = x_k$ for some $0 \le k \le N$, we are done. If $x_* \in (x_j, x_{j+1})$, we clearly have $|x_* - x_k| \le \frac{1}{2}\mathsf{h}$ where $k = j$ or $j+1$. Taylor expanding at x_*, we get

$$|p(x_k)| \ge |p(x_*)| - \frac{1}{2}\|p''\|_{L^\infty([a,b])} \cdot (\tfrac{1}{2}\mathsf{h})^2 \qquad \text{(note the saving of } \tfrac{1}{2}\mathsf{h})$$
$$\Rightarrow \|p\|_{L^\infty([a,b])}(1 - \frac{d^2(d^2-1)\mathsf{h}^2}{6(b-a)^2}) \le \max_{0\le k\le N} |p(x_k)|. \tag{1.26}$$

The argument for $\min_{x\in[a,b]} p$ is similar. We omit the details. □

The following easy corollary can be used to bound rational functions.

Corollary 1.5 *Let $\alpha > 0$. Let $f(x)$ be a polynomial of degree $m \ge 1$ and $g(x) > 0$ be a polynomial of degree $n \ge 1$ on an interval $[a, b]$. If $\mathsf{h} > 0$, $6(\frac{b-a}{\mathsf{h}})^2 > \max\{m^2(m^2-1), n^2(n^2-1)\}$,*

$$\mathcal{M}(\alpha g - f, [a,b], \mathsf{h}) > 0 \quad \textit{and} \quad \mathcal{M}(\alpha g + f, [a,b], \mathsf{h}) > 0, \tag{1.27}$$

then

$$\left\| \frac{f}{g} \right\|_{L^\infty([a,b])} \le \alpha.$$

Proof Note that $-\alpha \le \frac{f}{g} \le \alpha$ iff. $\alpha g - f \ge 0$ and $\alpha g + f \ge 0$. Then apply Lemma 1.4 to $\alpha g \pm f$. □

2 Outline of the proof of Theorem 1.1

We shall work with the case $\tau \in [1, \infty)$. The case $\tau \in (-\infty, -1]$ follows easily from the identity

$$\begin{cases} L_+ a = \tau b \\ L_- b = \tau a \end{cases} \Rightarrow \begin{cases} L_+ a = (-\tau)(-b) \\ L_-(-b) = (-\tau)a. \end{cases}$$

Step 1. Preliminary reduction. We argue by contradiction. Assume for some $\tau \in [1, \infty)$ and nontrivial $(\widetilde{f}^{(1)}, \widetilde{g}^{(1)})^\top \in L^2(\mathbb{R}^3, \mathbb{C}^2)$, it holds that $\mathcal{H}(\widetilde{f}^{(1)}, \widetilde{g}^{(1)})^\top = \tau(\widetilde{f}^{(1)}, \widetilde{g}^{(1)})^\top$. By elliptic theory, we have $\widetilde{f}^{(1)}, \widetilde{g}^{(1)} \in H^k(\mathbb{R}^3, \mathbb{C})$ for any integer $k \ge 1$. By using the spherical harmonics $\{Y_\ell^m : |m| \le \ell,\ \ell \ge 0\}$, we denote

$$\widetilde{f}_{\ell m}(r) = r \int_{\mathbb{S}^2} \widetilde{f}^{(1)}(r,\theta,\varphi)\overline{Y_\ell^m(\theta,\varphi)} d\sigma,$$
$$\widetilde{g}_{\ell m}(r) = r \int_{\mathbb{S}^2} \widetilde{g}^{(1)}(r,\theta,\varphi)\overline{Y_\ell^m(\theta,\varphi)} d\sigma. \tag{2.1}$$

For each integer $\ell \ge 0$ and m, we have

$$\partial_{rr} \widetilde{f}_{\ell m} = (\frac{\ell(\ell+1)}{r^2} + 1 - \tau - 2Q^2)\widetilde{f}_{\ell m} - Q^2 \widetilde{g}_{\ell m};$$

$$\partial_{rr}\widetilde{g}_{\ell m} = (\tfrac{\ell(\ell+1)}{r^2} + 1 + \tau - 2Q^2)\widetilde{g}_{\ell m} - Q^2 \widetilde{f}_{\ell m}. \tag{2.2}$$

Here both $\widetilde{f}_{\ell m}$ and $\widetilde{g}_{\ell m} \in H^k((0,\infty), \mathbb{C})$ for any integer $k \geq 1$, and as $r \to 0^+$,

$$\text{for } \ell = 0, \quad \widetilde{f}_{\ell m}(r) = \text{const} \cdot r + \mathcal{O}(r^3), \quad \widetilde{g}_{\ell m}(r) = \text{const} \cdot r + \mathcal{O}(r^3); \tag{2.3}$$

$$\text{for } \ell = 1, \quad \widetilde{f}_{\ell m}(r) = \text{const} \cdot r^2 + \mathcal{O}(r^4), \quad \widetilde{g}_{\ell m}(r) = \text{const} \cdot r^2 + \mathcal{O}(r^4); \tag{2.4}$$

$$\text{for } \ell \geq 2, \quad \widetilde{f}_{\ell m}(r) = \mathcal{O}(r^3), \quad \widetilde{g}_{\ell m}(r) = \mathcal{O}(r^3). \tag{2.5}$$

We then only need to rule out the existence of nontrivial $(\widetilde{f}_{\ell m}, \widetilde{g}_{\ell m})^\top$ for each $\ell \geq 0$. By considering separately real and imaginary parts, we only need to rule out nontrivial $\widetilde{f}_{\ell m}, \widetilde{g}_{\ell m} \in H^k((0,\infty), \mathbb{R}), \forall k \geq 1$.

Step 2. The case $\ell = 0$. We denote $U = \widetilde{f}_{0m}$, $V = -\widetilde{g}_{0m}$, $t = r$, and work with the system

$$U'' = (1 - \tau - 2Q^2)U + Q^2 V, \qquad V'' = Q^2 U + (1 + \tau - 2Q^2)V. \tag{2.6}$$

Here the initial condition is $(U, V, U', V')\big|_{t=0} = (0, 0, \cos\theta, \sin\theta)$, where $\theta \in [0, 2\pi)$. Moreover $U, V \in C^\infty([0,\infty))$ and decay exponentially (cf. Proposition 4.1) as $t \to \infty$. The goal is to show that U and V must be identically zero which contradicts the initial condition $(0, 0, \cos\theta, \sin\theta)$.

Genesis of the idea: harnessing hyperbolicity via weighted estimates. We first consider $\tau \gg 1$. The following insightful three-line computation is the starting point of our analysis. In the argument we use the fact that U, V, U' and V' decay sufficiently fast as $t \to \infty$. Using the identity $\partial_{tt} V V = \frac{1}{2}\partial_{tt}(V^2) - (V')^2$, we get

$$\begin{aligned} &\tfrac{1}{2}\partial_{tt}(V^2) - (V')^2 = Q^2 UV + (1 + \tau - 2Q^2)V^2 \\ &\Rightarrow \int_0^\infty \big((V')^2 + (1 + \tau - 2Q^2)V^2 + Q^2 UV\big) t\,dt = 0. \end{aligned} \tag{2.7}$$

Using $U''U' = \frac{1}{2}\big((U')^2\big)'$, we have

$$\begin{aligned} &\tfrac{1}{2}\int_0^\infty \big((U')^2\big)' t^2 dt + \tfrac{1}{2}\int_0^\infty (\tau + 2Q^2 - 1)t^2 (U^2)' dt = \int_0^\infty Q^2 U' V t^2 dt \\ &\Rightarrow \int_0^\infty \big((U')^2 t + \tfrac{1}{2}((\tau - 1 + 2Q^2)t^2)' U^2\big) dt + \int_0^\infty Q^2 U' V t^2 dt = 0. \end{aligned} \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$\begin{aligned} \int_0^\infty \Big(&(V')^2 + (1 + \tau - 2Q^2)V^2 + (U')^2 + (\tau - 1 + 2Q^2 + 2Q'Qt)U^2 \\ &+ Q^2 UV + tQ^2 U'V \Big) t\,dt = 0. \end{aligned} \tag{2.9}$$

Clearly, if $\tau \gg 1$, then the above implies $U \equiv 0$ and $V \equiv 0$.

Quantitative estimates for $\tau \geq 19.2$. Motivated by the above computation, we employ a suitable weight function $\beta' = \alpha > 0$ and derive the fundamental identity

$$\int_0^T \mathcal{E}_0 \alpha dt = \tfrac{\beta(T)}{2}(U')^2(T) - \tfrac{\beta(0)}{2}(U')^2(0) + \tfrac{\beta(T)}{2}(\tau - 1 + 2Q^2(T))U^2(T) + \alpha V V'(T) - \tfrac{\alpha' V^2(T)}{2}, \tag{2.10}$$

where

$$\mathcal{E}_0 := (\tfrac{\tau-1+2Q^2}{2} + \tfrac{\beta}{\beta'} 2Q'Q)U^2 + Q^2 U' V \tfrac{\beta}{\beta'} + \tfrac{(U')^2}{2} + (V')^2 + (\tau + 1 - 2Q^2 - \tfrac{\alpha''}{2\alpha})V^2 + Q^2 U V. \tag{2.11}$$

The strategy is to find suitable β so that we can send $T \to \infty$ and derive

$$\int_0^\infty \mathcal{E}_0(t)\alpha(t)dt = 0, \quad \mathcal{E}_0(t) \geq \epsilon_0(t) \cdot (U^2(t) + V^2(t)), \tag{2.12}$$

where $\epsilon_0(t) > 0$ for all $0 < t < \infty$. Clearly, (2.12) would imply the desired contradiction $U \equiv 0$, $V \equiv 0$.

To prove (2.12), we let[9] $\beta = \frac{t^2}{1+3.75t^2}$ and impose the stronger inequalities (below $\alpha_1(t) > 0$ is tunable)

$$\begin{aligned} &\tau \geq 2Q^2 - 1 + \tfrac{\alpha''}{2\alpha} + \tfrac{Q^2}{2\alpha_1} + \tfrac{Q^4}{2}(\tfrac{\beta}{\beta'})^2 =: F_{a,\alpha_1}; \\ &\tau \geq 1 - 2Q^2 - \tfrac{4\beta}{\beta'} Q'Q + \alpha_1 Q^2 =: F_{b,\alpha_1}; \\ &\max\{F_{a,\alpha_1}, F_{b,\alpha_1}\} \leq 19.2, \qquad \forall t \geq 0. \end{aligned} \tag{2.13}$$

The proof of (2.13) is elementary thanks to the quantitative characterization of Q. This concludes the case $\tau \geq 19.2$.

We then move on to the remaining regime $\tau \in [1, 19.2]$. We first derive a general criterion.

General strategy: refined weights. In Theorem 4.5 we employ a general weight β with $\beta' = \alpha > 0$ and derive[10]

$$\int_T^\infty \mathcal{E}_1 \alpha dt + \tfrac{\beta(T)}{2}(U')^2(T) + \beta(T) \cdot (\tfrac{\tau-1}{2} + Q^2(T)) \cdot U^2(T) + A\alpha(T) V V'(T) = 0, \tag{2.14}$$

where $A > 0$ is a parameter, $T > 0$ and

$$\mathcal{E}_1 = (\tfrac{\tau-1}{2} + Q^2 + \tfrac{\beta}{\beta'} 2Q'Q)U^2 + \tfrac{(U')^2}{2} + Q^2 U' V \tfrac{\beta}{\beta'} + A(\tau + 1 - 2Q^2)V^2$$

[9] This was found through iterative tuning based on the profile of Q.

[10] Here β and $\alpha = \beta'$ is assumed not to grow too fast as $t \to \infty$, e.g., $|\beta(t)| + |\alpha(t)| \lesssim e^{2.828t}$ suffices (cf. Theorem 4.5).

$$+ A(V')^2 + AV'\frac{\alpha'}{\alpha}V + AQ^2UV. \tag{2.15}$$

Depending on the size of τ, we make a judicious choice of $T > 0$, $A > 0$, β with $\alpha = \beta' > 0$ on (T, ∞) such that

$$\mathcal{E}_1(t) > \epsilon_1(t) \cdot (U^2(t) + V^2(t)), \qquad \forall t \in [T, \infty); \tag{2.16}$$

$$(U')^2(T) + (2Q^2(T) + \tau - 1)U^2(T) + \frac{2A\alpha(T)}{\beta(T)}VV'(T) \geq 0, \tag{2.17}$$

where $\epsilon_1(t) > 0$ for any $t \in [T, \infty)$. Note that if the above inequality holds, then $U \equiv 0$, $V \equiv 0$ for all $t \in [0, \infty)$.

To guarantee (2.16), we impose the following sufficient conditions (below $\alpha_1(t) > 0$ is a tunable-parameter)

$$\begin{cases} \tau \geq 1 - 2Q^2 - \frac{4\beta}{\alpha}Q'Q + A\alpha_1 Q^2, \qquad \forall t \geq T; \\ \tau \geq 2Q^2 - 1 + \frac{1}{4}\left(\frac{\alpha'}{\alpha}\right)^2 + \frac{Q^2}{2\alpha_1} + \frac{1}{2A}\left(\frac{\beta}{\alpha}\right)^2 Q^4, \qquad \forall t \geq T. \end{cases} \tag{2.18}$$

The existence of α_1 is guaranteed by the following inequalities:

$$\begin{cases} \tau - 1 + 2Q^2 + 4\frac{\beta}{\alpha}Q'Q > 0, \quad \forall t \geq T; \\ \frac{1}{2}AQ^4 < (\tau - 1 + 2Q^2 + 4\frac{\beta}{\alpha}Q'Q)(\tau + 1 - 2Q^2 - \frac{1}{4} \\ \qquad \times (\frac{\alpha'}{\alpha})^2 - \frac{1}{2A}(\frac{\beta}{\alpha})^2 Q^4), \quad \forall t \geq T. \end{cases} \tag{2.19}$$

Thus we reduce matters to finding suitable $T > 0$, $\beta(\cdot)$ with $\alpha = \beta' > 0$ on (T, ∞), $A > 0$, $\alpha_1(\cdot) > 0$ so that the conditions (2.17) and (2.19) are fulfilled. We shall adopt a divide-and-conquer approach and make a judicious[11] partition of the τ-interval $[1,\ 19.2]$ into five sub-intervals, namely: $[1,\ 19.2] = \bigcup_{j=1}^5 \mathcal{I}_j$, where

$$\mathcal{I}_1 = [1,\ 1.2], \quad \mathcal{I}_2 = [1.2,\ 2.1], \quad \mathcal{I}_3 = [2.1,\ 5.4];$$
$$\mathcal{I}_4 = [5.4,\ 12.4], \quad \mathcal{I}_5 = [12.4,\ 19.2]. \tag{2.20}$$

On each sub-interval $\mathcal{I}_j$, we choose[12] parameters A, T, $\beta(t)$ and $\alpha(t) = \beta'(t)$ according to Table 1 in Sect. 5. The rigorous justification of (2.19) is carried out in Appendix B.

It remains for us to justify (2.17) on each τ-interval $\mathcal{I}_j$. We first make some further simplification. Let $p_1 \in (0, 2Q(T)^2 + \tau - 1]$ and $p_2 \geq \frac{2A\alpha(T)}{\beta(T)}$. To ensure (2.17), it suffices for us to show

$$U'(T)^2 + p_1 U^2(T) + p_2 VV'(T) \geq 0. \tag{2.21}$$

[11]There is some flexibility in determining the partition and identifying the corresponding $(A, T, \beta(t))$ so that (2.17)–(2.19) hold. In principle a sufficiently fine partition of the τ-interval combined with some mediocre choices of $(A, T, \beta(t))$ will still yield acceptable results. However these will most likely lead to (much) lengthier arguments. In order to simplify the whole presentation we reach the current form after many trial-and-error attempts and several rounds of ad hoc optimization.

[12]If $\tau \in \mathcal{I}_j \cap \mathcal{I}_{j+1}$ (for example $\tau = 1.2$), we can choose (A, T, β) either as in $\mathcal{I}_j$ or $\mathcal{I}_{j+1}$.

Clearly $(U, V, U', V')^\top = Y_a \cos\theta + Y_b \sin\theta$, where $Y_a = Y_a(\tau, t)$ and $Y_b = Y_b(\tau, t)$ denote the exact solutions corresponding to initial data $(0, 0, 1, 0)^\top$ and $(0, 0, 0, 1)^\top$ respectively. To remove the "θ degrees of freedom", it suffices for us to check

$$\widetilde{H}_{p_1,p_2}(Y_a(\tau, T), Y_b(\tau, T)) \geq 0, \tag{2.22}$$

where $\widetilde{H}_{p_1,p_2}(\vec{x}, \vec{y})$ is a quartic polynomial in $\vec{x} = (x_1, x_2, x_3, x_4)^\top$ and $\vec{y} = (y_1, y_2, y_3, y_4)^\top \in \mathbb{R}^4$ obtained from computing the discriminant of a quadratic expression in $z = \tan\theta$ arising from (2.21). To verify (2.22), we first resort to Lemma 4.4, where we reduce matters to specifying $p_1 \in (0, 2Q(T)^2 + \tau - 1]$, $p_2 \geq \frac{2A\alpha(T)}{\beta(T)}$ and finding $x^\tau = (x_1^\tau, \ldots, x_4^\tau)^\top \in \mathbb{R}^4$, $y^\tau = (y_1^\tau, \ldots, y_4^\tau)^\top \in \mathbb{R}^4$, $\epsilon_s > 0$ such that

$$1) \|Y_a(\tau, T) - x^\tau\|_{l^\infty} \leq \epsilon_s, \;\; \|Y_b(\tau, T) - y^\tau\|_{l^\infty} \leq \epsilon_s; \tag{2.23}$$

$$2)\; x_2^\tau > \epsilon_s,\; x_4^\tau > \epsilon_s; \tag{2.24}$$

$$3)\; H\text{-criterion holds, namely (see (4.13))}\; H^{\epsilon_s}_{p_1,p_2}(x^\tau, y^\tau) \geq 0. \tag{2.25}$$

The advantage of $H^{\epsilon_s}_{p_1,p_2}$ is that it can accommodate l^∞-perturbations of size ϵ_s. The remaining task is to identify p_1, p_2 and ϵ_s on each $\mathcal{I}_j$, and justify (2.23)–(2.25). Remarkably, it is possible to construct explicit polynomials in τ and t such that the corresponding x^τ, y^τ have an explicit form. This technical part is carried out in Sect. 5 and Appendix C. We now wrap up the discussion for $\ell = 0$ and turn next to the case $\ell \geq 1$.

Step 3. The case $\ell = 1$. For simplicity we denote $t = r$, $F = \widetilde{f}_{\ell m}$, $G = \widetilde{g}_{\ell m}$ and work with the system

$$\begin{cases} \partial_{tt} F = (\frac{2}{t^2} + 1 - \tau - 2Q^2)F - Q^2 G \\ \partial_{tt} G = (\frac{2}{t^2} + 1 + \tau - 2Q^2)G - Q^2 F. \end{cases}$$

Here $F(t) = \text{const} \cdot t^2 + \mathcal{O}(t^4)$, $G(t) = \text{const} \cdot t^2 + \mathcal{O}(t^4)$ as $t \to 0^+$. Furthermore, $F, G \in H^k((0, \infty), \mathbb{R})$ for any integer $k \geq 1$. Denote $U = F + G$ and $V = F - G$. Then

$$\begin{cases} U'' = (\frac{2}{t^2} + 1 - 3Q^2)U - \tau V \\ V'' = (\frac{2}{t^2} + 1 - Q^2)V - \tau U. \end{cases} \tag{2.26}$$

Using $L_-(x_1 Q) = -2\partial_1 Q$, we deduce $\int_0^\infty U(t) t^2 Q(t) dt = 0$ which will be used momentarily.

By using the multiplier $\frac{1}{2}U + tU'$ for the U-equation and the multiplier $\frac{1}{2}V + tV'$ for the V-equation (see (2.26)), we arrive at the important identity (here H^1-integrability is enough, cf. Remark 9.5)

$$\int_0^\infty \Big((U')^2 + (\tfrac{2}{t^2} + 3tQ'Q)U^2\Big)dt + \int_0^\infty \Big((V')^2 + (\tfrac{2}{t^2} + tQ'Q)V^2\Big)dt = 0. \tag{2.27}$$

Denote

$$\mathcal{F} = \left\{ f \in H^2((0,\infty),\mathbb{R}) : \ f(t) = \text{const} \cdot t^2 + \mathcal{O}(t^4) \text{ as } t \to 0^+ \right\}.$$

To obtain the desired conclusion $U \equiv 0$ and $V \equiv 0$, it suffices to show the following: If $U \in \mathcal{F}$ and satisfies $\int_0^\infty U t^2 Q dt = 0$, then

$$\int_0^\infty \left((U')^2 + (\tfrac{2}{t^2} + 3tQ'Q)U^2 \right) dt \geq 0. \tag{2.28}$$

Also if $V \in \mathcal{F}$, then for some function $\widetilde{\eta}_2(t) > 0$, it holds that

$$\int_0^\infty \left((V')^2 + (\tfrac{2}{t^2} + tQ'Q)V^2 \right) dt \geq \int_0^\infty \widetilde{\eta}_2(t) V(t)^2 dt. \tag{2.29}$$

It turns out (2.29) follows easily from the inequality[13] $2 + t^3 Q'Q > 0$, $\forall t > 0$. On the other hand, to show (2.28) we work with the operator $S = -\partial_{tt} + \frac{2}{t^2} + 3tQ'Q$ and show the following:

(1) If φ solves $S\varphi = 0$ on $(0,\infty)$, and $\varphi(t) = t^2 + \mathcal{O}(t^6)$ as $t \to 0^+$, then φ must change its sign exactly once on $(0,\infty)$, and $\varphi(t) = -\gamma_1 t^2 + \mathcal{O}(t^{-1})$ for $t \geq 10$, where $\gamma_1 > 0$ is a constant;

(2) S has a principal eigenvalue which is negative, and $\int_0^\infty f S f dt \geq 0$ for any $f \in \mathcal{F}$ with $\int_0^\infty f f_0 dt = 0$, where f_0 is the principal eigenfunction;

(3) There exists some $f_* \in \mathcal{F}$ such that $Sf_* = t^2 Q$ and $\int_0^\infty f_* t^2 Q dt < 0$.

By a decomposition argument, the inequality (2.28) follows from statements (2) and (3). Statement (2) follows from statement (1), which in turn is proved by comparison arguments. On the other hand, to prove statement (3) we make the change of variable $g = -t^{-2} f_*$ and consider the g_α-system

$$g_\alpha'' = -\tfrac{4}{t} g_\alpha' + 3tQ'Qg_\alpha + Q, \qquad \text{with initial condition } g_\alpha(0) = \alpha, \ g_\alpha'(0) = 0.$$

The goal is then to locate a suitable constant α such that $|g_\alpha(t)| \lesssim t^{-3}$ as $t \to \infty$ and $\int_0^\infty g_\alpha(t) Q t^4 dt > 0$. For this we develop a "shooting" approach and complete the proof via the following:

(a) g_α is Lipschitz in α: $\|g_{\alpha_1} - g_{\alpha_2}\|_\infty \leq C|\alpha_1 - \alpha_2|$ for some constant $C > 0$. The limit $\lim_{t\to\infty} g_\alpha(t)$ exists and the map $\alpha \to \lim_{t\to\infty} g_\alpha(t)$ is continuous.

(b) For $\alpha_1 = 0$, we have $g_{\alpha_1}(t) > 0$ for all $t \in [0,\infty)$.

(c) For $\alpha_2 = 5.8$, we have $\lim_{t\to\infty} g_{\alpha_2}(t) < 0$ and $\int_0^\infty g_{\alpha_2} Q t^4 dt > 0$.

Via a quantitative comparison argument, we show g_{α_1} satisfies $g_{\alpha_1} > 0$, $g_{\alpha_1}' > 0$, $\forall t > 0$. The estimate of g_{α_2} is carried out via three intricate comparison arguments.

Step 4. The case $\ell \geq 2$. We denote $t = r$, $F = \widetilde{f}_{\ell m}$, $G = \widetilde{g}_{\ell m}$ and consider (below $c = \ell(\ell+1) \geq 6$)

$$\begin{cases} \partial_{tt} F = (\frac{c}{t^2} + 1 - \tau - 2Q^2)F - Q^2 G \\ \partial_{tt} G = (\frac{c}{t^2} + 1 + \tau - 2Q^2)G - Q^2 F. \end{cases}$$

[13] See statement (4) in Lemma A.2 in the Appendix.

Here $F(t), G(t) = \mathcal{O}(t^3)$ as $t \to 0^+$, and $F, G \in H^k((0,\infty),\mathbb{R})$ for any integer $k \geq 1$.

Denote $U = F + G$ and $V = F - G$. Then

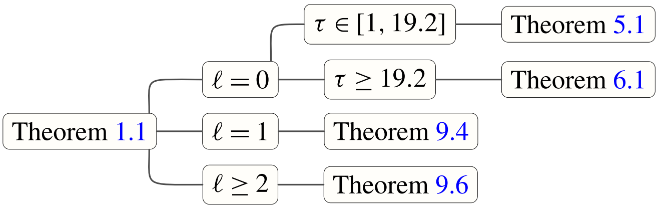
$$\begin{cases} U'' = (\frac{c}{t^2} + 1 - 3Q^2)U - \tau V \\ V'' = (\frac{c}{t^2} + 1 - Q^2)V - \tau U. \end{cases}$$

By using the multiplier $\frac{1}{2}U + tU'$ for the U-equation, and the multiplier $\frac{1}{2}V + tV'$ for the V-equation, we obtain

$$\int_0^\infty \Big((U')^2 + (\tfrac{c}{t^2} + 3tQ'Q)U^2\Big)dt + \int_0^\infty \Big((V')^2 + (\tfrac{c}{t^2} + tQ'Q)V^2\Big)dt = 0.$$

By Lemma A.2, we have $2 + t^3 Q'Q > 0$ for all $t > 0$. Since $c \geq 6$, it follows that $U \equiv 0$ and $V \equiv 0$.

A schematic description of the whole proof of Theorem 1.1 can be found below.



As detailed in later sections, the proofs of Theorems 5.1, 6.1, and 9.4 are partly computer-aided (cf. the proof of Proposition 5.2). These exact computations involve very light programming and can, in principle, be carried out in any suitable programming language using the explicit expressions constructed in our proofs. To facilitate verification, we provide a complete set of fully annotated MATHEMATICA programs, cross-referenced with the relevant proofs in this paper, available for download at [1].

3 Properties of L_+, L_-, $\widetilde{A}$, $\mathcal{H}$, and proof of Corollary 1.3

In this section, we collect a few important properties of L_+, L_-, $\widetilde{A}$ and $\mathcal{H}$. For the linearization of general NLS and related investigations we refer to the papers [15, 22, 33–35, 73, 74] and the references therein. Let Q be the ground state solution to (1.1) and recall

$$L_+ = -\Delta + 1 - 3Q^2, \qquad L_- = -\Delta + 1 - Q^2.$$

Lemma 3.1 (Properties of Q, L_+ and L_-) *The following hold true.*

(1) *Pohozaev's identities* ([60]): $\|Q\|_2^2 = \frac{1}{4}\|Q\|_4^4$, $\|\nabla Q\|_2^2 = \frac{3}{4}\|Q\|_4^4$.

(2) *If* $f \in H^1(\mathbb{R}^3,\mathbb{C})$ *and* $f \perp \Delta Q$ (*the condition* $f \perp \Delta Q$ *can be replaced by* $f \perp Q^3$), *then* $\langle L_+ f, f\rangle \geq 0$.

(3) *If* $f \in H^1(\mathbb{R}^3,\mathbb{C})$, *then* $\langle L_- f, f\rangle \geq 0$.

(4) $\sigma_{\mathrm{ess}}(L_+) = [1, \infty)$, $\sigma_{\mathrm{ess}}(L_-) = [1, \infty)$.
(5) $\sigma_{\mathrm{dis}}(L_+) = \{-c_*, 0\}$ *where* $c_* > 0$. *The eigenvalue* $-c_*$ *is simple with corresponding eigenfunction being radial, strictly positive,* C^∞ *and exponentially decaying. Also* $\ker(L_+) = \mathrm{span}\{\partial_1 Q,\ \partial_2 Q,\ \partial_3 Q\}$.
(6) $\sigma_{\mathrm{dis}}(L_-) = \{0\}$ *and* $\ker(L_-) = \mathrm{span}\{Q\}$.
(7) L_+ *and* L_- *have no resonances at the edge* $\lambda = 1$.

Proof (1) We use the equation $-\Delta Q + Q = Q^3$. Multiplying by Q and integrating gives us

$$\|\nabla Q\|_2^2 + \|Q\|_2^2 = \|Q\|_4^4. \tag{3.1}$$

By using the multiplier $x \cdot \nabla Q$, we get[14]

$$\int_{\mathbb{R}^3} (-\Delta Q) x \cdot \nabla Q dx = -\tfrac{1}{2}\|\nabla Q\|_2^2, \quad \int_{\mathbb{R}^3} Q x \cdot \nabla Q dx = -\tfrac{3}{2}\|Q\|_2^2$$
$$\int_{\mathbb{R}^3} (-Q^3) x \cdot \nabla Q dx = \tfrac{3}{4}\|Q\|_4^4. \tag{3.2}$$

These yield $\frac{1}{2}\|\nabla Q\|_2^2 + \frac{3}{2}\|Q\|_2^2 = \frac{3}{4}\|Q\|_4^4$. Thus $\|\nabla Q\|_2^2 = \frac{3}{4}\|Q\|_4^4$ and $\|Q\|_2^2 = \frac{1}{4}\|Q\|_4^4$.

(2) & *(3)*. We resort to the variational characterization of Q. Consider the expression

$$J = \tfrac{3}{2}\log(\|\nabla \widetilde{f}\|_2^2) + \tfrac{1}{2}\log(\|\widetilde{f}\|_2^2) - \log(\|\widetilde{f}\|_4^4), \tag{3.3}$$

where $\widetilde{f} = Q + \epsilon g$, $g_1 = \mathrm{Re}(g)$, $g_2 = \mathrm{Im}(g)$. Clearly the coefficient for $\mathcal{O}(\epsilon^2)$-term must be nonnegative.

Observe that $|\nabla Q + \epsilon \nabla g_1 + i\epsilon \nabla g_2|^2 = |\nabla Q + \epsilon \nabla g_1|^2 + \epsilon^2 |\nabla g_2|^2$. Thus

$$\begin{aligned} J = &\tfrac{3}{2}\log\Big(\|\nabla Q\|_2^2 + 2\epsilon\langle \nabla Q, \nabla g_1\rangle + \epsilon^2 \|\nabla g\|_2^2\Big) \\ &+ \tfrac{1}{2}\log\Big(\|Q\|_2^2 + 2\epsilon\langle Q, g_1\rangle + \epsilon^2\|g\|_2^2\Big) \\ &- \log\Big(\|Q\|_4^4 + 4\langle Q^3, g_1\rangle\epsilon + \epsilon^2\int_{\mathbb{R}^3}(6Q^2 g_1^2 + 2Q^2 g_2^2)dx + \mathcal{O}(\epsilon^3)\Big). \end{aligned} \tag{3.4}$$

Using $\log(1+x) = x - \frac{x^2}{2} + \cdots$ for $|x| \ll 1$, we find the coefficient of ϵ^2 term in J as:

$$\begin{aligned} &\tfrac{3}{2}\left(\tfrac{\|\nabla g\|_2^2}{\|\nabla Q\|_2^2} - \tfrac{1}{2}\big(\tfrac{2\langle \nabla Q, \nabla g_1\rangle}{\|\nabla Q\|_2^2}\big)^2\right) + \tfrac{1}{2}\left(\tfrac{\|g\|_2^2}{\|Q\|_2^2} - \tfrac{1}{2}\big(\tfrac{2\langle Q, g_1\rangle}{\|Q\|_2^2}\big)^2\right) \\ &\qquad - \Big(\tfrac{1}{\|Q\|_4^4}\int_{\mathbb{R}^3}(6Q^2 g_1^2 + 2Q^2 g_2^2)dx - \tfrac{1}{2}\big(\tfrac{4\langle Q^3, g_1\rangle}{\|Q\|_4^4}\big)^2\Big) \geq 0. \end{aligned} \tag{3.5}$$

[14] These can also be computed by using $\frac{d}{d\lambda}Q_\lambda\big|_{\lambda=1} = Q + x \cdot \nabla Q$, where $Q_\lambda = \lambda Q(\lambda x)$.

Using $\|\nabla Q\|_2^2 = \frac{3}{4}\|Q\|_4^4$, $\|Q\|_2^2 = \frac{1}{4}\|Q\|_4^4$, we deduce

$$2\|\nabla g\|_2^2 - \frac{3}{4}\|Q\|_4^4\left(\frac{2\langle \nabla Q, \nabla g_1\rangle}{\|\nabla Q\|_2^2}\right)^2 + 2\|g\|_2^2 - 16\frac{(\langle Q, g_1\rangle)^2}{\|Q\|_4^4}$$
$$- \int_{\mathbb{R}^3}(6Q^2 g_1^2 + 2Q^2 g_2^2)dx + 8\frac{\langle Q^3, g_1\rangle^2}{\|Q\|_4^4} \geq 0.$$

Now if $\int_{\mathbb{R}^3} g_1 \Delta Q dx = 0$, then clearly $\int_{\mathbb{R}^3} Q^3 g_1 dx = \int_{\mathbb{R}^3} Q g_1 dx$, and we obtain

$$2\langle L_+ g_1, g_1\rangle + 2\langle L_- g_2, g_2\rangle \geq 8\frac{\langle Q, g_1\rangle^2}{\|Q\|_4^4}. \tag{3.6}$$

If $\int_{\mathbb{R}^3} g_1 Q^3 dx = 0$, we also obtain $\langle L_+ g_1, g_1\rangle + \langle L_- g_2, g_2\rangle \geq 0$. The desired conclusion easily follows.

(4) This follows easily from Weyl's essential spectrum theorem (cf. Theorem XIII. 14 of [61]).

(5) Clearly $\langle L_+ Q, Q\rangle = -2\|Q\|_4^4 < 0$. Thus L_+ must have a negative eigenvalue $-c_* < 0$. By using variational characterization it is not difficult to extract a radial and non-negative eigenfunction. By using ODE arguments (cf. Lemma 7.3 for a similar case) one can deduce strict positivity and exponential decay. Denote this eigenfunction as h_* with $\|h_*\|_2 = 1$. Assume there is another negative direction, i.e., $L_+ h_1 = -c_1 h_1$, for some $c_1 > 0$ and h_1 with $\|h_1\|_2 = 1$. Without loss of generality we assume h_1 is real-valued and $h_1 \perp h_*$. Clearly, we can find $a, b \in \mathbb{R}$, not all zero such that

$$h = a h_* + b h_1, \qquad h \perp \Delta Q. \tag{3.7}$$

But then $\langle L_+ h, h\rangle < 0$ and h is not identically zero. This contradicts the property that $\langle L_+ \widetilde{h}, \widetilde{h}\rangle \geq 0$ for any $\widetilde{h} \perp \Delta Q$. Thus $-c_*$ must be simple and there are no eigenvalues in the interval $(-\infty, 0)$. For the radial case, the fact that L_+ has no eigenvalues in $(0, 1]$ and no resonances at $\lambda = 1$ is rigorously proved in [20] (see also [23]). For the nonradial case, the fact that L_+ has no eigenvalues in $(0, 1]$ is proved in [44].

(6) & (7). For the radial case, the fact that L_- has no eigenvalues in $(0, 1]$ and no endpoint resonances is proved in [20] (see also [23] for numerical verification). For the nonradial case, the fact that L_- has no eigenvalues in $(0, 1]$ is proved in [44]. It is also proved in [44] that L_+ and L_- have no resonances at $\lambda = 1$. The fact that $\ker(L_+) = \mathrm{span}\{\partial_1 Q, \partial_2 Q, \partial_3 Q\}$, and $\ker(L_-) = \mathrm{span}\{Q\}$ is first proved in Proposition 2.8 of [73]. □

Denote

$$Q^\perp = \{f \in L^2(\mathbb{R}^3, \mathbb{C}) : \langle Q, f\rangle = 0\}.$$

We denote by P the projection operator onto $Q^\perp$: namely for any $\widetilde{f} \in L^2(\mathbb{R}^3, \mathbb{C})$, define

$$P\widetilde{f} = \widetilde{f} - \langle Q, \widetilde{f}\rangle Q \cdot \|Q\|_2^{-2}.$$

We shall first study the operators $A = PL_+P$ and $K_s = PL_+P + sL_-^{-1}$, $s > 0$ on the Hilbert space $Q^\perp$. The motivation is as follows. Let $0 \neq \tau \in \mathbb{R}$. If $f = (f_1, f_2)^\top \in H^2(\mathbb{R}^3, \mathbb{C}^2)$ with $\|f\|_2 = 1$ solves

$$\begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = i\tau \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \text{i.e.,} \begin{cases} L_+ f_1 = i\tau f_2 \\ L_- f_2 = i\tau f_1 \end{cases}, \tag{3.8}$$

then $f_1 \perp Q$ and $(PL_+P + \tau^2 L_-^{-1}) f_1 = 0$. Furthermore $\|f_1\|_2 > 0$.

Conversely, let $s > 0$ and assume $f_1 \in H^2(\mathbb{R}^3, \mathbb{C}) \cap Q^\perp$ with $\|f_1\|_2 = 1$ solves

$$(PL_+P + sL_-^{-1}) f_1 = 0, \tag{3.9}$$

then for some c_1, we can rewrite $L_+ f_1 = -sL_-^{-1} f_1 + c_1 Q = i\tau \underbrace{\left(i\tau L_-^{-1} f_1 + \frac{c_1}{i\tau} Q \right)}_{=: f_2}$ ($\tau := \sqrt{s}$).

Thus for $0 \neq \tau \in \mathbb{R}$,

$$\begin{aligned} &\text{studying } \begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = i\tau \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \text{ on } L^2(\mathbb{R}^3, \mathbb{C}^2) \\ \Leftrightarrow \quad &\text{studying } (PL_+P + \tau^2 L_-^{-1}) f_1 = 0 \text{ on } Q^\perp. \end{aligned} \tag{3.10}$$

Lemma 3.2 *The operator PL_+P is a self-adjoint operator on $Q^\perp$ with $\mathcal{D}(PL_+P) = H^2(\mathbb{R}^3, \mathbb{C}) \cap Q^\perp$. The essential spectrum $\sigma_{\text{ess}}(PL_+P) = [1, \infty)$, and*

$$\sigma(PL_+P) \cap (-\infty, 1) = \{-\gamma_1, 0\}, \tag{3.11}$$

where $\gamma_1 > 0$. Furthermore the eigenvalue $-\gamma_1$ is simple with the corresponding eigenfunction being radial, and the eigenvalue 0 has multiplicity three[15] *with* $\ker(PL_+P) = \text{span}\{\partial_1 Q, \partial_2 Q, \partial_3 Q\}$.

Proof Denote $A = PL_+P$. We first show that A is self-adjoint on $Q^\perp$. It suffices to check $\mathcal{D}(A^*) = H^2(\mathbb{R}^3, \mathbb{C}) \cap Q^\perp$. Fix $g \in \mathcal{D}(A^*)$, i.e., $g \in Q^\perp$ and there exists $C > 0$ such that

$$\begin{aligned} &|\langle A\widetilde{f}, g\rangle| = |\langle PL_+P\widetilde{f}, g\rangle| = |\langle L_+\widetilde{f}, g\rangle| \leq C\|\widetilde{f}\|_2, \\ &\forall\, \widetilde{f} \in Q^\perp \cap H^2(\mathbb{R}^3, \mathbb{C}). \end{aligned} \tag{3.12}$$

The above implies for another constant $C_1 > 0$, $|\langle L_+ f, g\rangle| \leq C_1 \|f\|_2$, $\forall f \in H^2(\mathbb{R}^3, \mathbb{C})$. Since L_+ is self-adjoint, we obtain $g \in \mathcal{D}(L_+) = H^2(\mathbb{R}^3, \mathbb{C})$. Thus $\mathcal{D}(A^*) = H^2(\mathbb{R}^3, \mathbb{C}) \cap Q^\perp$ and A is self-adjoint.

Now suppose $\lambda \in \sigma_{\text{ess}}(A)$. By Weyl's criterion (cf. Theorem VII.12 of [61]) we can find $u_n \in \mathcal{D}(A)$ with $\|u_n\| = 1$, $u_n \rightharpoonup 0$ (i.e., u_n weakly tends to 0 in L^2) such that $(A - \lambda)u_n \to 0$. This implies

$$(L_+ - \lambda)u_n - \langle Q, (L_+ - \lambda)u_n\rangle Q\|Q\|_2^{-2} \to 0, \qquad \text{as } n \to \infty. \tag{3.13}$$

[15] Here we regard PL_+P as an operator on $Q^\perp$ so that Q is not counted as an eigenvector.

Since $u_n \rightharpoonup 0$, the term $\langle Q, (L_+ - \lambda)u_n\rangle$ clearly converges to zero. Thus $(L_+ - \lambda)u_n \to 0$, as $n \to \infty$. This implies that $\lambda \in \sigma_{\text{ess}}(L_+) = [1, \infty)$.

Conversely assume $\lambda \in \sigma_{\text{ess}}(L_+)$. Then we can find $u_n \in \mathcal{D}(L_+)$ with $\|u_n\| = 1$, $u_n \rightharpoonup 0$ such that $(L_+ - \lambda)u_n \to 0$. Define

$$v_n = u_n - \langle Q, u_n\rangle Q \cdot \|Q\|_2^{-2}.$$

Clearly $\|v_n\|_2 \to 1$ and $(L_+ - \lambda)v_n \to 0$, as $n \to \infty$. This implies $P(L_+ - \lambda)Pv_n \to 0$ as $n \to \infty$. Thus $\lambda \in \sigma_{\text{ess}}(A)$. Consequently $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(L_+) = [1, \infty)$.

We show that A has a negative eigenvalue. Since $\sigma_{\text{ess}}(A) = [1, \infty)$, it suffices for us to find $\phi \in \mathcal{D}(A)$ such that $\langle A\phi, \phi\rangle < 0$. Recall[16] $L_+ Q_1 = -2Q$ where $Q_1 = Q + rQ'$. Let $\phi = Q_1 + \frac{1}{2}Q$. Clearly $\phi \perp Q$ and

$$\begin{aligned}\langle A\phi, \phi\rangle &= \langle L_+\phi, \phi\rangle = \langle -2Q - Q^3, \phi\rangle \\ &= \langle -Q^3, \tfrac{3}{2}Q + rQ'\rangle = -\tfrac{3}{2}\|Q\|_4^4 + \tfrac{3}{4}\|Q\|_4^4 < 0. \end{aligned} \tag{3.14}$$

Thus A must have at least one negative eigenvalue.

Next we show that A has at most one negative direction. Assume this is not the case and

$$PL_+Pf_1 = -\lambda_1 f_1, \qquad PL_+Pf_2 = -\lambda_2 f_2, \tag{3.15}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, $f_1 \perp Q$, $f_2 \perp Q$. We may assume $f_1 \perp f_2$, and $\|f_1\|_2 = \|f_2\|_2 = 1$. Clearly for $a, b \in \mathbb{C}$,

$$\begin{aligned}\langle L_+(af_1 + bf_2), af_1 + bf_2\rangle &= \langle PL_+P(af_1 + bf_2), af_1 + bf_2\rangle \\ &= -|a|^2\lambda_1 - |b|^2\lambda_2. \end{aligned} \tag{3.16}$$

Choosing suitable a, b with $|a|^2 + |b|^2 \neq 0$, we get

$$af_1 + bf_2 \perp \Delta Q.$$

This contradicts $\langle L_+ f, f\rangle \geq 0, \forall f \perp \Delta Q$.

Next we show that the eigenfunction corresponding to the negative eigenvalue $-\gamma_1$ must be radial. Let

$$P(L_+ + \gamma_1)Pf = 0, \qquad f \perp Q, \ \|f\|_2 = 1.$$

If for $\ell \geq 1$, f_l is the projection of f into the l^{th} spherical harmonics, then apparently we must have $f_l \perp \varphi_0$, where $L_+\varphi_0 = -c_*\varphi_0$, $\|\varphi_0\|_2 = 1$ and is radial and $c_* > 0$ (i.e., $-c_*$ is the unique negative eigenvalue of L_+, see (5) of Lemma 3.1). This implies

$$0 = \langle P(L_+ + \gamma_1)Pf_l, f_l\rangle = \langle L_+ f_l, f_l\rangle + \gamma_1\|f_l\|_2^2 \geq \gamma_1\|f_l\|_2^2, \qquad \text{if } \ell \geq 1. \tag{3.17}$$

Thus f must be radial.

[16] Observe $\Delta Q_\lambda + Q_\lambda^3 - \lambda^2 Q_\lambda = 0$, $Q_\lambda = \lambda Q(\lambda x)$ and $Q_1 = \frac{\partial}{\partial\lambda} Q_\lambda\big|_{\lambda=1}$.

Next we classify the eigenfunctions of A corresponding to the zero eigenvalue. Suppose $Af = 0$ and $f \perp Q$. Then $L_+ Pf = cQ$ for some constant c. This implies (for some c_j)

$$Pf = \sum_{j=1}^{3} c_j \partial_j Q - \tfrac{1}{2} c Q_1,$$

where $Q_1 = Q + rQ'$ (note $L_+ Q_1 = -2Q$). Since $Pf \perp Q$, we get $c = 0$, $f = \sum_{j=1}^{3} c_j \partial_j Q$. Thus

$$\ker(A) = \operatorname{span}\{\partial_1 Q, \partial_2 Q, \partial_3 Q\}.$$

Finally we show that A has no eigenvalues in $(0, 1)$. It suffices to consider the function[17] $g(\lambda) = \langle (L_+ - \lambda)^{-1} Q, Q\rangle$, which is well-defined since $\lambda \in \rho(L_+)$ for $0 < \lambda < 1$. Clearly $g'(\lambda) > 0$ and $g(0) > 0$ (see Remark 3.3). Thus $g(\lambda) > 0$ for all $0 < \lambda < 1$. The desired conclusion follows. □

Remark 3.3 It is not difficult to check that L_+^{-1} exists as a bounded operator from $L^2_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{C})$ to $H^2_{\mathrm{rad}}(\mathbb{R}^3, \mathbb{C})$. Denote $u_\lambda = (L_+ - \lambda)^{-1} Q$. By using projection into spherical harmonics, we have u_λ is radial. Also denote $u_0 = L_+^{-1} Q = -\frac{1}{2} Q_1$ (recall $Q_1 = Q + rQ'$). Clearly

$$(L_+ - \lambda) u_\lambda = Q, \quad L_+ u_0 = Q$$

$$\Rightarrow (L_+ - \lambda)(u_\lambda - u_0) = \lambda u_0 \qquad \text{(note that } u_0 \text{ is radial)} \tag{3.18}$$

$$\Rightarrow (1 - \lambda L_+^{-1})(u_\lambda - u_0) = \lambda L_+^{-1} u_0. \tag{3.19}$$

Thus $\|u_\lambda - u_0\|_{H^2} = \mathcal{O}(\lambda)$. In particular $\lim_{\lambda \to 0^+} g(\lambda) = g(0) > 0$.

Proposition 3.4 *Let $s > 0$ and denote $K_s = PL_+P + sL_-^{-1}$. The following hold.*
(1) *K_s is a self-adjoint operator on $Q^\perp$ with $\mathcal{D}(K_s) = H^2(\mathbb{R}^3, \mathbb{C}) \cap Q^\perp$.*
(2) $\sigma_{\mathrm{ess}}(K_s) \subset [1, \infty)$.

Proof (1) Since L_-^{-1} is a bounded self-adjoint operator and PL_+P is self-adjoint, it follows that K_s is self-adjoint. Indeed denote $A = PL_+P$ and $B = L_-^{-1}$. Taking λ_0 sufficiently large, one can[18] solve $(A + B + i\lambda_0)u = f$ for any $f \in Q^\perp$. This in turn implies $A + B$ is self-adjoint.

[17] The function $g(\lambda)$ already appeared in [73] (see the discussion around formula (E.5) therein) and is also used in [64] (see the proof of Lemma 15 in [64]). The appearance of $g(\lambda)$ is very natural: if for some $\lambda_* \in (0, 1)$, it holds that $PL_+\phi = \lambda_* \phi$, then $(L_+ - \lambda_*)\phi = \mathrm{const} \cdot Q$ and $\phi \perp Q$. Thus one must have $\langle (L_+ - \lambda_*)^{-1} Q, Q\rangle = 0$.

[18] Here we invoke the following well-known criterion. Suppose T is a symmetric closed operator on a Hilbert space $\mathbb{H}$ with dense domain. Let $0 \neq \lambda \in \mathbb{R}$. If $(T + \lambda i)u = f$ has a unique solution for every $f \in \mathbb{H}$, then $T^* = T$. To justify this, one can assume $g \in \mathcal{D}(T^*)$, and show that $g \in \mathcal{D}(T)$. To this end, we let u be the solution to $(T + \lambda i)u = (T^* + \lambda i)g$. Since $\mathcal{D}(T) \subset \mathcal{D}(T^*)$, we have $(T^* + \lambda i)(u - g) = 0$. The fact that $\ker(T^* + \lambda i) = \{0\}$ follows easily from $\operatorname{ran}(T - \lambda i) = \mathbb{H}$. Hence $g = u \in \mathcal{D}(T)$.

(2) Suppose $\lambda \in \sigma_{\text{ess}}(K_s)$. By Weyl we can find $u_n \in \mathcal{D}(K_s)$ with $\|u_n\|_2 = 1$, $u_n \rightharpoonup 0$ such that $(K_s - \lambda)u_n \to 0$ as $n \to \infty$. Denote

$$\widetilde{v}_n = u_n - \sum_{j=0}^{3} \langle e_j, u_n \rangle e_j, \qquad v_n = \widetilde{v}_n / \|\widetilde{v}_n\|_2,$$

where e_j are the normalized discrete eigenfunctions of PL_+P. For n sufficiently large, clearly $\|\widetilde{v}_n\| \sim \|u_n\|_2 = 1$ (since $\langle e_j, u_n \rangle \to 0$ as $n \to \infty$) and $\langle PL_+Pv_n, v_n \rangle \geq 1$. It is easy to check $(K_s - \lambda)v_n \to 0$, i.e.,

$$\langle (PL_+P + sL_-^{-1} - \lambda)v_n, v_n \rangle \to 0.$$

Thus $\lambda \geq 1$. □

Lemma 3.5 *Define* $X_1 := H^1(\mathbb{R}^3, \mathbb{C}) \cap Q^{\perp}$ *and* $K_s = PL_+P + sL_-^{-1}$. *For* $s > 0$, *we have*

$$\begin{aligned} \mu_2(s) := \sup_{f \in Q^{\perp}} \inf_{\substack{g \perp f \\ g \in X_1, \|g\|_2 = 1}} \langle K_s g, g \rangle \\ \geq \inf_{g \perp e_0,\, g \in X_1, \|g\|_2 = 1} \langle K_s g, g \rangle > 0. \end{aligned} \tag{3.20}$$

Proof Denote the eigenfunctions of $A = PL_+P$ on $Q^{\perp}$ as $(e_j)_{j=0}^3$, namely:

$$Ae_0 = -\gamma_1 e_0, \quad Ae_j = 0, \qquad 1 \leq j \leq 3; \quad \|e_j\|_2 = 1, \quad \forall\, 0 \leq j \leq 3. \tag{3.21}$$

For $g \in X_1$ with $g \perp e_0$, we write $g = \sum_{j=1}^3 \langle e_j, g \rangle e_j + g^{\perp}$, where $g^{\perp} \perp e_j$, $0 \leq j \leq 3$. Clearly

$$\begin{aligned} &\langle Ag, g \rangle = \langle Ag^{\perp}, g^{\perp} \rangle \geq \|g^{\perp}\|_2^2 \\ &\textstyle\sum_{j=1}^3 |\langle e_j, g \rangle|^2 = \sum_{j=0}^3 |\langle L_-^{\frac{1}{2}} e_j, L_-^{-\frac{1}{2}} g \rangle|^2 \leq C_0 \|L_-^{-\frac{1}{2}} g\|_2^2, \end{aligned}$$

where $C_0 > 0$ is a constant. It follows that

$$\begin{aligned} \langle PL_+Pg, g \rangle + s\langle L_-^{-1} g, g \rangle &\geq \|g^{\perp}\|_2^2 + s\|L_-^{-\frac{1}{2}} g\|_2^2 \\ &\geq \|g^{\perp}\|_2^2 + s \cdot \tfrac{1}{C_0} \textstyle\sum_{j=1}^3 |\langle e_j, g \rangle|^2 \\ &\geq \min\{1, \tfrac{s}{C_0}\} \|g\|_2^2 = \min\{1, \tfrac{s}{C_0}\} > 0. \end{aligned} \tag{3.22}$$

Thus

$$\mu_2(s) \geq \inf_{\substack{g \perp e_0 \\ g \in X_1, \|g\|_2 = 1}} \langle K_s g, g \rangle > 0.$$

□

Lemma 3.6 *Recall* $X_1 := H^1(\mathbb{R}^3, \mathbb{C}) \cap Q^{\perp}$ *and* $K_s = PL_+P + sL_-^{-1}$. *For* $s > 0$, *consider*

$$\mu_1(s) = \inf_{g \in X_1,\, \|g\|_2 = 1} \langle K_s g, g\rangle. \tag{3.23}$$

Then the following hold true.

(1) $\mu_1(s) < 0$ *if* $0 < s \ll 1$, *and* $\mu_1(s) > 0$ *if* $s \gg 1$.

(2) *For each* $s > 0$, $\mu_1(s)$ *is attained at some* $g_*(s) \in X_1$ *with* $\|g_*(s)\|_2 = 1$.

(3) $\mu_1(s)$ *is a Lipschitz-continuous and strictly monotonically increasing function of* s.

(4) *There exists a unique* s_0 *such that* $\mu_1(s_0) = 0$. *Furthermore* $\mu_1(s_0) = 0$ *is attained by some* $g_*(s_0) \in X_1$ *which is radial and* $\|g_*(s_0)\|_2 = 1$. *The function* $g_*(s_0)$ *is smooth, in particular* $g_*(s_0) \in H^m(\mathbb{R}^3, \mathbb{C})$ *for any integer* $m \geq 1$. *If* $\widetilde{g}(s_0)$ *also attains the infimum* $\mu_1(s_0) = 0$, *then* $\widetilde{g}(s_0) = e^{i\theta} g_*(s_0)$, *where* $\theta \in [0, 2\pi)$ *is a constant.*

Moreover, the following uniqueness statement holds: if $g \in X_1$ *solves* $K_{s_0} g = 0$, *then* g *must be a scalar multiple of* $g_*(s_0)$. *In particular, we can take* $g_*(s_0)$ *as radial, and* $g_*(s_0) \in H^m(\mathbb{R}^3, \mathbb{R})$, $\forall m \in \mathbb{N}$.

Proof It is easy to check that $\mu_1(s)$ is a finite number for any $s > 0$. We shall use e_j as defined in (3.21).

(1) To show $\mu_1(s) < 0$ if $0 < s \ll 1$ we take $g = e_0$. Clearly if $s > 0$ is sufficiently small, then

$$\langle K_s e_0, e_0\rangle = -\gamma_1 + s\|L_-^{-\frac{1}{2}} e_0\|_2^2 < -\tfrac{1}{2}\gamma_1. \tag{3.24}$$

To show $\mu_1(s) > 0$ when $s \gg 1$, we first observe that for some constant $C_1 > 0$,

$$\textstyle\sum_{j=0}^3 |\langle e_j, g\rangle|^2 = \sum_{j=0}^3 |\langle L_-^{\frac{1}{2}} e_j, L_-^{-\frac{1}{2}} g\rangle|^2 \leq C_1 \|L_-^{-\frac{1}{2}} g\|_2^2. \tag{3.25}$$

Using $g = \sum_{j=0}^3 \langle e_j, g\rangle e_j + g^{\perp}$ with $g^{\perp} \perp e_j$, we get for $s \gg 1$:

$$\begin{aligned} \langle PL_+Pg, g\rangle + s\langle L_-^{-1} g, g\rangle &\geq -\gamma_1 |\langle e_0, g\rangle|^2 + \|g^{\perp}\|_2^2 + s\|L_-^{-\frac{1}{2}} g\|_2^2 \\ &\geq -\gamma_1 |\langle e_0, g\rangle|^2 + \|g^{\perp}\|_2^2 + s \cdot \tfrac{1}{C_1} \textstyle\sum_{j=0}^3 |\langle e_j, g\rangle|^2 \\ &\geq \|g\|_2^2 = 1. \end{aligned} \tag{3.26}$$

(2) Fix $s > 0$. Let $(g_n)_{n \geq 1}$ be a minimizing sequence, i.e., $g_n \in X_1$, $\|g_n\|_2 = 1$, and $\langle K_s g_n, g_n\rangle \to \mu_1(s)$. Without loss of generality we assume

$$\mu_1(s) \leq \langle K_s g_n, g_n\rangle \leq \mu_1(s) + 1, \qquad \text{for all } n.$$

Since $s > 0$ and $Pg_n = g_n$, we have $\langle L_+ g_n, g_n\rangle \leq \mu_1(s) + 1$ which together with $\|g_n\|_2 = 1$ yields $\|g_n\|_{H^1} \leq c_3$, where $c_3 > 0$ is a constant. By passing to a subsequence, we have $g_{n_j} \rightharpoonup g_*$, as $j \to \infty$ for some $g_* \in X_1$. Observe (below $A =$

PL_+P)

$$\langle Ag_{n_j}, g_{n_j}\rangle = \langle Ag_*, g_*\rangle + \langle A(g_{n_j} - g_*), g_{n_j} - g_*\rangle + \langle A(g_{n_j} - g_*), 2g_*\rangle$$
$$\geq \langle Ag_*, g_*\rangle - \gamma_1|\langle e_0, g_{n_j} - g_*\rangle|^2 + \langle A(g_{n_j} - g_*), 2g_*\rangle. \tag{3.27}$$

Thus $\liminf_{j\to\infty}\langle Ag_{n_j}, g_{n_j}\rangle \geq \langle Ag_*, g_*\rangle$. Also

$$\langle L_-^{-1}g_{n_j}, g_{n_j}\rangle = \langle L_-^{-1}g_*, g_*\rangle + \langle L_-^{-1}(g_{n_j} - g_*), g_{n_j} - g_*\rangle + \langle L_-^{-1}(g_{n_j} - g_*), 2g_*\rangle$$
$$\geq \langle L_-^{-1}g_*, g_*\rangle + \langle g_{n_j} - g_*, 2L_-^{-1}g_*\rangle. \tag{3.28}$$

This implies $\liminf_{j\to\infty}\langle L_-^{-1}g_{n_j}, g_{n_j}\rangle \geq \langle L_-^{-1}g_*, g_*\rangle$ and $\mu_1(s) = \lim_{j\to\infty}\langle K_s g_{n_j}, g_{n_j}\rangle \geq \langle K_s g_*, g_*\rangle$.

(3) By definition of $\mu_1(s)$, it is clear that $\mu_1(s)$ is monotonically increasing in s. Fix $\widetilde{s} > 0$ and let $g_*(\widetilde{s})$ be a minimizer for $\mu_1(\widetilde{s})$ with $\|g_*(\widetilde{s})\|_2 = 1$. By the minimality of $\mu_1(s)$, we have

$$\mu_1(s) \leq \langle K_s g_*(\widetilde{s}), g_*(\widetilde{s})\rangle = \mu_1(\widetilde{s}) + (s - \widetilde{s})\langle L_-^{-1}g_*(\widetilde{s}), g_*(\widetilde{s})\rangle. \tag{3.29}$$

Let $g_*(s)$ be a minimizer for $\mu_1(s)$. Then

$$\mu_1(\widetilde{s}) \leq \langle K_{\widetilde{s}}g_*(s), g_*(s)\rangle = \mu_1(s) + (\widetilde{s} - s)\langle L_-^{-1}g_*(s), g_*(s)\rangle. \tag{3.30}$$

Using $\|L_-^{-1}g_*(\widetilde{s})\|_2 \leq \|g_*(\widetilde{s})\|_2 = 1$ and $\|L_-^{-1}g_*(s)\| \leq \|g_*(s)\|_2 = 1$, we get

$$|\mu_1(s) - \mu_1(\widetilde{s})| \leq |s - \widetilde{s}|. \tag{3.31}$$

Strict monotonicity follows from the fact that $\langle L_-^{-1}g_*(s), g_*(s)\rangle > 0$.

(4) The existence and uniqueness of s_0 is clear. Minimality of the variational characterization implies

$$K_{s_0}g_*(s_0) = Ag_*(s_0) + s_0 L_-^{-1}g_*(s_0) = 0. \tag{3.32}$$

By elliptic estimates, we have $g_*(s_0) \in H^m(\mathbb{R}^3, \mathbb{C})$ for any integer $m \geq 1$. By using projection into spherical harmonics, it is not difficult to check that $g_*(s_0)$ must be radial ($(\langle Ag_*^{(l)}, g_*^{(l)}\rangle \geq 0 \Rightarrow \|L_-^{-\frac{1}{2}}g_*^{(l)}\|_2^2 = 0$ where $g_*^{(l)}$ is the l^{th} spherical harmonic projection of $g_*(s_0)$, $\ell \geq 1$). Furthermore we can show uniqueness of $g_*(s_0)$ up to multiplication by $e^{i\theta}$. Indeed suppose g_1 and g_2 both attain the infimum $\mu_1(s_0) = 0$ and are radial. We may assume $g_1 \perp g_2$. Then clearly $K_{s_0}g_1 = 0$, $K_{s_0}g_2 = 0$. Then for some constants c_1, c_2 not all zero we can have $c_1 g_1 + c_2 g_2 \perp e_0$ (where we recall $Ae_0 = -\gamma_1 e_0$). This clearly implies

$$\langle A(c_1 g_1 + c_2 g_2), c_1 g_1 + c_2 g_2\rangle \geq 0. \tag{3.33}$$

On the other hand,

$$\langle L_-^{-1}(c_1 g_1 + c_2 g_2), c_1 g_1 + c_2 g_2\rangle > 0.$$

This contradicts $K_{s_0}(c_1 g_1 + c_2 g_2) = 0$.

Note that the argument above also proved the uniqueness of solutions to the equation $Af + s_0 L_-^{-1} f = 0$ for $f \in H^1(\mathbb{R}^3, \mathbb{C}) \cap Q^\perp$. In particular, by taking real or imaginary parts of $g_*(s_0)$, we obtain the conclusion that $g_*(s_0)$ can be written as a scalar multiple of a real-valued function. □

Let $s_0 > 0$ be the unique parameter as specified in Lemma 3.6. Denote $\tau_0 = \sqrt{s_0}$. Let $\phi_1 \in Q^\perp \cap H^2(\mathbb{R}^3, \mathbb{R})$ be the unique solution to

$$(PL_+P + \tau_0^2 L_-^{-1})\phi_1 = 0$$

satisfying $\|\phi_1\|_2 = 1$, ϕ_1 being radial. Note that for some unique constant $c_1 \in \mathbb{R}$, we have

$$L_+\phi_1 = -s_0 L_-^{-1}\phi_1 + c_1 Q = i\tau_0\phi_2, \quad (\phi_2 = i\tau_0 L_-^{-1}\phi_1 + \frac{c_1}{i\tau_0} Q). \tag{3.34}$$

Recall $K_s = PL_+P + sL_-^{-1}$. With these notations, we have the following corollary.

Corollary 3.7 *The following hold true.*

(1) *Assume* $s > 0$ *and for some* $\phi \in H^2(\mathbb{R}^3, \mathbb{C}) \cap Q^\perp$ *with* $\|\phi\|_2 = 1$, *it holds that*

$$PL_+P\phi + sL_-^{-1}\phi = 0.$$

Then we must have $s = s_0$ *and* $\phi = \mathrm{const} \cdot \phi_1$.

(2) $\phi_1 \in H^m(\mathbb{R}^3, \mathbb{R})$, $\forall m \in \mathbb{N}$. *Moreover* ϕ_1, $L_-^{-1}\phi_1$ *and all their derivatives have exponential decay.*

(3) $\ker(\begin{pmatrix} -i\tau_0 & L_- \\ L_+ & -i\tau_0 \end{pmatrix}) = \left\{ d_1 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} : \quad d_1 \in \mathbb{C} \right\}$.

(4) $\sigma_{\mathrm{dis}}(\widetilde{A}) \cap i\mathbb{R} = \{0,\ i\sqrt{s_0},\ -i\sqrt{s_0}\}$.

Proof (1) By Proposition 3.4, for each $s > 0$, we have $\sigma_{\mathrm{ess}}(K_s) \subset [1, \infty)$. Thus only discrete eigenvalues of K_s may appear in $(-\infty, 1)$. Lemma 3.5 implies $\mu_2(s) > 0$. Thus by the min-max principle (cf. Theorem XIII.1 of [61]), the operator K_s has exactly one eigenvalue below $\mu_2(s)$, which is given by $\mu_1(s)$ defined in Lemma 3.6. In yet other words, if $K_s\phi = 0$, then we must have $\mu_1(s) = 0$. By Lemma 3.6, we get $s = s_0$ and $\phi = \mathrm{const} \cdot \phi_1$.

(2) Lemma 3.6 implies $\phi_1 \in H^m(\mathbb{R}^3, \mathbb{R})$. To show exponential decay, we denote $h = L_-^{-1}\phi_1$ and observe

$$L_+L_-h + s_0 h = \mathrm{NICE}, \tag{3.35}$$

where NICE denotes terms which have exponential decay. Rewriting the above gives us

$$((1-\Delta)^2 + s_0)h = \mathrm{NICE}. \tag{3.36}$$

Since $s_0 > 0$, the desired exponential decay follows.

(3) The identity follows from (3.10) and the uniqueness statement in Lemma 3.6.

(4) By (3.10), the discrete spectrum of $\widetilde{A}$ on $i\mathbb{R}$ can be reduced to studying the equation ($s=\tau^2$)

$$(PL_+P + sL_-^{-1})f = 0, \qquad \text{on } Q^{\perp}. \tag{3.37}$$

By statement (1) proved earlier, we have a unique solution $s=s_0$, $f=\text{const}\cdot\phi_1$. □

Remark 3.8 (Grillakis's argument reloaded) We now sketch an alternative[19] strategy (see Theorem 2.1 of Grillakis [33]) to prove the following:

Claim: There exists a unique $s_0>0$ such that the equation $(A+s_0L_-^{-1})\phi=0$ admits a nontrivial solution in $Q^{\perp}\cap H^2(\mathbb{R}^3,\mathbb{C})$.

We sketch the proof of the above claim in the following steps. Our exposition here follows broadly Grillakis's strategy. To streamline the proof, we shall incorporate some new technical estimates that we established earlier. Note that by Proposition 3.4 we have $\sigma_{\text{ess}}(A+sL_-^{-1})\subset[1,\infty)$. The proof consists of the following steps.

(1) For $s\ge\Lambda\gg1$, one can verify for any nontrivial $g\in Q^{\perp}\cap H^2(\mathbb{R}^3,\mathbb{C})$ that

$$\langle (A+sL_-^{-1})g, g\rangle > 0 \qquad \text{(by (3.26))}. \tag{3.38}$$

(2) For $0<s=\epsilon\ll1$, we have

$$\mu_1(s) = \inf_{\substack{g\in H^1(\mathbb{R}^3,\mathbb{C})\cap Q^{\perp}\\ \|g\|_2=1}} \langle (A+sL_-^{-1})g, g\rangle < 0, \qquad \text{(by (3.24))}. \tag{3.39}$$

(3) For $s\in[\epsilon,\Lambda]$, we assume that the equation $(A+sL_-^{-1})\phi=0$ admits no nontrivial solutions in $Q^{\perp}\cap H^2(\mathbb{R}^3,\mathbb{C})$ (later in Step 4 we will arrive at a contradiction). For $s\in[\epsilon,\Lambda]$, one can find[20] sufficiently small $\delta>0$ such that the equation $(A+sL_-^{-1}+\delta)\phi=0$ has no nontrivial solutions in $Q^{\perp}\cap H^2(\mathbb{R}^3,\mathbb{C})$. Equivalently we are asserting

$$-\delta\in\rho(A+sL_-^{-1}), \quad \forall s\in[\epsilon,\Lambda].$$

(4) By examining the Riesz projection (the contour Γ is depicted in Fig. 2)

$$P_s = \frac{1}{2\pi i}\int_{\Gamma}(z-A-sL_-^{-1})^{-1}dz,$$

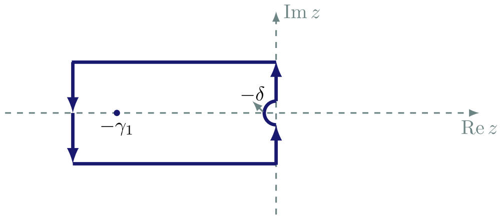
one deduces that $\dim(\text{Ran}(P_s))$ must be a constant for $s\in[\epsilon,\Lambda]$. But this is a contradiction since $\dim(\text{Ran}(P_\Lambda))=0$ whilst $\dim(\text{Ran}(P_\epsilon))\ge1$.

(5) Uniqueness of s_0 follows easily from the strict monotonicity of $\mu_1(s)$ shown in Lemma 3.6.

[19] See Proposition D.1 for yet another proof based on Riesz projection.

[20] If such δ does not exist, then we can find $\delta_j\to0^+$, $s_j\in[\epsilon,\Lambda]$, such that $(A+s_jL_-^{-1}+\delta_j)\phi_j=0$ for some nontrivial ϕ_j. This in turn yields $(A+s_\infty L_-^{-1})\phi_\infty=0$ for some $s_\infty\in[\epsilon,\Lambda]$ and some nontrivial ϕ_∞.

Fig. 2 The contour Γ



The following proposition collects several well known results in the literature (cf. Lemma 3 of [24, 33], Sect. 4 of [64], Lemmas 12.7, 12.8, Proposition 12.10 of [62], and [37]). We include a proof here for the sake of completeness. It should be noted that there is some novelty in the proof of statements (3) & (4).

Proposition 3.9 *The following hold true.*

(1) *The essential spectrum* $\sigma_{\text{ess}}(\mathcal{H}) = \sigma_{\text{ess}}(\widetilde{A}) = (-\infty, -1]\bigcup[1, \infty)$. *For some* $1 \le n_0 \le \infty$, $\sigma_{\text{dis}}(\mathcal{H}) = \bigcup_{j=1}^{n_0}\{z_j\}$, *where each* z_j *is a discrete eigenvalue with* $\ker(\mathcal{H} - z_j)$ *being nontrivial.*

(2) $\bigcup_{j=1}^{n_0}\{z_j\} \subset \mathbb{R}\bigcup i\mathbb{R}$.

(3) *The geometric and algebraic multiplicity of each* ***nonzero*** z_j *coincide, i.e., each nonzero* z_j *has no generalized eigenfunctions.*

(4) *For some* $\lambda_0 > 0$, *we have* $\sigma(\widetilde{A}) \cap i\mathbb{R} = \{0, i\lambda_0, -i\lambda_0\}$. *The eigenvalues* $i\lambda_0$ *and* $-i\lambda_0$ *are simple. The corresponding eigenfunctions of* $i\lambda_0$ *and* $-i\lambda_0$ *are* C^∞ *and exponentially decaying.*

Remark 3.10 The statement $\sigma_{\text{ess}}(\mathcal{H}) = \mathbb{R} \setminus (-1, 1)$ can also be easily proved via Weyl's essential spectrum theorem (cf. Theorem XIII. 14 of [61]). In the notation therein, regard $A = \mathcal{H}_0 = \begin{pmatrix} -\Delta + 1 & 0 \\ 0 & \Delta - 1 \end{pmatrix}$ which is clearly a self-adjoint operator, and $B = \mathcal{H}$ as a closed operator. By (3.46), we see for some $|z| \gg 1$,

$$(\mathcal{H} - z)^{-1} - \mathcal{H}_0(z)^{-1} = \text{some compact operator}, \tag{3.40}$$

and there are points in the upper and lower half plane for $\rho(B)$. Thus $\sigma_{\text{ess}}(\mathcal{H}) = \sigma_{\text{ess}}(\mathcal{H}_0) = \mathbb{R} \setminus (-1, 1)$.

Remark 3.11 An alternative way to prove statement (4) is to use the scalar operator $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ (see Lemma 12.11 of [62]). Note that the operator $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ was already used in [72]. The idea goes as follows:

$$\begin{aligned} &\begin{pmatrix} -z & L_- \\ L_+ & -z \end{pmatrix}\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = 0 \quad \Rightarrow \quad L_+ L_- b_0 = z^2 b_0 \\ &\Rightarrow \quad L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} (L_-^{\frac{1}{2}} b_0) = z^2 L_-^{\frac{1}{2}} b_0; \\ &\begin{pmatrix} -z & L_- \\ L_+ & -z \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \quad \Rightarrow \quad L_+ L_- b_1 = z^2 b_1 + 2z b_0 \end{aligned} \tag{3.41}$$

$$\Rightarrow \quad L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} (L_-^{\frac{1}{2}} b_1) = z^2 L_-^{\frac{1}{2}} b_1 + 2z L_-^{\frac{1}{2}} b_0. \tag{3.42}$$

Thus (assuming $\ker(\widetilde{A} - z)$ is nontrivial and $z \neq 0$), if $\widetilde{A}$ has a generalized eigenspace at z, then $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ has a generalized eigenspace at z^2. This in turn contradicts the self-adjointness of the scalar operator $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$. To show $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ with domain $H^4(\mathbb{R}^3, \mathbb{C})$ is self-adjoint, one can directly check that the domain of the adjoint of $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$ is $H^4(\mathbb{R}^3, \mathbb{C})$ by using the Fredholm-ness of $L_-^{\frac{1}{2}}$ and L_+. This is done in [62]. Alternatively (see footnote [18]) one can write

$$L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} = L_-^2 - 2L_-^{\frac{1}{2}} Q^2 L_-^{\frac{1}{2}},$$

and check for $|\lambda| \gg 1$, the equation

$$(L_-^2 - 2L_-^{\frac{1}{2}} Q^2 L_-^{\frac{1}{2}} + \lambda i) u = f$$

admits a unique solution $\forall f \in L^2(\mathbb{R}^3, \mathbb{C})$. The solvability follows from

$$\| L_-^{\frac{1}{2}} Q^2 L_-^{\frac{1}{2}} (L_-^2 + \lambda i)^{-1} \|_{L^2 \to L^2} \ll 1.$$

The latter can be verified via the identity

$$\begin{aligned} &\langle L_-^{\frac{1}{2}} Q^2 L_-^{\frac{1}{2}} (L_-^2 + \lambda i)^{-1} u,\ L_-^{\frac{1}{2}} Q^2 L_-^{\frac{1}{2}} (L_-^2 + \lambda i)^{-1} u \rangle \\ =&\langle L_- Q^2 L_-^{\frac{1}{2}} (L_-^2 + \lambda i)^{-1} u, Q^2 L_-^{\frac{1}{2}} (L_-^2 + \lambda i)^{-1} u \rangle. \end{aligned} \tag{3.43}$$

Proof of Proposition 3.9 (1) We first show $\sigma_{\mathrm{ess}}(\mathcal{H}) \supset F := (-\infty, -1] \bigcup [1, \infty)$. Note that, by using the definition of essential spectrum, it suffices to show that $F \subset \sigma(\mathcal{H})$ (since each $\lambda \in F$ will not be isolated). Consider $\lambda_0 \geq 1$ and define $k_0 = \sqrt{\lambda_0 - 1}$. Let $\phi \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ be such that $\phi(z) \equiv 1$ for $|z| \leq 1$ and $\phi(z) \equiv 0$ for $|z| \geq 2$. For $n \geq 1$, we define (below $r = |x|$)

$$g_n(x) = \begin{cases} \frac{\sin(k_0 r)}{r} \frac{1}{\sqrt{n}} \phi(\frac{r - 2^n}{n}), & \text{if } k_0 > 0; \\ \frac{1}{\sqrt{n} r} \phi(\frac{r - 2^n}{n}), & \text{if } k_0 = 0. \end{cases} \tag{3.44}$$

Clearly for $n \gg 1$, $\|g_n\|_{L^2(\mathbb{R}^3)} \sim 1$ and $\|(\mathcal{H} - \lambda_0 I) \begin{pmatrix} g_n \\ 0 \end{pmatrix} \|_2 \to 0$ as $n \to \infty$. Thus $\lambda_0 \in \sigma(\mathcal{H})$. Similarly one can show $-\lambda_0 \in \sigma(\mathcal{H})$. Thus $\sigma(\mathcal{H}) \supset (-\infty, -1] \bigcup [1, \infty)$. It follows that $\sigma_{\mathrm{ess}}(\mathcal{H}) \supset F$.

Next for $z \in \mathbb{C}$, $z \notin F$, denote

$$\mathcal{H}_0(z) = \begin{pmatrix} -\Delta + 1 - z & 0 \\ 0 & \Delta - 1 - z \end{pmatrix}, \qquad V = Q^2 \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}. \tag{3.45}$$

Note that for $z \notin F$, $\mathcal{H}_0(z)^{-1}$ exists and the operator $V \mathcal{H}_0(z)^{-1}$ is compact. Write

$$\mathcal{H} - z = (1 + V \mathcal{H}_0(z)^{-1}) \mathcal{H}_0(z), \qquad z \notin F. \tag{3.46}$$

Analytic Fredholm (see Theorem VI. 14 of [61]) implies there are at most countably many isolated $z_j \in \mathbb{C} \setminus F$, $1 \le j \le n_0 \le \infty$ (with no accumulation points), such that each $\ker(1 + V\mathcal{H}_0(z_j)^{-1})$ is nontrivial, and the algebraic multiplicity of $1 + V\mathcal{H}_0(z_j)^{-1}$ is finite. The operator $1 + V\mathcal{H}_0(z)^{-1}$ is invertible for all $z \notin F \bigcup \left(\bigcup_{j=1}^{n_0} \{z_j\} \right)$. Note that the above argument also showed that there is no essential spectrum of $\mathcal{H}$ outside F, i.e., $\sigma_{\text{ess}}(\mathcal{H}) = F$.

(2) If $z_j \neq 0$, then for some $(f_1, f_2)^\top$ with $\|f_1\|_2^2 + \|f_2\|_2^2 = 1$ we have

$$L_- f_2 = z_j f_1, \qquad L_+ f_1 = z_j f_2.$$

This implies $f_1 \perp Q$ and $PL_+Pf_1 - z_j^2 L_-^{-1} f_1 = 0$. It follows that

$$z_j^2 \langle L_-^{-1} f_1, f_1 \rangle = \langle PL_+Pf_1, f_1 \rangle.$$

In particular z_j^2 must be real-valued. Thus $z_j \in \mathbb{R} \bigcup i\mathbb{R}$.

(3) Fix $z_j \neq 0$. It suffices for us to show that there are no solutions to the system

$$\begin{cases} L_-\phi_2 - z_j\phi_1 = f_1 \\ L_+\phi_1 - z_j\phi_2 = f_2, \end{cases}$$

where $(f_1, f_2)^\top$ is an eigenfunction, i.e.,

$$\begin{cases} L_- f_2 = z_j f_1 \\ L_+ f_1 = z_j f_2. \end{cases} \tag{3.47}$$

Assume $(\phi_1, \phi_2)^\top$ exists and we shall derive a contradiction. Observe that

$$\begin{aligned} &\langle L_-\phi_2 - z_j\phi_1, \overline{f_2} \rangle + \langle L_+\phi_1 - z_j\phi_2, \overline{f_1} \rangle \\ =&\langle z_j\phi_2, \overline{f_1} \rangle - \langle z_j\phi_1, \overline{f_2} \rangle + \langle z_j\phi_1, \overline{f_2} \rangle - \langle z_j\phi_2, \overline{f_1} \rangle = 0. \end{aligned} \tag{3.48}$$

This implies $\int_{\mathbb{R}^3} f_1 f_2 dx = 0$. By using $L_- f_2 = z_j f_1$ and $f_1 \perp Q$, we obtain

$$\int_{\mathbb{R}^3} f_1 L_-^{-1} f_1 dx = 0. \tag{3.49}$$

Subcase 1: $0 \neq z_j \in \mathbb{R}$. By taking real or imaginary part of (3.47), we may assume that f_1 and f_2 are both real-valued and nontrivial. Then we obtain $\|L_-^{-\frac{1}{2}} f_1\|_2^2 = 0$ which is the desired contradiction.

Subcase 2: $0 \neq z_j \in i\mathbb{R}$, i.e., z_j is purely imaginary. By taking real or imaginary part of (3.47), we may assume that f_1 is real-valued, if_2 is real-valued and both f_1 and f_2 are not identically zero. Then (3.49) implies $\|L_-^{-\frac{1}{2}} f_1\|_2^2 = 0$ which is the desired contradiction.

(4) Since $\sigma_{\text{ess}}(\widetilde{A}) = (-\infty, -1] \bigcup [1, \infty)$, only discrete eigenvalues may appear in $i\mathbb{R}$. By Corollary 3.7, we have $\sigma(\widetilde{A}) \cap i\mathbb{R} = \sigma_{\text{dis}}(\widetilde{A}) \cap i\mathbb{R} = \{0, i\lambda_0, -i\lambda_0\}$, where $\lambda_0 = \sqrt{s_0}$. The exponential decay of the corresponding eigenfunction also follows from Corollary 3.7. □

The next proposition asserts that $\widetilde{A}$ (equivalently $\mathcal{H}$) does not have any nonzero eigenvalues in the interval $[-1, 1]$. This fact was first proved in Lemma 15 of [64]. The argument therein is based on an adaptation of Proposition 2.1.2 in [58] (it relies on a min-max approach using the fact that L_+ has no eigenvalues in $(0, 1)$). We give a streamlined proof here without appealing to the min-max formulation. See also Remark 3.16 for a further simplification.

Proposition 3.12 *The operator* $\widetilde{A}$ (*and the operator* $\mathcal{H}$) *does not have any eigenvalue in* $[-1, 0)\bigcup(0, 1]$.

Proof Recall $A = PL_+P$ and we denote for $0 < s \le 1$,

$$K_s f = Af - sL_-^{-1} f, \quad f \in Q^\perp.$$

It suffices for us to show that K_s has no nontrivial kernel for $s \in (0, 1]$. To this end we argue by contradiction. Assume for some $s \in (0, 1]$, and $f \in H^2(\mathbb{R}^3, \mathbb{R}) \cap Q^\perp$ with $\|f\|_2 = 1$ that $Af = sL_-^{-1} f$.

Recall (see Lemma 3.2) $Ae_0 = -\gamma_1 e_0$ where $e_0 \in H^2(\mathbb{R}^3, \mathbb{R}) \cap Q^\perp$ is radial and $-\gamma_1 < 0$ is the unique negative eigenvalue of A, and $Ae_j = 0$ for $1 \le j \le 3$, and

$$\langle A\widetilde{f}, \widetilde{f}\rangle \ge \|\widetilde{f}\|_2^2, \quad \text{if } \widetilde{f} \in Q^\perp \cap H^2(\mathbb{R}^3, \mathbb{C}) \text{ and } \widetilde{f} \perp \operatorname{span}\{e_0, e_1, e_2, e_3\}. \tag{3.50}$$

Here we assume $\|e_j\|_2 = 1$ for $0 \le j \le 3$. We shall write $f = \sum_{j=0}^3 c_j e_j + f^\perp$ with $c_j := \langle e_j, f\rangle$. Clearly

$$\begin{aligned} \langle Af^\perp, f^\perp\rangle &= \langle A(f - \textstyle\sum_{j=0}^3 c_j e_j), f - \sum_{j=0}^3 c_j e_j\rangle = \langle A(f - c_0 e_0), f - c_0 e_0\rangle \\ &= \langle sL_-^{-1} f, f\rangle - 2\mathrm{Re}(\langle sL_-^{-1} f, c_0 e_0\rangle) - \gamma_1 \|c_0 e_0\|_2^2; \end{aligned} \tag{3.51}$$

$$\begin{aligned} \langle sL_-^{-1} f^\perp, f^\perp\rangle &= \langle sL_-^{-1}(f - \textstyle\sum_{j=0}^3 c_j e_j), f - \sum_{j=0}^3 c_j e_j\rangle \\ &= \langle sL_-^{-1} f, f\rangle - 2\mathrm{Re}(\langle sL_-^{-1} f, c_0 e_0\rangle) \\ &\quad + \langle sL_-^{-1}(\textstyle\sum_{j=0}^3 c_j e_j), \sum_{j=0}^3 c_j e_j\rangle. \end{aligned} \tag{3.52}$$

In the last equality, we used the identity $\langle sL_-^{-1} f, e_j\rangle = \langle Af, e_j\rangle$ to simplify the cross terms.

It follows from (3.50), (3.51) and (3.52) that

$$\begin{aligned} &\|f^\perp\|_2^2 \le \langle Af^\perp, f^\perp\rangle \le \langle sL_-^{-1} f^\perp, f^\perp\rangle \le \|f^\perp\|_2^2 \\ &\Rightarrow \langle Af^\perp, f^\perp\rangle = \langle sL_-^{-1} f^\perp, f^\perp\rangle = \langle f^\perp, f^\perp\rangle. \end{aligned}$$

It follows that $c_j = 0$ for all $0 \le j \le 3$. Since $\|L_-^{-1} h\|_2 < \|h\|_2$ for $h \perp Q$ and h not being identically zero (see Remark 3.13), we conclude $f^\perp \equiv 0$. □

Remark 3.13 The fact that $\|L_-^{-1} h\|_2 < \|h\|_2$ for $h \perp Q$ and h not being identically zero can be justified as follows. Firstly by using the inequality $\|h\|_2^2 \ge \langle L_-^{-1} h, h\rangle \ge$

$\|L_-^{-1}h\|_2^2$ for $h \perp Q$, we see that for any $\widetilde{h} \perp Q$:

$$\|L_-^{-1}\widetilde{h}\|_2 = \|\widetilde{h}\|_2 \Rightarrow \langle L_-^{-1}\widetilde{h}, \widetilde{h}\rangle = \|\widetilde{h}\|_2^2 = \|L_-^{-1}\widetilde{h}\|_2^2$$
$$\Rightarrow \|(1 - L_-^{-1})\widetilde{h}\|_2^2 = \|\widetilde{h}\|_2^2 + \|L_-^{-1}\widetilde{h}\|_2^2 - 2\langle L_-^{-1}\widetilde{h}, \widetilde{h}\rangle = 0. \tag{3.53}$$

Thus $\widetilde{h} = L_-^{-1}\widetilde{h}$. Since L_- has no eigenfunctions or resonances at $\lambda = 1$, this would imply $\widetilde{h} \equiv 0$.

The next proposition asserts that $\widetilde{A}$ (and the operator $\mathcal{H}$) does not admit any resonances at the edge $\lambda = \pm 1$. This fact was first proved in Lemma 16 of [64]. The argument therein is an adaptation of an argument in Appendix 1 of [58], using the fact that the scalar operator L_- does not have a resonance at $\lambda = 1$. Our proof here appears to be slightly simpler than [64]. As it turns out, Proposition 3.12 and Proposition 3.14 can be proved at one stroke. See Remark 3.16.

Proposition 3.14 *The operator $\widetilde{A}$ (and the operator $\mathcal{H}$) does not have any resonances at $\lambda = 1$ or $\lambda = -1$.*

Proof Consider $\lambda = 1$. The case $\lambda = -1$ is analogous (observe that $L_+\widetilde{a} = -\widetilde{b}$, $L_-\widetilde{b} = -\widetilde{a} \Rightarrow L_+a = b$, $L_-b = a$ with $a = \widetilde{a}$, $b = -\widetilde{b}$). We argue by contradiction. Suppose there exists a resonance of $\mathcal{H}$ at $\lambda = 1$, i.e., for some $a, b : \mathbb{R}^3 \to \mathbb{C}$ satisfying $(a, b)^\top \notin L^2(\mathbb{R}^3, \mathbb{C}^2)$ and $\int_{\mathbb{R}^3}(|a|^2 + |b|^2) \cdot (1 + |x|)^{-\gamma} dx < \infty$ for all $\gamma > 1$, it holds that

$$L_+a = b, \qquad L_-b = a. \tag{3.54}$$

By taking real or imaginary parts, we can further assume the pair is real-valued. Also it is easy to see $a \notin L^2$ (since if $a \in L^2$, then (3.54) implies $b \in L^2$, contradicting the assumption that $(a, b)^\top \notin L^2$).

Let $\phi \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$ be a radial bump function such that $\phi(x) = 1$ for $|x| \le 1$ and $\phi(x) = 0$ for $|x| \ge 2$. For N which later will be taken sufficiently large, we define $\phi_N = \phi(x/N)$ and

$$b_N(x) := P(b(x)\phi_N) = b\phi_N - \frac{\langle Q, b\phi_N\rangle}{\|Q\|_2^2} Q, \qquad a_N = L_- b_N. \tag{3.55}$$

Write

$$a_N = \sum_{j=0}^{3} \underbrace{\langle e_j, a_N\rangle}_{=:c_j^N} e_j + a_N^\perp, \qquad a = \sum_{j=0}^{3} \underbrace{\langle e_j, a\rangle}_{=:c_j} e_j + a^\perp.$$

Here we extend the definition of $\langle e_j, a\rangle$ (recall $a \notin L^2$!) as $\langle e_j, a\rangle := \int_{\mathbb{R}^3} \overline{e_j} a dx$ which is well-defined thanks to the exponential decay of e_j. The same convention will be used later without explicit mention. We evaluate

$$\langle Aa_N^\perp, a_N^\perp\rangle = \langle A(a_N - \textstyle\sum_{j=0}^3 c_j^N e_j), a_N - \sum_{j=0}^3 c_j^N e_j\rangle$$

$$
\begin{aligned}
&= \langle A(a_N - c_0^N e_0), a_N - c_0^N e_0\rangle \\
&= \langle Aa_N, a_N\rangle + 2\gamma_1 \mathrm{Re}(\langle a_N, c_0^N e_0\rangle) - \gamma_1 |c_0^N|^2 \\
&= \langle Aa_N, a_N\rangle + \gamma_1 |c_0^N|^2. \qquad (3.56)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\langle a_N^\perp, a_N^\perp\rangle &= \langle a_N - \textstyle\sum_{j=0}^3 c_j^N e_j, a_N - \sum_{j=0}^3 c_j^N e_j\rangle \\
&= \langle a_N, a_N\rangle - \sum_{j=0}^3 |c_j^N|^2.
\end{aligned}
$$

By taking $N \to \infty$ and using $\langle Aa_N^\perp, a_N^\perp\rangle \ge \langle a_N^\perp, a_N^\perp\rangle$, we obtain (see Remark 3.15)

$$
\langle (A-1)a, a\rangle \ge -\gamma_1 |c_0|^2 - \textstyle\sum_{j=0}^3 |c_j|^2. \qquad (3.57)
$$

Write $\langle (A-1)a, a\rangle = -\|(A-1)a\|_2^2 + \langle (A-1)a, Aa\rangle$. Recall $L_- Aa = a$ (see (3.54)). Clearly[21]

$$
\langle (A-1)a, Aa\rangle = \lim_{N\to\infty} \Big(\langle Aa_N, Aa_N\rangle - \langle L_-(Aa_N), Aa_N\rangle \Big) \le 0. \qquad (3.58)
$$

It follows that $\|(A-1)a\|_2^2 - \gamma_1 |c_0|^2 - \sum_{j=0}^3 |c_j|^2 \le 0$. On the other hand, we have

$$
\|(A-1)a\|_2^2 \ge (\gamma_1 + 1)^2 |c_0|^2 + \textstyle\sum_{j=1}^3 |c_j|^2 + \|(A-1)a^\perp\|_2^2. \qquad (3.59)
$$

Thus we obtain $c_0 = 0$ and $Aa^\perp = a^\perp$. Now note that

$$
\begin{aligned}
&\langle L_-^{-1} a_N^\perp, a_N^\perp\rangle \\
=&\langle L_-^{-1}(a_N - \textstyle\sum_{j=0}^3 c_j^N e_j),\ a_N - \sum_{j=0}^3 c_j^N e_j\rangle \\
=&\langle (L_-^{-1} - 1)a_N, a_N\rangle + \textstyle\sum_{j=0}^3 |c_j^N|^2 + \langle a_N^\perp, a_N^\perp\rangle \\
&+ \langle L_-^{-1}(\sum_{j=0}^3 c_j^N e_j), \sum_{j=0}^3 c_j^N e_j\rangle + \text{Cross terms}. \qquad (3.60)
\end{aligned}
$$

Using the fact that $L_-^{-1} a = Aa$, $c_0 = 0$ and $Ae_j = 0$ for $1 \le j \le 3$, it is not difficult to check that the "Cross terms" in the above goes to zero as N tends to infinity. On the other hand, using

$$
(L_-^{-1} - 1)a = Aa - a = Aa^\perp - a = a^\perp - a = -\textstyle\sum_{j=1}^3 c_j e_j,
$$

we get $\langle (L_-^{-1} - 1)a, a\rangle = -\sum_{j=1}^3 |c_j|^2$.

[21] It's tempting to invoke the formal identity $\langle (A-1)a, Aa\rangle = \underbrace{\langle Aa, Aa\rangle}_{\text{formal expression!}} - \langle L_- Aa, Aa\rangle$; but the issue is that $Aa \notin L^2$.

By using $\langle a_N^\perp, a_N^\perp\rangle \geq \langle L_-^{-1} a_N^\perp, a_N^\perp\rangle$ and sending $N\to\infty$, we deduce from (3.60) that

$$\langle L_-^{-1}(\textstyle\sum_{j=1}^3 c_j e_j),\ \sum_{j=1}^3 c_j e_j\rangle \leq 0. \tag{3.61}$$

These imply $c_j = 0$, $\forall\, 1\leq j\leq 3$ and $L_-^{-1}a = a$, contradicting L_- has no resonances/eigenvalues at $\lambda = 1$. □

Remark 3.15 We now sketch the proof for the needed regularities of $(a,b)^\top$ in (3.54). Clearly

$$\begin{cases}(1-\Delta-3Q^2)a = b\\ (1-\Delta-Q^2)b = a\end{cases} \Rightarrow \begin{cases}(2-\Delta)(a-b) = 3Q^2a - Q^2b\\ -\Delta(a+b) = 3Q^2a + Q^2b.\end{cases}$$

It follows that $\partial^m(a-b)$ has exponential decay for any integer $m\geq 0$, $\Delta(a+b)$ has exponential decay, and $\partial^m(a+b)\in L^2$ for any integer $m\geq 1$. In particular $(A-1)a = P(L_+-1)Pa$ has exponential decay and all the computations carried out in (3.57)–(3.59) are rigorous.

Remark 3.16 The proof of Proposition 3.14 is more general. In particular, it yields another proof of Proposition 3.12. For simplicity we discuss only the eigenvalue case. The gist of the whole proof can be summarized as follows. Firstly, denote $a^\perp = a - \sum_{j=0}^3 \langle e_j, a\rangle e_j$ and observe (below $c_j = \langle e_j, a\rangle$)

$$\begin{aligned}-\|(A-1)a\|_2^2 + \langle (A-1)a, Aa\rangle &= \langle (A-1)a, a\rangle\\ &= \langle (A-1)a^\perp, a^\perp\rangle - (\gamma_1+1)|c_0|^2 - \sum_{j=1}^3 |c_j|^2.\end{aligned}$$

Then we obtain the fundamental identity

$$\langle (A-1)a, Aa\rangle = \langle (A-1)a^\perp, a^\perp\rangle + \|(A-1)a^\perp\|_2^2 + (\gamma_1+1)\gamma_1|c_0|^2. \tag{3.62}$$

If $Aa = sL_-^{-1}a$, $s\in(0,1]$, then

$$\langle (A-1)a, Aa\rangle \leq 0.$$

Since $\langle (A-1)a^\perp, a^\perp\rangle \geq 0$, we get $c_0 = 0$, $Aa^\perp = a^\perp$ and

$$\begin{aligned}0 = \langle (A-1)a, Aa\rangle &= \langle (sL_-^{-1}-1)a, L_-^{-1}a\rangle\\ &\leq \langle (L_-^{-1}-1)a, L_-^{-1}a\rangle.\end{aligned} \tag{3.63}$$

Thus $\langle (L_-^{-1}-1)a, L_-^{-1}a\rangle = 0$. Let $\widetilde{a} = L_-^{-1}a$. Clearly $\langle \widetilde{a},\widetilde{a}\rangle = \langle L_-\widetilde{a},\widetilde{a}\rangle = \|L_-^{\frac12}\widetilde{a}\|_2^2$ $(\Rightarrow \|L_-^{\frac14}\widetilde{a}\|_2 = \|\widetilde{a}\|_2)$ and

$$\|(L_-^{\frac12}-1)\widetilde{a}\|_2^2 = 2\|\widetilde{a}\|_2^2 - 2\langle L_-^{\frac12}\widetilde{a},\widetilde{a}\rangle = 0.$$

This clearly implies $L_-\widetilde{a} = \widetilde{a}$. Since L_- has no eigenvalue at 1, we conclude $\widetilde{a}\equiv 0$ and consequently $a\equiv 0$.

We now complete the proof of Corollary 1.3. Here we assume Theorem 1.1 holds.

Proof of Corollary 1.3 Propositions 3.9–3.12 imply

$$\sigma_{\mathrm{ess}}(\widetilde{A}) = (-\infty, -1]\bigcup[1, \infty), \qquad \sigma_{\mathrm{dis}}(\widetilde{A}) = \{0,\ i\lambda_0,\ -i\lambda_0\}.$$

Both $i\lambda_0$ and $-i\lambda_0$ are simple eigenvalues with C^∞ exponentially decaying eigenfunctions. Proposition 3.14 excludes any resonances at $\lambda = \pm 1$. Theorem 1.1 asserts the absence of embedded eigenvalues in $(-\infty, -1]\bigcup[1, \infty)$. Thus statement (3) holds. Regarding the root space of $\widetilde{A}$ at zero, it is easy to check the spaces $\ker(\widetilde{A})$ and $\ker(\widetilde{A}^2)$. The system

$$\begin{cases} L_- b = Q_1 \\ L_+ a = 0 \end{cases}$$

clearly has no H^2-solutions since $\langle Q, Q_1\rangle \neq 0$. By a similar reasoning, the system

$$\begin{cases} L_- b = 0 \\ L_+ a = x_j Q \end{cases}$$

has no H^2-solutions. Thus $\ker(\widetilde{A}^n) = \ker(\widetilde{A}^2)$ for any $n \geq 3$. □

4 Preliminary analysis for the case $\ell = 0$, $\tau \geq 1$

In this section, we gather a few preliminary results for the case $\ell = 0$, $\tau \geq 1$. Consider

$$\begin{cases} U'' = (1 - \tau - 2Q^2)U + Q^2 V \\ V'' = Q^2 U + (1 + \tau - 2Q^2)V \\ (U, V, U', V')\big|_{t=0} = (0, 0, \cos\theta, \sin\theta), \end{cases} \tag{4.1}$$

where $\theta \in [0, 2\pi)$ is fixed. We first derive a fundamental energy estimate. Note that

$$\begin{aligned} \big((U')^2\big)' &= (1 - \tau - 2Q^2)(U^2)' + 2Q^2 V U'; \\ \big((V')^2\big)' &= 2Q^2 U V' + (1 + \tau - 2Q^2)(V^2)'. \end{aligned}$$

Summing these identities gives

$$\begin{aligned} &\big((U')^2 + (V')^2\big)' = 2Q^2(UV)' + (1 - 2Q^2)(U^2 + V^2)' + \tau(V^2 - U^2)' \\ &\quad\Longrightarrow \quad E_0' = 2(Q^2)'\,(U^2 + V^2 - UV), \end{aligned}$$

with

$$E_0 := (U')^2 + (V')^2 + (\tau - 1)U^2 - (\tau + 1)V^2 + 2Q^2(U^2 + V^2 - UV). \tag{4.2}$$

Since $(Q^2)' < 0$ for $t > 0$, E_0 is monotonically decreasing in time. The structure of E_0, specifically the negative term $-(\tau+1)V^2$ which prevents full coercivity, is itself a reflection of the hyperbolic character of the system. As we now show, this energy identity allows us to relate the L^∞-norm of V to the onset of exponential instability in the solution. The following proposition quantifies this idea.

Proposition 4.1 *Let $\tau \geq 1$ and (U, V) be a smooth solution to (4.1) on $[0,\infty)$. If $U \in L^2([0,\infty), dt)$ and*

$$\sup_{t>T_0} |V(t)| < \infty, \qquad \text{for some } T_0 > 0, \tag{4.3}$$

then in fact the solution remains uniformly bounded:

$$\sup_{t\geq 0}\big(\sqrt{\tau-1}|U(t)| + |U'(t)| + |V(t)| + |V'(t)|\big) < \infty. \tag{4.4}$$

Furthermore, for some constant $C_\tau > 0$ depending only on τ, we have the exponential decay estimate

$$|U(t)| + |U'(t)| + |V(t)| + |V'(t)| \leq C_\tau e^{-\sqrt{\tau+1}\,t}, \quad t \geq 1.$$

Proof By using (4.2) and the fact that E_0 is uniformly bounded, it is clear that (4.4) follows from (4.3). To obtain exponential decay, we consider two separate cases.

Case 1). $\tau = 1$. We first consider the V-equation and denote $F_1 := Q^2(U - 2V)$. Note $Q^2(t) \lesssim \frac{e^{-2t}}{t^2}$, $t > 1$ and $V(0) = 0$. We have (below note $\frac{C_1}{2} + \frac{1}{2\sqrt{2}}\int_0^\infty e^{-\sqrt{2}s}F_1(s)ds = 0$ since otherwise $V \notin L^\infty$)

$$\begin{aligned}
V(t) &= C_1 \sinh\sqrt{2}t + \int_0^t \frac{\sinh\sqrt{2}(t-s)}{\sqrt{2}} F_1(s)ds \\
&= e^{\sqrt{2}t}\Big(\frac{C_1}{2} + \frac{1}{2\sqrt{2}}\int_0^t e^{-\sqrt{2}s}F_1(s)ds\Big) + e^{-\sqrt{2}t}\Big(-\frac{C_1}{2} - \frac{1}{2\sqrt{2}}\int_0^t e^{\sqrt{2}s}F_1(s)ds\Big) \\
&= e^{\sqrt{2}t}\Big(-\frac{1}{2\sqrt{2}}\int_t^\infty e^{-\sqrt{2}s}F_1(s)ds\Big) + e^{-\sqrt{2}t}\Big(-\frac{C_1}{2} - \frac{1}{2\sqrt{2}}\int_0^t e^{\sqrt{2}s}F_1(s)ds\Big). \quad (4.5)
\end{aligned}$$

Thus $|V(t)| + |V'(t)| \lesssim e^{-\sqrt{2}t}$ for all $t \geq 1$. For the U-equation, we re-write it as

$$U'' = Q^2 \cdot (-2U + V) =: F_2.$$

Clearly $U'(t) = -\int_t^\infty F_2(s)ds$ which implies $|U'(t)| \lesssim e^{-\sqrt{2}t}$ for $t \geq 1$. Since $U \in L^2$, the exponential decay of U follows easily.

Case 2). $\tau > 1$. This is similar to the $\tau = 1$ case and we sketch the minor changes. We write

$$\begin{aligned}
&V'' - k^2 V = Q^2(U - 2V) =: F_1 \qquad (k := \sqrt{\tau+1}) \\
&\Rightarrow V(t) = C_1 \sinh kt + \int_0^t \frac{\sinh k(t-s)}{k} F_1(s)ds
\end{aligned}$$

$$= e^{kt}\left(\frac{C_1}{2} + \frac{1}{2k}\int_0^\infty e^{-ks}F_1(s)ds\right) - e^{kt}\frac{1}{2k}\left(\int_t^\infty e^{-ks}F_1(s)ds\right)$$
$$+ e^{-kt}\cdot\left(-\frac{C_1}{2} - \frac{1}{2k}\int_0^t e^{ks}F_1(s)ds\right).$$

For the U-equation, we rewrite it as $U'' + k_1^2 U = F_2 := (-2U + V)Q^2$ ($k_1 = \sqrt{\tau - 1}$). Thus

$$U(t) = \int_t^\infty \frac{\sin k_1(s-t)}{k_1} F_2(s)ds.$$

We now view $(U, V)^\top$ as solutions solving

$$\begin{pmatrix} U \\ V \end{pmatrix}(t) = \Theta\left(\begin{pmatrix} U \\ V \end{pmatrix}\right)(t)$$
$$:= \begin{pmatrix} \int_t^\infty \frac{\sin k_1(s-t)}{k_1}(-2U + V)Q^2 ds \\ -\frac{1}{2k}\int_t^\infty e^{-k(s-t)}(U - 2V)Q^2 ds - \frac{1}{2k}\int_0^t e^{k(s-t)}(U - 2V)Q^2 ds \end{pmatrix}$$
$$- \frac{C_1}{2}e^{-kt}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Firstly, it is easy to obtain $|U(t)| + |V(t)| \lesssim e^{-c_1 t}$, $\forall t > 1$ for some small exponent $c_1 > 0$. Clearly $\Theta\left(\begin{pmatrix} U \\ V \end{pmatrix}\right)$ produces an enhanced decay. A bootstrap argument then yields the desired decay. □

The next lemma provides a useful quantitative growth estimate for ODE solutions.

Lemma 4.2 *Suppose $m = m(t) : [0, \infty) \to \mathbb{R}$ is a C^1 smooth function such that $m(t) \geq 0$ and $m'(t) \leq 0$ for all $t \in [0, \infty)$. Let $\tau \geq 1$ and consider (below $a_1, a_2 \in \mathbb{R}$, and $F_i : [0, \infty) \to \mathbb{R}$ are continuous)*

$$\begin{cases} \widetilde{f}'' = (1 - \tau - 2m)\widetilde{f} + m\widetilde{g} + F_1, \\ \widetilde{g}'' = (1 + \tau - 2m)\widetilde{g} + m\widetilde{f} + F_2; \\ (\widetilde{f}, \widetilde{g}, \widetilde{f}', \widetilde{g}')\big|_{t=0} = (0, 0, a_1, a_2). \end{cases} \tag{4.6}$$

Denote $F(s) = \sqrt{F_1(s)^2 + F_2(s)^2}$, $a = \sqrt{a_1^2 + a_2^2}$ and $k = \sqrt{\tau + 1}$. For $t > 0$ it holds that

$$(\widetilde{f}'(t)^2 + \widetilde{g}'(t)^2 + (\tau - 1)\widetilde{f}^2 + 2m(t)\cdot(\widetilde{f}^2 + \widetilde{g}^2 - \widetilde{f}\widetilde{g}))^{\frac{1}{2}}$$
$$\leq a\cosh(kt) + \int_0^t \cosh(k(t-s))F(s)ds;$$
$$(\widetilde{f}(t)^2 + \widetilde{g}(t)^2)^{\frac{1}{2}} \leq a\frac{\sinh(kt)}{k} + \int_0^t \frac{\sinh(k(t-s))}{k}F(s)ds. \tag{4.7}$$

Proof Define $E_1 = (\widetilde{f}')^2 + (\widetilde{g}')^2 + (\tau-1)\widetilde{f}^2 + 2m\cdot(\widetilde{f}^2+\widetilde{g}^2-\widetilde{f}\widetilde{g})$. We have (note below that $\sqrt{E_1}\ge|\widetilde{g}'|$)

$$\begin{aligned}\partial_t E_1 &= (\tau+1)2\widetilde{g}\widetilde{g}' + 2m'(\widetilde{f}^2+\widetilde{g}^2-\widetilde{f}\widetilde{g}) + 2F_1\widetilde{f}' + 2F_2\widetilde{g}' \\ &\le (\tau+1)2\widetilde{g}\widetilde{g}' + 2F\cdot\sqrt{E_1} \\ &\le (\tau+1)2\Big(\int_0^t \sqrt{E_1}(s)ds\Big)\cdot\sqrt{E_1} + 2F\cdot\sqrt{E_1}. \end{aligned} \tag{4.8}$$

Thus $\partial_t\sqrt{E_1} \le (\tau+1)\int_0^t\sqrt{E_1}(s)ds + F$ which yields the bound on $\sqrt{E_1}$ and $\int_0^t\sqrt{E_1}ds$. The bound for $(\widetilde{f}^2+\widetilde{g}^2)$ follows from the identity $\partial_t(\widetilde{f}^2+\widetilde{g}^2) = 2\widetilde{f}\widetilde{f}' + 2\widetilde{g}\widetilde{g}'$. □

In the following, Q_*^2 is a polynomial approximation of Q^2 to be specified later. Consider for $I=a$ or b:

$$\begin{cases} U_I'' = (1-\tau-2Q^2)U_I + Q^2V_I, \\ V_I'' = (1+\tau-2Q^2)V_I + Q^2U_I; \\ (U_I,V_I,U_I',V_I')\big|_{t=0} = \begin{cases}(0,0,1,0), & \text{if } I=a;\\ (0,0,0,1), & \text{if } I=b.\end{cases}\end{cases}$$

$$\begin{cases} \widetilde{U}_I'' = (1-\tau-2Q_*^2)\widetilde{U}_I + Q_*^2\widetilde{V}_I + \widetilde{F}_I, \\ \widetilde{V}_I'' = (1+\tau-2Q_*^2)\widetilde{V}_I + Q_*^2\widetilde{U}_I + \widetilde{G}_I; \\ (\widetilde{U}_I,\widetilde{V}_I,\widetilde{U}_I',\widetilde{V}_I')\big|_{t=0} = \begin{cases}(0,0,1,0), & \text{if } I=a;\\ (0,0,0,1), & \text{if } I=b.\end{cases}\end{cases} \tag{4.9}$$

Here $\widetilde{U}_I(\tau,t)$, $\widetilde{V}_I(\tau,t)$ ($I=a$ or b) are explicit polynomial functions in τ and t whose explicit form will be specified later. The terms $\widetilde{F}_I(\tau,t)$, $\widetilde{G}_I(\tau,t)$ collect the corresponding residual error, namely

$$\begin{cases}\widetilde{F}_I := \partial_{tt}\widetilde{U}_I - (1-\tau-2Q_*^2)\widetilde{U}_I - Q_*^2\widetilde{V}_I \\ \widetilde{G}_I := \partial_{tt}\widetilde{V}_I - (1+\tau-2Q_*^2)\widetilde{V}_I - Q_*^2\widetilde{U}_I.\end{cases} \tag{4.10}$$

Note that $\widetilde{F}_I$ and $\widetilde{G}_I$ are (by definition) polynomials in τ and t. As we shall show later, this allows us to derive explicit, quantifiable upper bounds. Specifically, upon integrating out the variable t in the L^2 integrals, we obtain explicit τ-dependent bounds. This will be instrumental in the divide-and-conquer scheme detailed later.

For the ease of notation, we denote

$$\begin{aligned} &Y_I = (U_I,V_I,U_I',V_I')^\top, \qquad \widetilde{Y}_I = (\widetilde{U}_I,\widetilde{V}_I,\widetilde{U}_I',\widetilde{V}_I')^\top, \quad I=a \text{ or } b; \\ &z_I(\tau,T) := \int_0^T (\widetilde{F}_I(\tau,s)^2 + \widetilde{G}_I(\tau,s)^2)ds. \end{aligned} \tag{4.11}$$

Lemma 4.3 *Let* $\tau \geq 1$ *and* $T > 0$. *Assume* $\frac{d}{dt}(Q_*^2) \leq 0$ *and denote* $\epsilon_Q := \sup_{0\leq t\leq T} |Q_*^2 - Q^2|$. *We have*

$$X_{\tau,T} := \max_{I=a,b} \max_{0\leq t\leq T} \|Y_I - \widetilde{Y}_I\|_{l^\infty}$$
$$\leq \frac{1}{2}\sqrt{\frac{2kT+\sinh(2kT)}{k}} \max_{I=a,b} \sqrt{z_I(\tau,T)} + \frac{3T\sinh(kT)}{2k}\epsilon_Q, \quad (k=\sqrt{\tau+1}). \quad (4.12)$$

Proof By Lemma 4.2, we have $\max_{I=\text{a or b}}(U_I(t)^2 + V_I(t)^2)^{\frac{1}{2}} \leq \frac{\sinh(kt)}{k}$ for any $t > 0$.

Denote $\eta = (\eta_1, \eta_2) = (f - \widetilde{f}, g - \widetilde{g})$, where $(f, g, \widetilde{f}, \widetilde{g}, \widetilde{F}, \widetilde{G}) = (U_I, V_I, \widetilde{U}_I, \widetilde{V}_I, \widetilde{F}_I, \widetilde{G}_I)$, $I = a$ or b. Clearly

$$\begin{cases} \eta_1'' = (1-\tau-2Q_*^2)\eta_1 + Q_*^2\eta_2 + F_1 - \widetilde{F}, & F_1 = (Q^2 - Q_*^2)(-2f+g); \\ \eta_2'' = Q_*^2\eta_1 + (1+\tau-2Q_*^2)\eta_2 + F_2 - \widetilde{G}, & F_2 = (Q^2 - Q_*^2)(-2g+f); \\ (\eta_1, \eta_2, \eta_1', \eta_2')\big|_{t=0} = 0. \end{cases}$$

By Lemma 4.2, we obtain (note $5f^2 + 5g^2 - 8fg \leq 9(f^2+g^2)$)

$$X_{\tau,T} \leq \int_0^T \cosh(k(T-s))\Big((\widetilde{F}(\tau,s)^2 + \widetilde{G}(\tau,s)^2)^{\frac{1}{2}} + \epsilon_Q$$
$$\cdot (5f^2 + 5g^2 - 8fg)^{\frac{1}{2}}(s)\Big)ds$$
$$\leq \left(\int_0^T \cosh^2(k(T-s))ds\right)^{\frac{1}{2}} \sqrt{z_I(\tau,T)} + \epsilon_Q \int_0^T \cosh(k(T-s)) \cdot 3\frac{\sinh(ks)}{k} ds$$
$$\leq \frac{1}{2}\sqrt{\frac{2kT+\sinh(2kT)}{k}}\sqrt{z_I(\tau,T)} + \frac{3T\sinh(kT)}{2k}\epsilon_Q. \qquad \square$$

Next, we prove an elementary lemma to be used in Theorem 4.5. First we need some notation.

Let $p_1 > 0$, $p_2 > 0$. For $\vec{x} = (x_1, x_2, x_3, x_4)^\top \in \mathbb{R}^4$, $\vec{y} = (y_1, y_2, y_3, y_4)^\top \in \mathbb{R}^4$, we define

$$\begin{aligned}\widetilde{H}_{p_1,p_2}(\vec{x},\vec{y}) = {} & 4p_1x_3^2y_1^2 + 4p_1p_2x_2x_4y_1^2 - 4p_1p_2x_1x_4y_1y_2 - p_2^2x_4^2y_2^2 \\ & - 8p_1x_1x_3y_1y_3 - 4p_2x_3x_4y_2y_3 \\ & + 4p_2x_2x_4y_3^2 + 4p_1x_1^2y_3^2 - 4p_1p_2x_1x_2y_1y_4 \\ & + 4p_1p_2x_1^2y_2y_4 + 4p_2x_3^2y_2y_4 \\ & + 2p_2^2x_2x_4y_2y_4 - 4p_2x_2x_3y_3y_4 - p_2^2x_2^2y_4^2; \\ \widetilde{H}^M_{p_1,p_2}(\vec{x},\vec{y}) = {} & 4p_1x_3^2y_1^2 + 4p_1p_2x_2x_4y_1^2 + 4p_1p_2x_1x_4y_1y_2 \\ & + p_2^2x_4^2y_2^2 + 8p_1x_1x_3y_1y_3 + 4p_2x_3x_4y_2y_3 \\ & + 4p_2x_2x_4y_3^2 + 4p_1x_1^2y_3^2 + 4p_1p_2x_1x_2y_1y_4\end{aligned}$$

$$+ 4p_1p_2x_1^2y_2y_4 + 4p_2x_3^2y_2y_4$$
$$+ 2p_2^2x_2x_4y_2y_4 + 4p_2x_2x_3y_3y_4 + p_2^2x_2^2y_4^2;$$

Note that $\widetilde{H}^M_{p_1,p_2}$ "majorizes" $\widetilde{H}_{p_1,p_2}$. For $\epsilon_s > 0$, we define

$$H^{\epsilon_s}_{p_1,p_2}(\vec{x}^*,\vec{y}^*) = \widetilde{H}_{p_1,p_2}(\vec{x}^*,\vec{y}^*) - \Big(\widetilde{H}^M_{p_1,p_2}(|x_1^*|+\epsilon_s,\ldots,|y_4^*|+\epsilon_s) - \widetilde{H}^M_{p_1,p_2}(|x_1^*|,|x_2^*|,\ldots,|y_4^*|)\Big). \tag{4.13}$$

Lemma 4.4 (H-criterion) *Fix* $\epsilon_s > 0$, $\vec{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)^\top \in \mathbb{R}^4$, *and* $\vec{y}^* = (y_1^*, y_2^*, y_3^*, y_4^*)^\top \in \mathbb{R}^4$. *Suppose* (*see* (4.13))

$$x_2^* > \epsilon_s,\ x_4^* > \epsilon_s, \qquad \textit{and } H^{\epsilon_s}_{p_1,p_2}(\vec{x}^*,\vec{y}^*) > 0.$$

Then $\forall\,\vec{x} = (x_1,\ldots,x_4)^\top \in \mathbb{R}^4$ *with* $\|\vec{x}-\vec{x}^*\|_{l^\infty} \le \epsilon_s$, $\vec{y} = (y_1,\ldots,y_4)^\top \in \mathbb{R}^4$ *with* $\|\vec{y}-\vec{y}^*\|_{l^\infty} \le \epsilon_s$, *we have*

$$(x_3\cos\theta + y_3\sin\theta)^2 + p_1(x_1\cos\theta + y_1\sin\theta)^2 + p_2(x_2\cos\theta + y_2\sin\theta)(x_4\cos\theta + y_4\sin\theta) \ge 0, \quad \forall\theta \in [0,2\pi). \tag{4.14}$$

Proof Clearly $x_2x_4 > 0$. Note that (4.14) (after re-arranging terms) is equivalent to:

$$as^2 + bs + c \ge 0, \qquad \forall s \in \mathbb{R},$$

where

$$a = x_3^2 + p_1x_1^2 + p_2x_2x_4, \qquad b = 2x_3y_3 + 2p_1x_1y_1 + p_2(x_2y_4 + x_4y_2);$$
$$c = y_3^2 + p_1y_1^2 + p_2y_2y_4.$$

Note $x_2x_4 > 0 \Rightarrow a > 0$. Therefore, it suffices to show that the discriminant $\mathfrak{D} = b^2 - 4ac$ is non-positive. A direct but lengthy computation reveals that

$$\mathfrak{D} \le 0 \iff \widetilde{H}_{p_1,p_2}(\vec{x},\vec{y}) \ge 0,$$

where $\widetilde{H}_{p_1,p_2}$ is defined preceding (4.13). On the other hand,

$$\begin{aligned}\widetilde{H}_{p_1,p_2}(\vec{x},\vec{y}) &\ge \widetilde{H}_{p_1,p_2}(\vec{x}^*,\vec{y}^*) - \Big(\widetilde{H}^M_{p_1,p_2}(|x_1^*|+\epsilon_s,|x_2^*|+\epsilon_s,\ldots,|y_4^*|+\epsilon_s) \\ &\quad - \widetilde{H}^M_{p_1,p_2}(|x_1^*|,|x_2^*|,\ldots,|y_4^*|)\Big) \\ &= H^{\epsilon_s}_{p_1,p_2}(\vec{x}^*,\vec{y}^*) \ge 0.\end{aligned}$$

Here we use the majorant $\widetilde{H}^M_{p_1,p_2}$ to control the maximal deviation of $|\widetilde{H}_{p_1,p_2}(\vec{x},\vec{y}) - \widetilde{H}_{p_1,p_2}(\vec{x}^*,\vec{y}^*)|$. □

We are now ready to state a general criterion for the nonexistence of nontrivial solutions for the case $\ell = 0$, $\tau \geq 1$. We will apply this criterion primarily for $\tau \in [1,\ 19.2]$, as detailed later.

Fix $\tau \in [1, \infty)$. Assume for some $T > 0$, $A > 0$, there exists a smooth function $\beta : [T, \infty) \to \mathbb{R}$ such that:

(I) $\beta > 0$, $\alpha = \beta' > 0$ for all $t > T$, and $\lim_{t\to\infty} e^{-2.8281t}(|\alpha(t)| + |\beta(t)|) = 0$.

(II) For all $t \geq T$, it holds that

$$\begin{cases} \tau - 1 + 2Q^2 + 4\frac{\beta}{\alpha} Q' Q > 0; \\ \frac{1}{2} A Q^4 < \left(\tau - 1 + 2Q^2 + 4\frac{\beta}{\alpha} Q' Q\right) \\ \qquad\qquad \times \left(\tau + 1 - 2Q^2 - \frac{1}{4}(\frac{\alpha'}{\alpha})^2 - \frac{1}{2A}(\frac{\beta}{\alpha})^2 Q^4\right). \end{cases} \tag{4.15}$$

(III) There exist $\epsilon_s > 0$, $\vec{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)^\top \in \mathbb{R}^4$, $\vec{y}^* = (y_1^*, y_2^*, y_3^*, y_4^*)^\top \in \mathbb{R}^4$ satisfying (see (4.11))

$$\|Y_a(\tau, T) - \vec{x}^*\|_{l^\infty} \leq \epsilon_s, \qquad \|Y_b(\tau, T) - \vec{y}^*\|_{l^\infty} \leq \epsilon_s, \qquad x_2^* > \epsilon_s,\ x_4^* > \epsilon_s;$$

and for some $p_1 \in [0, 2Q(T)^2 + \tau - 1]$ and $p_2 \geq \frac{2A\alpha(T)}{\beta(T)}$ (see (4.13))

$$H^{\epsilon_s}_{p_1,p_2}(\vec{x}^*, \vec{y}^*) > 0. \tag{4.16}$$

A few technical remarks are now in order. First, condition (II) choreographs a subtle dance between the putative eigenvalue τ and the far-field behavior of Q. For instance, taking $\tau \geq 2$, $\beta(t) = t$, and $A = 1$, the exponential decay of Q allows us to choose $T > 0$ large enough so that (4.15) holds. On the other hand, in the limiting case $\tau = 1$, the first condition in (4.15) reduces to

$$\beta' Q^2 + 2\beta Q' Q > 0 \quad \Longleftrightarrow \quad (\beta Q^2)' > 0.$$

From this, it is immediate that the weight β must grow at least like $e^{(2+\varepsilon)t}$ as $t \to \infty$. In any case, it is intuitively clear that, as far as condition (II) is concerned, the set of admissible choices (T, A, β) is nonempty.

Second, condition (III) stems from practical considerations. The rationale for this formulation is as follows. In principle, as will become clear during the proof of Theorem 4.5 (see in particular (4.22)), one could work with the simpler "ideal" condition

$$\widetilde{H}_{p_1,p_2}(Y_a(\tau, T), Y_b(\tau, T)) > 0.$$

In practice, however, the ground state Q is known only up to a certain numerical precision, and the computed solutions $Y_I(\tau, T)$ will inherit a comparable level of error. We therefore need a criterion that is stable under small ℓ^∞ perturbations and can be rigorously verified in finite-precision arithmetic. For this reason, we impose the stronger but more robust sufficient condition (III).

Theorem 4.5 *Fix $\tau \geq 1$ and assume the above conditions (I), (II), (III) hold. If $(U, V)^\top \in C^2([0,\infty), \mathbb{R}^2)$ is a smooth solution to the ODE system (4.1), then $(U, V)^\top \notin L^2([0,\infty), \mathbb{R}^2)$.*

Proof Assume $(U, V)^\top \in L^2$. Using $(\frac{1}{2}(U')^2)' + (\tau - 1 + 2Q^2)\Big(\frac{U^2}{2}\Big)' = Q^2 V U'$, we get

$$
\begin{aligned}
&(\beta \tfrac{1}{2}(U')^2)' + \Big(\beta \cdot (\tau - 1 + 2Q^2)\tfrac{U^2}{2}\Big)' \\
&\quad = \beta' \tfrac{1}{2}(U')^2 + Q^2 V U' \beta + (\beta \cdot (\tau - 1 + 2Q^2))' \tfrac{U^2}{2}. \qquad (4.17)
\end{aligned}
$$

Integrating on the time interval $[T_1, T_2]$, we obtain (assuming $\beta' > 0$)

$$
\begin{aligned}
&\beta \tfrac{1}{2}(U')^2\Big|_{T_1}^{T_2} + \beta \cdot (\tau - 1 + 2Q^2)\tfrac{U^2}{2}\Big|_{T_1}^{T_2} \\
&= \int_{T_1}^{T_2} \Big((\tfrac{\tau-1}{2} + Q^2 + 2\tfrac{\beta}{\beta'} Q' Q) U^2 + Q^2 U' V \tfrac{\beta}{\beta'} + \tfrac{(U')^2}{2} \Big) \beta' dt. \qquad (4.18)
\end{aligned}
$$

On the other hand, for V we recall $(\frac{V^2}{2})'' = (V')^2 + Q^2 U V + (\tau + 1 - 2Q^2) V^2$ and obtain

$$
\begin{aligned}
&\int_{T_1}^{T_2} ((V')^2 + Q^2 U V + (\tau + 1 - 2Q^2) V^2) \alpha dt \\
&= \int_{T_1}^{T_2} (\tfrac{V^2}{2})'' \alpha dt = \alpha (\tfrac{V^2}{2})'\Big|_{T_1}^{T_2} - \int_{T_1}^{T_2} V' \tfrac{\alpha'}{\alpha} V \alpha dt.
\end{aligned}
$$

This implies (below $A > 0$ is introduced as a flexible parameter)

$$
A\alpha(\tfrac{V^2}{2})'\Big|_{T_1}^{T_2} = A \int_{T_1}^{T_2} \Big((V')^2 + V' \tfrac{\alpha'}{\alpha} V + Q^2 U V + (\tau + 1 - 2Q^2) V^2 \Big) \alpha dt. \qquad (4.19)
$$

Denote $\alpha = \beta' > 0$ and define

$$
\begin{aligned}
\mathcal{E}_1 = &(\tfrac{\tau-1}{2} + Q^2 + 2\tfrac{\beta}{\beta'} Q' Q) U^2 + \tfrac{(U')^2}{2} + Q^2 U' V \tfrac{\beta}{\beta'} \\
&+ A(\tau + 1 - 2Q^2) V^2 + A(V')^2 + A V' \tfrac{\alpha'}{\alpha} V + A Q^2 U V. \qquad (4.20)
\end{aligned}
$$

Clearly $\frac{\beta}{2}(U')^2\big|_{T_1}^{T_2} + \beta \cdot (\frac{\tau-1}{2} + Q^2) U^2\big|_{T_1}^{T_2} + A\alpha(\frac{V^2}{2})'\big|_{T_1}^{T_2} = \int_{T_1}^{T_2} \mathcal{E}_1 \alpha dt$.

Setting $T_1 = T$ and $T_2 \to \infty$ (here we use[22] $\lim_{t\to\infty} e^{-2.8281t}(|\beta(t)| + |\alpha(t)|) = 0$), we obtain

$$
\Big(\int_T^\infty \mathcal{E}_1 \alpha dt\Big) + \tfrac{\beta}{2}(U')^2(T) + \beta \cdot (\tfrac{\tau-1}{2} + Q^2) U^2(T) + A\alpha V V'(T) = 0. \qquad (4.21)
$$

[22] By Proposition 4.1, $|U(T)| + |U'(T)| \lesssim e^{-\sqrt{2}T}$. Thus the threshold decay is $e^{-2\sqrt{2}T}$.

By using Cauchy-Schwartz, we have

$$Q^2 U' V \frac{\beta}{\beta'} \geq -\frac{1}{2}(U')^2 - \frac{1}{2} V^2 Q^4 \left(\frac{\beta}{\beta'}\right)^2;$$

$$Q^2 U V \geq -\frac{\alpha_1 Q^2 U^2}{2} - \frac{Q^2 V^2}{2\alpha_1}; \qquad V' \frac{\alpha'}{\alpha} V \geq -(V')^2 - \frac{1}{4}\left(\frac{\alpha'}{\alpha}\right)^2 V^2.$$

For $\tau \geq 1$, we impose the following constraints (for $t \geq T$)

$$\frac{\tau-1}{2} + Q^2 + 2\frac{\beta}{\beta'} Q' Q - \frac{A\alpha_1 Q^2}{2} > 0; \qquad \text{(Coeff of } U^2\text{)}$$

$$\tau + 1 - 2Q^2 - \frac{1}{4}\left(\frac{\alpha'}{\alpha}\right)^2 - \frac{Q^2}{2\alpha_1} - \frac{1}{2A}\left(\frac{\beta}{\beta'}\right)^2 Q^4 > 0. \qquad \text{(Coeff of } V^2\text{)}.$$

The existence of α_1 is guaranteed by (4.15). Thus to arrive at the desired contradiction, it suffices to check

$$U'(T)^2 + (2Q^2 + \tau - 1)U^2(T) + \frac{2A\alpha}{\beta} V V'(T) \geq 0.$$

Let $p_1 \in (0, 2Q(T)^2 + \tau - 1]$ and $p_2 \geq \frac{2A\alpha(T)}{\beta(T)}$. It suffices to check[23]

$$U'(T)^2 + p_1 U^2(T) + p_2 V V'(T) \geq 0. \tag{4.22}$$

Clearly $(U, V, U', V')^\top = Y_a \cos\theta + Y_b \sin\theta$. By Lemma 4.4, (4.22) is ensured by (4.16). □

Remark 4.6 Condition (4.22) provides an essentially sharp criterion for detecting hyperbolic instability. Indeed, a minimal scenario in which (4.22) could fail (i.e., hyperbolic instability does not occur) is when $VV'(t) \leq 0$ for all sufficiently large t. In that case, we would have $\sup_{t \geq T_0} V^2(t) < \infty$ for some $T_0 > 0$. If in addition $U \in L^2([0, \infty), dt)$, then Proposition 4.1 forces exponential decay of both U and V, making them a candidate eigenfunction. Hence (4.22) is nearly optimal and intrinsically linked to hyperbolic instability.

5 The case $\ell = 0$ and $\tau \in [1, 19.2]$

In this section, we tackle the case $\tau \in [1, 19.2]$ for $\ell = 0$.

Theorem 5.1 *Fix $\tau \in [1, 19.2]$. Let $(U, V)^\top \in C^2([0, \infty), \mathbb{R}^2)$ be the smooth solution to the ODE system (4.1). Then $(U, V)^\top \notin L^2([0, \infty), \mathbb{R}^2)$.*

To prove Theorem 5.1, we shall apply Theorem 4.5 with a divide-and-conquer strategy. We split the interval [1, 19.2] into 5 sub-intervals, and choose parameters A, T, $\beta(t)$ according to Table 1.

[23] Here it suffices to consider $VV'(T) < 0$. Clearly in this case $\frac{2A\alpha}{\beta} VV' \geq p_2 VV'$.

Table 1 Choice of $[\tau_L, \tau_R]$, A, T, p_1, p_2, $\beta(t)$ and $\Delta\tau$

$\tau \in [\tau_L, \tau_R]$	A	T	β	p_1	p_2	ϵ_s	$\Delta\tau$
$[1, 1.2]$	$3 \cdot 10^{-4}$	1.6	$e^{2.6t} - 63$	0.2	0.1	0.026	$2 \cdot 10^{-3}$
$[1.2, 2.1]$	0.002	1.1	$e^{1.4t} - 4.61$	1	0.5	0.023	10^{-2}
$[2.1, 5.4]$	0.01	0.856	$t - 0.845$	4	1.82	0.02	10^{-2}
$[5.4, 12.4]$	0.02	2/3	$t - 0.655$	9.6	3.43	0.026	10^{-2}
$[12.4, 19.2]$	0.032	0.475	$t - 0.46$	0.1	4.27	0.045	$2 \cdot 10^{-2}$

Clearly, condition (I) in Theorem 4.5 is satisfied. The verification of condition (II) is presented in full detail in Appendix B. We emphasize that condition (II) involves an explicit check on Q that is conceptually quite straightforward.

To verify condition (III), more detailed analysis is required. In light of Lemma 4.3, we must construct accurate approximate solutions that stay close to the exact ODE solutions $Y_I(\tau, t)$, $I = a, b$. For each of the five regimes listed in Table 1, we provide explicit[24] approximations $\widetilde{Y}_I(\tau, t)$. Notably, for $\tau \in [1, 12.4]$, we are able to express the terminal vector $\widetilde{Y}_I(\tau, T)$ as an explicit **quadratic** polynomial in τ. For $\tau \in [12.4, 19.4]$, it even reduces to an explicit **linear** function in τ. This makes checking condition (III) relatively pedestrian.

Below, we first treat the representative case $\tau \in [12.4, 19.2]$, where low-degree polynomials yield remarkable accuracy. We introduce a highly accurate approximation $\widetilde{Y}_I$ of Y_I on the short time interval $[0, T]$ with $T = 0.475$. These approximants are polynomials of degree only 7 in t and **linear** in τ. While not the only possible choice, the current expressions offer a good balance of simplicity and accuracy. Analogous formulas for other ranges of τ become a bit lengthy, and are therefore deferred to Appendix C.

To this end, for $\tau \in [12.4, 19.2]$, we define (we adopt the same notation as in (4.9)–(4.11))

$$
\begin{aligned}
\widetilde{U}_a(\tau, t) &= \tfrac{4550t^7}{131} - \tfrac{5787t^6}{91} + \tfrac{3986t^5}{105} - \tfrac{334t^4}{211} - \tfrac{727t^3}{120} + \tfrac{t^2}{242} \\
&\quad + t + \left(\tfrac{58t^7}{49} - \tfrac{305t^6}{129} + \tfrac{90t^5}{59} - \tfrac{6t^4}{61} - \tfrac{10t^3}{63} - \tfrac{t^2}{3170}\right)\tau; \\
\widetilde{V}_a(\tau, t) &= -\tfrac{2251t^7}{81} + \tfrac{4841t^6}{99} - \tfrac{11{,}060t^5}{403} + \tfrac{89t^4}{102} + \tfrac{429t^3}{136} - \tfrac{t^2}{140} \\
&\quad + \left(-\tfrac{9t^7}{92} + \tfrac{17t^6}{162} + \tfrac{5t^5}{111} - \tfrac{3t^4}{133} + \tfrac{t^3}{260} - \tfrac{t^2}{3409}\right)\tau; \\
\widetilde{U}_b(\tau, t) &= -\tfrac{3423t^7}{124} + \tfrac{4105t^6}{84} - \tfrac{3577t^5}{130} + \tfrac{72t^4}{79} + \tfrac{129t^3}{41} - \tfrac{t^2}{155} \\
&\quad + \left(\tfrac{5t^7}{83} - \tfrac{t^6}{52} - \tfrac{4t^5}{95} + \tfrac{t^4}{61} - \tfrac{t^3}{382} + \tfrac{t^2}{5233}\right)\tau; \\
\widetilde{V}_b(\tau, t) &= \tfrac{5468t^7}{155} - \tfrac{8018t^6}{121} + \tfrac{3078t^5}{79} - \tfrac{406t^4}{225} - \tfrac{1417t^3}{235} + \tfrac{t^2}{461}
\end{aligned}
$$

[24] These were obtained after several rounds of ad hoc optimization aimed at identifying polynomials of adequately low degree in both τ and t for each regime. In a much earlier draft, we employed discretized numerical schemes (with rigorous error bounds) to approximate $Y(\tau, T)$. Through careful interpolation and extensive iterative refinement, we arrived at the current continuous form.

$$+t + \left(-\frac{241t^7}{443} + \frac{288t^6}{253} - \frac{95t^5}{148} + \frac{15t^4}{374} + \frac{61t^3}{372} + \frac{t^2}{15{,}381}\right)\tau.$$

Observe that $\widetilde{Y}_I(\tau, T)$ is an explicit **linear** function in τ by construction. The next proposition shows it approximates $Y_I(\tau, T)$ well enough to make the H-criterion straightforward to check.

Proposition 5.2 *Let* $T = \frac{475}{10^3}$, $\tau_L = \frac{124}{10}$, $\tau_R = \frac{192}{10}$, $\Delta\tau = \frac{2}{10^2}$ *and recall* (4.9)–(4.11). *The following hold.*

(*i*) *For any* $\tau, \widetilde{\tau} \in [\tau_L, \tau_R]$ *with*[25] $|\tau - \widetilde{\tau}| \le \frac{1}{2}\Delta\tau$, *we have (here* $\|x\|_{l^\infty} = \max_{1\le i\le 4} |x_i|$ *for* $x \in \mathbb{R}^4$)

$$\max_{I=a,b} \|Y_I(\tau, T) - \widetilde{Y}_I(\widetilde{\tau}, T)\|_{l^\infty} \le \epsilon_s := \frac{45}{10^3}. \tag{5.1}$$

(*ii*) *For* $p_1 = \frac{1}{10}$, $p_2 = \frac{47}{10^2}$, *it holds that (below* $\widetilde{Y_{a,l}}$ *denotes the* l^{th} *component of* $\widetilde{Y}_a$)

$$\min_{0\le j\le N} \min\{\widetilde{Y_{a,2}}(\tau_j, T),\ \widetilde{Y_{a,4}}(\tau_j, T)\} > \epsilon_s,$$
$$\min_{0\le j\le N} H^{\epsilon_s}_{p_1,p_2}(\widetilde{Y}_a(\tau_j, T), \widetilde{Y}_b(\tau_j, T)) > 0, \tag{5.2}$$

where $H^{\epsilon_s}_{p_1,p_2}$ *is defined in* (4.13), *and* $\tau_j = \tau_L + j\Delta\tau$ $(0 \le j \le N)$ *with* $N = \frac{\tau_R - \tau_L}{\Delta\tau}$.

Proof We first control $z_I(\tau, T)$ (see (4.11)) with Q_*^2 defined in Lemma A.2. Note that $z_I(\tau, T)$ is an **explicit** polynomial of degree[26] $d = 4$ in the variable τ with rational coefficients.

Using Lemma 1.4, we find

$$\max_{I=a,b} \max_{\tau\in[\tau_L,\tau_R]} z_I(\tau, T) \le \frac{6N^2}{6N^2 - d^2(d^2-1)} \cdot \max_{I=a,b} \max_{0\le j\le N} z_I(\tau_j, T)$$
$$\le \frac{14{,}039}{18{,}384} \times 10^{-3}. \tag{5.3}$$

By Lemma 4.3 and Lemma A.2 (to control ϵ_Q), we get for $T = \frac{475}{10^3}$, $k = \sqrt{\frac{202}{10}}$ and all $\tau \in [\frac{124}{10}, \frac{192}{10}]$:

$$\max_{I=a,b} \|Y_I(\tau, T) - \widetilde{Y}_I(\tau, T)\|_{l^\infty} \le \frac{1}{2}\sqrt{\frac{2kT + \sinh(2kT)}{k}} \max_{I=a,b} (z_I(\tau, T))^{\frac{1}{2}} + \frac{3T\sinh(kT)}{2k}\epsilon_Q$$
$$\le \frac{418}{10^4}. \tag{5.4}$$

[25] Here we invoke the same trick as in Lemma 1.4: the $\frac{1}{2}\Delta\tau$-neighborhoods of the points $\{\tau_j\}_{j=0}^N$ suffice to cover the entire interval $[\tau_L, \tau_R]$.

[26] Recall $\widetilde{F}_I(\tau, t)$ contains terms such as $(1 + \tau - 2Q_*^2)\widetilde{U}_I$ and thus is of degree 2 in τ.

Since each component of $\widetilde{Y}_I(\tau, T)$ is linear in τ with (absolute value of) slope less than $1/10$, we have

$$\max_{I=a,b} \sup_{|\tau-\widetilde{\tau}|\leq\frac{1}{2}\Delta\tau} \|\widetilde{Y}_I(\tau, T) - \widetilde{Y}_I(\widetilde{\tau}, T)\|_{l^\infty} \leq \tfrac{1}{20}\Delta\tau \leq \tfrac{1}{10^3}. \tag{5.5}$$

Thus (5.1) follows. On the other hand, notice that $\widetilde{Y}_I(\tau, T)$, $I = a, b$ are **explicit** linear functions of τ. Thus (5.2) can be proved via a completely rigorous and exact computation. □

Proof of Theorem 5.1 The main point is to verify condition (III) in Theorem 4.5. Let A, T, β be chosen as in Table 1. For example for $\tau \in [12.4, 19.2]$ with $T = 0.475$, $A = 0.032$, $\beta(t) = t - 0.46$, $\epsilon_s = 0.045$, we have $2Q(T)^2 + \tau - 1 \geq p_1 := 0.1$, $\frac{2A\alpha(T)}{\beta(T)} \leq p_2 := 4.27$. This case clearly follows from Proposition 5.2. The other cases are rigorously proved via Proposition C.1 in Appendix C with minor changes in numerology. □

6 The case $\ell = 0$, $\tau \geq 19.2$

Theorem 6.1 *Let $\tau \geq 19.2$. Let $(U, V) \in C^2([0,\infty), \mathbb{R}^2)$ be the smooth solution to the ODE system (4.1). Then $(U, V)^\top \notin L^2([0,\infty), \mathbb{R}^2)$.*

Proof Assume $(U, V)^\top \in L^2$. The exponential decay of U, V follows from Proposition 4.1. Using $(\frac{1}{2}(U')^2)' + (\tau - 1 + 2Q^2)\big(\frac{U^2}{2}\big)' = Q^2VU'$, we have (below β is a smooth weight function to be specified later)

$$\begin{aligned}&(\beta\frac{1}{2}(U')^2)' + \big(\beta\cdot(\tau - 1 + 2Q^2)\tfrac{U^2}{2}\big)'\\&\quad= \beta'\tfrac{1}{2}(U')^2 + Q^2VU'\beta + (\beta\cdot(\tau - 1 + 2Q^2))'\tfrac{U^2}{2}.\end{aligned}$$

Integrating on the time slab $[0, T]$, we obtain

$$\begin{aligned}&\int_0^T \Big((\tfrac{\beta'}{2}(\tau - 1 + 2Q^2) + 2\beta Q'Q)U^2 + Q^2U'V\beta + \tfrac{\beta'(U')^2}{2}\Big)dt\\&= \tfrac{\beta(T)}{2}U'(T)^2 - \tfrac{\beta(0)}{2}U'(0)^2 + \tfrac{\beta}{2}(\tau - 1 + 2Q^2)U(T)^2.\end{aligned} \tag{6.1}$$

Note here we used the condition $U(0) = 0$.

Using $(\frac{V^2}{2})'' = (V')^2 + Q^2UV + (\tau + 1 - 2Q^2)V^2$, we obtain (below $\alpha = \beta' > 0$)

$$\begin{aligned}&\int_0^T ((V')^2 + Q^2UV + (\tau + 1 - 2Q^2)V^2)\alpha dt\\&\quad= \int_0^T (\tfrac{V^2}{2})''\alpha dt = \alpha(\tfrac{V^2}{2})'\Big|_0^T - \int_0^T \alpha'(\tfrac{V^2}{2})'dt\\&\quad= (\alpha VV' - \alpha'\tfrac{V^2}{2})\Big|_0^T + \int_0^T \tfrac{V^2}{2}\alpha''dt.\end{aligned}$$

$$\Rightarrow \quad \int_0^T \Big((V')^2 + Q^2 UV + (\tau + 1 - 2Q^2 - \tfrac{\alpha''}{2\alpha}) V^2 \Big) \alpha dt$$
$$= \alpha V V'(T) - \tfrac{\alpha' V^2(T)}{2}, \tag{6.2}$$

where we used $V(0) = 0$. By (6.1) and (6.2), we get

$$\int_0^T \mathcal{E}_0(t)\alpha(t) dt = \tfrac{\beta(T)}{2} U'(T)^2 - \tfrac{\beta(0)}{2} U'(0)^2 + \tfrac{\beta(T)}{2}(\tau - 1 + 2Q(T)^2) U(T)^2$$
$$+ \alpha V V'(T) - \tfrac{\alpha' V(T)^2}{2},$$

where[27]

$$\mathcal{E}_0 := (\tfrac{\tau - 1 + 2Q^2}{2} + 2\tfrac{\beta}{\beta'} Q' Q) U^2 + Q^2 U' V \tfrac{\beta}{\beta'} + \tfrac{(U')^2}{2}$$
$$+ (V')^2 + (\tau + 1 - 2Q^2 - \tfrac{\alpha''}{2\alpha}) V^2 + Q^2 UV. \tag{6.3}$$

By using Cauchy-Schwartz, we bound the cross terms in $\mathcal{E}_0$ as

$$Q^2 U' V \tfrac{\beta}{\beta'} \geq -\tfrac{1}{2}(U')^2 - \tfrac{1}{2} V^2 Q^4 \big(\tfrac{\beta}{\beta'}\big)^2, \qquad Q^2 UV \geq -\tfrac{\alpha_1 Q^2 U^2}{2} - \tfrac{Q^2 V^2}{2\alpha_1},$$

where $\alpha_1 = \alpha_1(t) > 0$ is another flexible parameter.

We impose the following constraints:

1). $\beta(0) \geq 0,\ \alpha(t) = \beta'(t) > 0, \forall t > 0,$ and for some $m_0 > 0$,
$\sup_{t \geq 0} \frac{|\beta(t)| + |\alpha(t)| + |\alpha'(t)|}{(1+t)^{m_0}} < \infty;$

2). $\tau + 1 - 2Q^2 - \frac{\alpha''}{2\alpha} - \frac{Q^2}{2\alpha_1} - \frac{1}{2}\big(\frac{\beta}{\beta'}\big)^2 Q^4 > 0, \quad$ (Coeff of V^2); (6.4)

3). $\tau - 1 + 2Q^2 + 4\frac{\beta}{\beta'} Q' Q - \alpha_1 Q^2 > 0, \quad$ (Coeff of U^2). (6.5)

Under the above conditions, we can send $T \to \infty$ and derive

$$\int_0^\infty \mathcal{E}_0(t)\alpha(t) dt = 0, \quad \mathcal{E}_0(t) \geq \epsilon_0(t) \cdot (U(t)^2 + V(t)^2),$$

where $\epsilon_0(t) > 0, \forall t > 0$. This implies $U \equiv 0,\ V \equiv 0$, contradicting $(U', V')|_{t=0} = (\cos\theta, \sin\theta)$.

In the rest of this proof, we take $\beta = \frac{t^2}{1+3.75t^2}$. Clearly $\frac{\alpha''}{2\alpha} = \frac{90(15t^2-4)}{(15t^2+4)^2}$ and $\frac{\beta}{\beta'} = \frac{1}{8} t(4 + 15t^2)$.

It suffices for us to check for $\tau \geq 19.2$:

$$F_a := \tfrac{202}{10} - (2 + \tfrac{1}{2\alpha_1}) Q^2 - \tfrac{\alpha''}{2\alpha} - \tfrac{1}{2}\big(\tfrac{\beta}{\beta'}\big)^2 Q^4 > 0; \tag{6.6}$$
$$F_b := \tfrac{182}{10} + (2 - \alpha_1) Q^2 + 4\tfrac{\beta}{\beta'} Q' Q > 0. \tag{6.7}$$

[27] A comparison with (4.20) shows that the key nuance is in the derivation of (6.2).

Case 1: $t \geq 1$. Take $\alpha_1 = 0.4$. Using $Q \leq Q_p(t) := \frac{2714}{1000} \cdot \frac{1}{t} e^{-t}$ (Lemma A.1), we get for $t \geq \frac{5}{2}$:

$$F_a \geq \frac{202}{10} - \left(\frac{325}{100} Q_p(t)^2 + \frac{90(15t^2-4)}{(15t^2+4)^2} + \frac{1}{128} (\frac{2714}{1000})^4 \cdot \frac{(4+15t^2)^2}{t^2} e^{-4t} \right) \Big|_{t=\frac{5}{2}} > 0,$$

where we used $(\frac{15t^2-4}{15t^2+4})' < 0$ and $(\frac{4+15t^2}{t} e^{-2t})' < 0$ for $t \geq 5/2$. Using[28] $Q < \frac{1}{t}$ for $t \geq 1$, we get

$$F_a \geq \frac{202}{10} - \frac{325}{100t^2} - \frac{90(15t^2-4)}{(15t^2+4)^2} - \frac{1}{128} \frac{(4+15t^2)^2}{t^2} > 0, \qquad \forall t \in [1,\ 2.5].$$

Since $QQ' > -2t^{-3}$ for $t > 0$ (Lemma A.2), we get

$$F_b \geq \frac{182}{10} - t^{-2}(4 + 15t^2) > 0, \qquad \forall t \geq 1.12. \tag{6.8}$$

For $t \in [1,\ 1.12]$, easy to check via Lemma A.2 that $Q(1.12) \geq 0.75$. Thus

$$F_b \geq \frac{182}{10} + \frac{8}{5} \cdot (\frac{3}{4})^2 - t^{-2}(4 + 15t^2) > 0, \qquad \forall t \in [1,\ 1.12].$$

Case 2: $t \in [0.6,\ 1]$. Take $\alpha_1 = 0.4$. Easy to check that $\frac{90(15t^2-4)}{(15t^2+4)^2} < 3t + \frac{1}{2}$, $\forall t \in [0.6, 1]$.

By Lemma A.2, we have $Q^2(t) \leq Q_*^2(t) + \frac{75}{10^5} =: W(t)$. Then

$$\begin{aligned} F_a \geq Z_1 &:= \frac{202}{10} - \frac{325}{100} W - (3t + \frac{1}{2}) - \frac{1}{128} t^2 (4 + 15t^2) W^2 \\ \Rightarrow F_a &\geq \mathcal{M}(Z_1, [\frac{3}{5},\ 1], \frac{1}{10^4}) > 0. \qquad \text{(By Lemma 1.4)}. \end{aligned} \tag{6.9}$$

For F_b we also apply Lemma A.2 and Lemma 1.4 to get (see (B.2) for P_2)

$$\begin{aligned} F_b \geq Z_2 &:= \frac{182}{10} + \frac{8}{5}(Q_*^2 - \frac{75}{10^5}) + \frac{1}{2} t (4 + 15t^2) P_2 \\ &\geq \mathcal{M}(Z_2, [\frac{3}{5},\ 1], \frac{1}{10^4}) > 0. \end{aligned} \tag{6.10}$$

Case 3: $t \in [0,\ 0.6]$. Take $\alpha_1 = -\frac{25t^2}{3} + \frac{25t}{27} + \frac{54}{19}$. Denote

$$\begin{aligned} Z_3 &= (4 + 15t^2)^2 \Big(\frac{202}{10} \alpha_1 - (2\alpha_1 + \frac{1}{2})(Q_*^2 + \frac{75}{10^5}) \Big) - 90(15t^2 - 4)\alpha_1 \\ &\quad - \frac{\alpha_1}{128} t^2 (4 + 15t^2)^4 (Q_*^2 + \frac{75}{10^5})^2; \\ Z_4 &= \frac{182}{10} + 2(Q_*^2 - \frac{75}{10^5}) + P_2 \cdot \frac{1}{2} t (4 + 15t^2) - \alpha_1 \cdot (Q_*^2 + \frac{75}{10^5}). \end{aligned}$$

The desired result follows from

$$\mathcal{M}(Z_3, [0, \frac{6}{10}], \frac{1}{10^4}) > 0, \qquad \mathcal{M}(Z_4, [0, \frac{6}{10}], \frac{1}{10^4}) > 0. \tag{6.11}$$

□

[28] When $t \geq 1$, $(tQ)'' = t(Q - Q^3) > 0 \Rightarrow (tQ)' < 0 \Rightarrow Q(t) \leq t^{-1} Q(1) < t^{-1}$.

7 Preliminary analysis for the case $\ell = 1$

In this section we gather a few preliminary results for the case $\ell = 1$. We will develop several lemmas progressively, starting with a simple one on the asymptotic behavior of a basic ODE.

Lemma 7.1 (Asymptotics of Φ) *Consider the ODE*:

$$\Phi''(t) = (\tfrac{2}{t^2} + a_1(t))\Phi(t) + F(t), \qquad t \geq 1,$$

where a_1 and F are continuous and for some $\gamma_1 > 0$, $\sup_{t\geq 1} e^{\gamma_1 t}(|a_1(t)| + |F(t)|) < \infty$. We have (below η_i, Φ_{S} are C^2, and c_i are constants)

$$\Phi(t) = c_1 \cdot (t^2 + \eta_1(t)) + c_2 \cdot (\tfrac{1}{t} + \eta_2(t)) + \Phi_{\mathrm{S}}(t), \tag{7.1}$$

where for some $0 < \gamma_2 < \gamma_1$,

$$\sup_{t\geq 1} e^{\gamma_2 t}(|\eta_1(t)| + |\eta_2(t)| + |\Phi_{\mathrm{S}}(t)|) < \infty.$$

In particular, $\lim_{t\to\infty} \frac{\Phi(t)}{t^2}$ *exists.*

Proof Write $\Phi = \Phi_{\mathrm{H}} + \Phi_{\mathrm{S}}$, where Φ_{S} is a decaying special solution and Φ_{H} solves the homogeneous equation.

1) Existence of an exponentially decaying special solution Φ_{S}. Consider the integral equation

$$\Phi_{\mathrm{S}}(t) = t^2 \int_t^\infty \tau^{-4} \int_\tau^\infty s^2(a_1\Phi_{\mathrm{S}} + F)\,ds\,d\tau, \qquad (T_0 \text{ is sufficiently large}).$$

A simple contraction on $[T_0, \infty)$ yields existence. We then solve Φ_{S} backwards from $t = T_0$ to $t = 1$.

2) The homogeneous equation. Firstly we find $\Phi_{\mathrm{H}}^{(1)} = t^2 + \eta_1$ with η_1 solving the integral equation

$$\eta_1(t) = \int_t^\infty (s-t)\left(a_1(s)s^2 + (\tfrac{2}{s^2} + a_1(s))\eta_1(s)\right) ds, \qquad t \geq T_1,$$
$$(T_1 \text{ is sufficiently large}).$$

Clearly we can obtain contraction using the norm $\|e^{\gamma_2 t}\eta_1(t)\|_{L_t^\infty([T_1,\infty))}$. Solving it backwards until $t = 1$ then yields a smooth solution on $[1, \infty)$. Similarly we find another solution $\Phi_{\mathrm{H}}^{(2)}(t) = \frac{1}{t} + \eta_2$ with η_2 solving

$$\eta_2(t) = \int_t^\infty (s-t)\left(a_1(s)\tfrac{1}{s} + (\tfrac{2}{s^2} + a_1(s))\eta_2(s)\right) ds, \qquad t \geq T_2,$$
$$(T_2 \text{ is sufficiently large}). \tag{7.2}$$

Independence follows from checking the Wronskian of $\Phi_{\mathrm{H}}^{(1)}$, $\Phi_{\mathrm{H}}^{(2)}$. ☐

In later arguments, we shall use the following Sturm comparison often without explicit mention.

Lemma 7.2 (Sturm comparison) *Let* $0 < l_0 < \infty$. *Suppose* $G = G(t)$, $g = g(t)$: $[0, l_0] \to \mathbb{R}$ *are Lipschitz functions satisfying* $G(t) \geq g(t)$, $\forall\, 0 \leq t \leq l_0$. *Assume* F, f *are* C^2 *functions satisfying*

$$\begin{cases} F'' \geq GF, \quad 0 < t < l_0; \\ f'' \leq gf, \quad 0 < t < l_0; \\ F(0) \geq f(0) > 0, \ \frac{F'(0)}{F(0)} \geq \frac{f'(0)}{f(0)}; \end{cases} \tag{7.3}$$

and $f(t) > 0$ *for all* $0 < t < l_0$. *Then*

$$F(t) \geq f(t) > 0, \quad \frac{F'(t)}{F(t)} \geq \frac{f'(t)}{f(t)}, \qquad \forall\, 0 < t < l_0. \tag{7.4}$$

Moreover, (7.4) *holds under the alternative initial condition* $F(0) = f(0) = 0$, $F'(0) \geq f'(0) > 0$.

Proof Use $R = \frac{F'(t)}{F(t)}$, $r = \frac{f'(t)}{f(t)}$ and consider $(R - r)'$. If $R(0) \geq r(0)$ and $F(0) \geq f(0) > 0$, then one can easily infer

$$R - r \geq 0, \qquad F - f \geq 0, \qquad \forall\, 0 \leq t < l_0.$$

If $F(0) = f(0) = 0$ and $F'(0) \geq f'(0) > 0$, then clearly for some $t_1 > 0$ sufficiently small, we have

$$F(t) \geq f(t) > 0, \quad F'(t) > 0, \quad f'(t) > 0, \qquad \forall\, 0 < t \leq t_1.$$

We now define $A = F/F'$, $B = f/f'$ on the interval $[0, t_1]$. Observe that $A(0) = B(0) = 0$, and

$$A' \leq 1 - GA^2, \qquad B' \geq 1 - gB^2,$$
$$(A - B)' \leq -(G - g)A^2 - g(A + B)(A - B).$$

Thus $(A - B)(t) \leq 0$ for all $0 \leq t \leq t_1$. Switching to the ODE for $R - r$ with $(R - r)\big|_{t=t_1} \geq 0$, we obtain $(R - r)(t) \geq 0$ for all $0 < t < l_0$. It also follows easily that $F(t) \geq f(t) > 0$ for all $0 < t < l_0$. □

The next lemma is concerned with the existence of a negative direction for the operator $S = -\partial_{tt} + \frac{2}{t^2} + 3tQ'Q$ to be used later.

Lemma 7.3 (Existence of a negative direction for S) *Let* $S = -\partial_{tt} + \frac{2}{t^2} + 3tQQ'$. *There exists* (e_0, f_0), $e_0 > 0$, $f_0 \in C^\infty((0,\infty), \mathbb{R})$, $f_0(t) = t^2 + 0.1e_0t^4 + \mathcal{O}(t^6)$ *as* $t \to 0^+$, *and* $f_0(t)$ *decays exponentially as* $t \to \infty$, *such that* $Sf_0 = -e_0 f_0$. *Furthermore* $f_0(t) > 0$ *for all* $0 < t < \infty$.

Proof In this proof we denote $\|u\|_2 := \left(\int_0^\infty u^2 dt\right)^{\frac{1}{2}}$. Consider the variational problem

$$I_1 = \inf_{\substack{u\in H_0^1((0,\infty),\mathbb{R})\\ \|u\|_2=1}} \underbrace{\left(\int_0^\infty (u')^2 dt + 2\int_0^\infty \frac{u^2}{t^2} dt + 3\int_0^\infty u^2 t Q' Q dt\right)}_{=:I(u)}. \tag{7.5}$$

Note that the singular term $\int_0^\infty \frac{u^2}{t^2} dt$ is not a problem[29] since we are working with the space $H_0^1((0,\infty),\mathbb{R})$. Clearly I_1 exists as a finite number. We first show that $I_1 < -l_0$ for some $l_0 > 0$. We proceed in two sub-steps.

Substep 1. Let $\mathcal{Y}_0 = -tQ'$. We show that

$$\int_0^\infty \mathcal{Y}_0 S \mathcal{Y}_0 dt = \int_0^\infty (3Q^2 - 1 + 3tQ'Q)\mathcal{Y}_0^2 dt = 0. \tag{7.6}$$

Indeed the first identity follows from $-\mathcal{Y}_0'' + (2t^{-2} + 1 - 3Q^2)\mathcal{Y}_0 = 0$. We now focus on the second.

Integrating $(t^2 Q')' = t^2(Q - Q^3)$ with the weight $-\frac{1}{3}Q^3$, we get

$$\int_0^\infty t^2 Q^2 (Q')^2 dt = -\frac{1}{3}\int_0^\infty t^2 Q^4 dt + \frac{1}{3}\int_0^\infty t^2 Q^6 dt. \tag{7.7}$$

Integrating $t^3 Q'' + 2t^2 Q' = t^3(Q - Q^3)$ with the weight $-Q^2 Q'$, we get

$$\begin{aligned} &\int_0^\infty t^3 (Q')^3 Q dt = \frac{1}{2}\int_0^\infty t^2 Q^2 (Q')^2 dt + \int_0^\infty t^2\left(\frac{3}{4}Q^4 - \frac{1}{2}Q^6\right) dt \\ \Rightarrow &\int_0^\infty t^3 (Q')^3 Q dt = \frac{7}{12}\int_0^\infty t^2 Q^4 dt - \frac{1}{3}\int_0^\infty t^2 Q^6 dt. \end{aligned} \tag{7.8}$$

The second identity in (7.6) follows from (7.7), (7.8) and $\|\nabla Q\|_2^2 = \frac{3}{4}\|Q\|_4^4$ (cf. (1) of Lemma 3.1).

Substep 2. Clearly $S\mathcal{Y}_0 = (3Q^2 - 1 + 3tQ'Q)\mathcal{Y}_0$. We can pick some $\widetilde{t_0} \gg 1$ such that $3Q^2 - 1 + 3tQ'Q < 0$, $\forall t \in \widetilde{O} = (\frac{1}{2}\widetilde{t_0}, \widetilde{t_0})$. Pick a sufficiently small function $\widetilde{\eta} \in C_c^\infty(\widetilde{O})$. Clearly one can choose $\widetilde{\eta}$ such that $\mathcal{Y}_1 = \mathcal{Y}_0 + \widetilde{\eta}$ satisfies

$$\int_0^\infty \mathcal{Y}_1 S \mathcal{Y}_1 dt < 0. \tag{7.9}$$

Thus $I_1 < -l_0$ for some $l_0 > 0$.

Now suppose $(u_n)_{n\ge 1}$ is a minimizing sequence, i.e., $\|u_n\|_2 = 1$, $I(u_n) \to I_1$ as $n \to \infty$. Since $I(u_n)$ is bounded from above and $\int_0^\infty (\widetilde{f}')^2 dt \ge \frac{1}{4}\int_0^\infty \frac{(\widetilde{f})^2}{t^2} dt$, we have $\sup_n \|u_n'\|_2 \lesssim 1$ and $\sup_n \|u_n\|_\infty \lesssim 1$. Thus by passing to a subsequence and relabelling if necessary (i.e., denoting u_{n_j} as u_n), we may assume

$$u_n \rightharpoonup v \text{ in } H_0^1((0,\infty),\mathbb{R}), \quad \text{for some } v \in H_0^1((0,\infty),\mathbb{R}). \tag{7.10}$$

[29] Here we use the inequality: $\int_0^\infty (\widetilde{f}')^2 dt \ge \frac{1}{4}\int_0^\infty \frac{(\widetilde{f})^2}{t^2} dt$, $\forall \widetilde{f} \in C_c^\infty((0,\infty),\mathbb{R})$, which can be proved via $\int_0^\infty (\frac{c}{t}\widetilde{f} - \widetilde{f}')^2 dt \ge 0$ with $c = \frac{1}{2}$.

Furthermore $u_n \to v$ locally uniformly (i.e., on compact subsets) and $u_n(x) \to v(x)$ for all x. Clearly

$$\liminf_{n\to\infty} \|u_n'\|_2^2 \ge \|v'\|_2^2, \quad \liminf_{n\to\infty} 2\|\tfrac{u_n}{t}\|_2^2 \ge 2\|\tfrac{v}{t}\|_2^2.$$

By Lebesgue Dominate Convergence we have

$$\lim_{n\to\infty} 3\int_0^\infty tQQ'u_n^2 dt = 3\int_0^\infty tQQ'v^2 dt.$$

Since $I(u_n) < -\frac{1}{2}l_0$ for all n sufficiently large, we obtain $I(v) \le -\frac{1}{2}l_0$. In particular v is nontrivial. Note that $I(|v|) \le I(v)$. Thus we may assume $v \ge 0$. Denote $\mathbb{R}^+ = (0, \infty)$. It follows from the minimality of v that

$$\int_{\mathbb{R}^+} \left(v'\widetilde{\phi}' + 2\tfrac{v}{t} \cdot \tfrac{\widetilde{\phi}}{t} + 3tQQ'v\widetilde{\phi} - I_1 v\widetilde{\phi} \right) dt = 0, \qquad \forall \widetilde{\phi} \in H_0^1(\mathbb{R}^+, \mathbb{R}), \tag{7.11}$$

where $I_1 < 0$. Thus we have (in distributional sense)

$$-v'' + 2t^{-2}v + 3tQ'Qv - I_1 v = 0, \qquad \text{in } \mathcal{D}'(\mathbb{R}^+). \tag{7.12}$$

Since $v \in H_0^1(\mathbb{R}^+, \mathbb{R})$, a simple bootstrap argument yields that $v \in C^\infty(\mathbb{R}^+)$ and the above identity holds in the classical sense for $t > 0$. Now note that the coefficient $tQ'Q$ can be expanded into a convergent power series near $t = 0$. By using the standard Frobenius method (cf. p. 521 of [2]) together with the condition $v \in H_0^1(\mathbb{R}^+)$, we obtain that $v(t)$ has a power series expansion of the form $v(t) = c_0(t^2 + c_1 t^4 + c_2 t^6 + \cdots)$ near $t = 0^+$. The exponential decay of $v(t)$ as $t \to \infty$ follows from the fact that $I_1 < 0$ and $v \in H_0^1(\mathbb{R}^+)$. We can choose the normalization that $c_0 = 1$. It is not difficult to check that $c_2 = 0.1e_0$ (here $e_0 = -I_1$).

Finally, since $v(t) \ge 0$ for $0 < t < \infty$, it follows that $v(t) > 0$ for $0 < t < \infty$. Indeed if $v(t_0) = 0$ for some $t_0 \in (0, \infty)$, then $v'(t_0) = 0$. Uniqueness of ODE implies $v \equiv 0$ which contradicts $I(v) < 0$. □

As a direct application of Lemma 7.2, we examine in the next lemma a solution to an auxiliary ODE system $S\varphi = 0$. The analysis reveals that any regular solution must change sign exactly once and grows quadratically as $t \to \infty$. This behavior will be essential in the subsequent proof of Proposition 9.1.

Lemma 7.4 *Let $S = -\partial_{tt} + \frac{2}{t^2} + 3tQ'Q$. Consider φ solving (note that $Q''(0) < 0$)*

$$\begin{cases} S\varphi = 0, & 0 < t < \infty; \\ \varphi(t) = t^2 + \frac{3}{28}Q''(0)Q(0)t^6 + \mathcal{O}(t^8), & t \to 0^+. \end{cases} \tag{7.13}$$

Then φ must change its sign exactly once on $(0, \infty)$, and for some constants $\gamma_1 > 0$, $c > 0$ we have

$$\left|\varphi(t) + \gamma_1 t^2\right| \le ct^{-1}, \quad t \ge 1. \tag{7.14}$$

Proof We begin by noting that φ must change its sign since it is orthogonal to the negative direction f_0 in Lemma 7.3 (recall $f_0 > 0$, $\forall t > 0$). We proceed in several steps.

Step 1. We show that the first positive zero of φ must occur at some $t_0 \geq 1$. Note that[30]

$$1 - 3Q^2 \leq 3tQ'Q, \qquad \forall t \in [0, 0.5]. \tag{7.15}$$

Let $\varphi_1 = -c_1 t Q'$ ($c_1 > 0$ is a suitable normalization constant) and compare

$$\begin{cases} \varphi_1'' = (\frac{2}{t^2} + 1 - 3Q^2)\varphi_1, \\ \varphi_1 = t^2 + \frac{1-3Q^2(0)}{10} t^4 + \mathcal{O}(t^6), \end{cases} \qquad \begin{cases} \varphi'' = (\frac{2}{t^2} + 3tQ'Q)\varphi \\ \varphi = t^2 + \mathcal{O}(t^6), \quad t \to 0^+. \end{cases} \tag{7.16}$$

Lemma 7.2 yields $\varphi \geq \varphi_1 > 0$ on the interval $(0, 0.5]$ and $\varphi'(0.5) > 0$.

For $t > 0.5$ we consider U_A and U_B solving

$$\begin{cases} U_A'' = (\frac{2}{t^2} + 3tQ'Q)U_A \\ (U_A, U_A')\big|_{t=0.5} = (1, 0); \end{cases} \qquad \begin{cases} U_B'' = (\frac{2}{t^2} + 3tQ'Q)U_B \\ (U_B, U_B')\big|_{t=0.5} = (0, 1). \end{cases} \tag{7.17}$$

It is not difficult to check that[31]

$$\frac{2}{t^2} + 3tQ'Q > -k_0^2, \quad \forall t \in [0.5, 1], \tag{7.18}$$

where $k_0 = \sqrt{8.67}$. Let $\widetilde{U}_A = \cos(k_0(t-0.5))$ and $\widetilde{U}_B = \frac{1}{k_0}\sin(k_0(t-0.5))$ solve

$$\begin{cases} \widetilde{U}_A'' = -k_0^2 \widetilde{U}_A \\ (\widetilde{U}_A, \widetilde{U}_A')\big|_{t=0.5} = (1, 0); \end{cases} \qquad \begin{cases} \widetilde{U}_B'' = -k_0^2 \widetilde{U}_B \\ (\widetilde{U}_B, \widetilde{U}_B')\big|_{t=0.5} = (0, 1). \end{cases} \tag{7.19}$$

Simple comparison yields that $U_A \geq \widetilde{U}_A > 0$ and $U_B \geq \widetilde{U}_B > 0$ on the interval $[0.5, 1]$. Since φ is a positive linear combination of U_A and U_B, we conclude that φ must stay positive on $[0.5, 1]$.

Step 2. We show φ changes its sign exactly once. Consider $t_0 \geq 1$ and ($\varphi(t_0) = 0$, $h = \frac{\varphi(t)}{\varphi'(t_0)}$)

$$\begin{cases} h'' = (\frac{2}{t^2} + 3tQ'Q)h, & t > t_0; \\ h(t_0) = 0, \ h'(t_0) = 1. \end{cases} \qquad \begin{cases} h_1'' = -3h_1, & t > t_0; \\ h_1(t_0) = 0, \ h_1'(t_0) = 1. \end{cases} \tag{7.20}$$

Lemma A.2 yields $2t^{-2} + 3tQ'Q > 0$, $\forall t \geq 1.7$. Easy to check that[32]

$$\frac{2}{t^2} + 3tQ'Q > -3, \quad \forall t \in [1, 1.7]. \tag{7.21}$$

[30] See the toy example in (1.23).

[31] This follows from $\mathcal{M}(\widetilde{p}, [\frac{1}{2}, 1], \frac{1}{10^5}) > 0$, where $\widetilde{p} = 2 + \frac{867}{100}t^2 + 3t^3 P_2$ and P_2 is defined in (B.2).

[32] This follows from $\mathcal{M}(\widetilde{p}_1, [1, \frac{17}{10}], \frac{5}{10^5}) > 0$, where $\widetilde{p}_1 = 2 + 3t^2 + 3t^3 P_2$.

Note $h_1(t) = \frac{1}{\sqrt{3}} \sin(\sqrt{3}(t - t_0))$. Comparing h with h_1 easily yields

$$h(t) > h_1(t) > 0, \qquad \frac{h'(t)}{h(t)} > \frac{h_1'(t)}{h_1(t)} > 0, \qquad \forall\, t \in (t_0,\ \tfrac{17}{10}]. \tag{7.22}$$

Clearly $h > 0$, $h' > 0$ for $t \geq \frac{17}{10}$, and $h(t) \to \infty$, $h'(t) \to \infty$ as $t \to \infty$.

Step 3. We characterize the asymptotic behavior of φ as $t \to \infty$. By Lemma 7.1, we have

$$\varphi(t) = -\widetilde{C}_1(t^2 + \eta_1(t)) + \widetilde{C}_2(\tfrac{1}{t} + \eta_2(t)), \qquad t \geq 1, \tag{7.23}$$

where η_i decays exponentially fast. Note that by Step 2, the constant $\widetilde{C}_1$ must be positive. □

Lemma 7.5 *Let f_0 be the same as in Lemma 7.3. We have*

$$\int_0^\infty f S f \geq 0, \qquad \forall\, f \in H_0^1((0,\infty), \mathbb{R}) \text{ with } f \perp f_0, \text{ i.e.}, \int_0^\infty f f_0 dt = 0. \tag{7.24}$$

Proof We consider the variational problem (recall $\mathbb{R}^+ := (0,\infty)$ and $\|u\|_2 := (\int_0^\infty u^2)^{\frac{1}{2}}$)

$$I_2 = \inf_{\substack{u \in H_0^1(\mathbb{R}^+,\mathbb{R}) \\ u \perp f_0 \\ \|u\|_2 = 1}} \underbrace{\left(\int_0^\infty (u')^2 dt + 2\int_0^\infty \frac{u^2}{t^2} dt + 3\int_0^\infty u^2 t Q' Q dt\right)}_{=: I(u)}. \tag{7.25}$$

If $I_2 \geq 0$ we are done. Now assume $I_2 < 0$. Note that $I_2 \geq I_1$. By arguing similarly as in the proof of Lemma 7.3, we can extract a minimizer $v_1 \in H_0^1$ with $v_1 \perp f_0$. Furthermore,

$$\int_0^\infty \left(v_1' \varphi' + 2\frac{v_1}{t} \cdot \frac{\varphi}{t} + 3t Q Q' v_1 \varphi - I_2 v_1 \varphi\right) dt = 0, \\ \forall\, \varphi \in H_0^1(\mathbb{R}^+, \mathbb{R}) \text{ with } \varphi \perp f_0. \tag{7.26}$$

Now note that for general $\phi \in H_0^1$, we have $\phi - c_\phi f_0 \perp f_0$ where $c_\phi = \|f_0\|_2^{-2} \times \int_0^\infty \phi f_0 dt$. Thus

$$\int_0^\infty \left(v_1' \phi' + 2\frac{v_1}{t} \cdot \frac{\phi}{t} + 3t Q Q' v_1 \phi - I_2 v_1 \phi\right) dt = \int_0^\infty v_1 (S f_0 - I_2 f_0) dt c_\phi = 0, \\ \forall\, \phi \in H_0^1(\mathbb{R}^+, \mathbb{R}), \tag{7.27}$$

where we used $\int_0^\infty v_1 (S f_0 - I_2 f_0) dt = (I_1 - I_2) \int_0^\infty v_1 f_0 dt = 0$. By a similar analysis as in the paragraph around (7.12), we obtain $v_1 \in C^\infty(\mathbb{R}^+)$, decays exponentially in t as $t \to \infty$, and (by choosing the normalization $c_0 = 1$) $v_1(t) = t^2 + c_1 t^4 + c_2 t^6 + \cdots$ near $t = 0^+$.

Since f_0 does not change its sign and $v_1 \perp f_0$, clearly v_1 must change its sign. Now we consider

$$\begin{cases} v_1'' = (\frac{2}{t^2} + 3tQ'Q - I_2)v_1, & t > 0; \\ v_1 = t^2 + 0.1(-I_2)t^4 + \mathcal{O}(t^6), & t \to 0^+; \end{cases}$$

$$\begin{cases} U'' = (\frac{2}{t^2} + 3tQ'Q)U, & t > 0; \\ U = t^2 + \mathcal{O}(t^6), & t \to 0^+. \end{cases} \tag{7.28}$$

By the proof of Lemma 7.4, we know the first positive zero of U must occur at some $t_0 \geq 1.5$. Thus the first positive zero of v_1 must occur at some $t_1 \geq 1.5$. By using an argument similar to Step 3 in the proof of Lemma 7.4, we then obtain v_1 changes its sign exactly once, and $v_1(t) \to -\infty$ as $t \to \infty$. This clearly contradicts the exponential decay of v_1 established earlier. Thus we conclude $I_2 \geq 0$. □

Our next technical lemma is concerned with an inhomogeneous ODE. More precisely, consider

$$g_\alpha'' = -\frac{4}{t} g_\alpha' + 3tQQ' g_\alpha + Q,$$
$$\text{with initial condition } g_\alpha(0) = \alpha, \; g_\alpha'(0) = 0. \tag{7.29}$$

Denote $J_{\alpha_1} = t^2 g_{\alpha_1}$ with $\alpha_1 = 0$. It is not difficult to check that (note $Q''(0) < 0$)

$$J_{\alpha_1}'' = (\frac{2}{t^2} + 3tQ'Q)J_{\alpha_1} + t^2 Q; \qquad J_{\alpha_1}(t) = \frac{1}{10}Q(0)t^4 + \frac{1}{56}Q''(0)t^6 + \mathcal{O}(t^8),$$
$$t \to 0^+. \tag{7.30}$$

Define[33] a super solution (note $\frac{81}{185} > \frac{1}{10}Q(0)$ and $0 < J_{\alpha_1}(t) < y(t)$, $0 < J_{\alpha_1}'(t) < y'(t)$ as $t \to 0^+$)

$$y(t) = t^2 y_1(t), \quad y_1(t) = -\frac{3t^8}{341} + \frac{46t^7}{525} - \frac{487t^6}{1404} + \frac{1801t^5}{2640} - \frac{2669t^4}{4312} + \frac{81t^2}{185}. \tag{7.31}$$

Proposition 7.6 *For all* $0 < t \leq 1.7$, *it holds that* $y'' \geq \frac{2}{t^2}y + t^2 Q$, *and* $0 < J_{\alpha_1}(t) \leq y(t)$. *Moreover*

$$(\frac{2}{t^2} + 3tQ'Q) \cdot y_1 + Q > 0, \qquad \forall 0 < t \leq 1.7. \tag{7.32}$$

Thus $J_{\alpha_1}(t) > 0$ *and* $J_{\alpha_1}'(t) > 0$ *for all* $t > 0$. *In particular,* $g_{\alpha_1}(t) > 0$ *for all* $t > 0$, *and*

$$\lim_{t \to \infty} g_{\alpha_1}(t) = \lim_{t \to \infty} t^{-2} J_{\alpha_1}(t) = \kappa_1 > 0, \quad \textit{where } \kappa_1 \textit{ is a constant.} \tag{7.33}$$

Proof We proceed in the following steps. For simplicity we abbreviate J_{α_1} as J.

[33] Another way is to solve the equation $I'' = \frac{2}{t^2}I + t^2(W_1 + \frac{75}{10^6})$. Setting $I = t^2 y_2$ yields $y_2 = \int_0^t s^{-4} \int_0^s \tilde{s}^4 (W_1 + \frac{75}{10^6}) d\tilde{s} ds$ which is an explicit polynomial. We prefer y_1 over y_2 for its lower degree.

Step 1. Clearly $y_1 > 0$ for $0 < t \le 1.7$. It is not difficult to verify (see Lemma A.2 for W_1 and (B.2) for P_2)

$$y'' \ge \frac{2}{t^2} y + t^2 (W_1 + \frac{75}{10^6}) \ge \frac{2}{t^2} y + t^2 Q, \qquad \forall 0 < t \le 1.7; \tag{7.34}$$

$$(\frac{2}{t^2} + 3t \cdot P_2) \cdot y_1 + W_1 - \frac{75}{10^6} > 0, \qquad \forall 0 < t \le 1.7. \tag{7.35}$$

Both can be easily checked via Lemma 1.4 since we are dealing with polynomial functions.

Step 2. We show $J > 0$, $J' > 0$ on $(0, 1.7]$. By using the expansion of J near $t = 0^+$, it is clear that if $t_1 > 0$ is sufficiently small, then $J(t) > 0$, $J'(t) > 0$ for all $t \in (0, t_1]$.

Claim: Let $t_* \in (0, 1.7]$. If $J(t) \ge 0$, $J'(t) \ge 0$ for all $t \in (0, t_*]$, then $J(t) > 0$, $J'(t) > 0$ for all $t \in (0, t_*]$. Proof of the Claim: Since $3tQ'Q \le 0$, we have

$$y'' \ge \frac{2}{t^2} y + t^2 Q, \qquad J'' \le \frac{2}{t^2} J + t^2 Q, \qquad \forall 0 < t < t_*. \tag{7.36}$$

Since $0 < J(t) < y(t)$, $0 < J'(t) < y'(t)$ for $t = 0^+$, (by examining the difference $y - J$) we obtain $J(t) \le y(t)$ for all $0 < t \le t_*$. We now show that $J'' > 0$ for all $t \in (0, t_*]$, i.e.,

$$(\frac{2}{t^2} + 3tQ'Q)J + t^2 Q > 0, \qquad \forall t \in (0, t_*]. \tag{7.37}$$

We may assume $\frac{2}{t^2} + 3tQ'Q < 0$, whence clearly (see (B.2) for P_2)

$$\begin{aligned}
&(\frac{2}{t^2} + 3tQ'Q)J + t^2 Q \\
\ge &(\frac{2}{t^2} + 3tQ'Q)y + t^2 Q = \frac{2}{t^2} y + 3tQ'Qy + t^2 Q \qquad \text{(recall } J \le y) \\
\ge &\frac{2}{t^2} y + 3t \cdot P_2 \cdot y + t^2 (W_1 - \frac{75}{10^6}) > 0, \qquad \text{(by (7.35))}.
\end{aligned}$$

Thus $J'' > 0$ for all $t \in (0, t_*]$. Positivity of J' and J follows from integrating in t. The claim is proved.

Fix $t_1 > 0$ sufficiently small such that $J(t) > 0$, $J'(t) > 0$ for all $t \in (0, t_1]$. Consider the set

$$\mathcal{P} = \{s \in [\frac{t_1}{2}, 1.7] : J > 0, \ J' > 0 \text{ for all } t \in [\frac{t_1}{2}, s]\}.$$

Clearly $t_1 \in \mathcal{P}$ and $\mathcal{P}$ is clopen in $[\frac{t_1}{2}, 1.7]$ (by the claim). Thus $\mathcal{P} = [\frac{t_1}{2}, 1.7]$, and $J, J' > 0$ on $(0, 1.7]$.

Step 3. For $t > 1.7$, we use $2t^{-2} + 3tQ'Q > 0$ (Lemma A.2) to conclude that $J, J' > 0$.

Finally (7.33) follows from Lemma 7.1. □

8 A technical estimate for the case $\ell = 1$

In this section, we estimate the upper solution $g_{\alpha_2}(t)$ defined in (7.29) with $\alpha_2 = 5.8$. The main result is Proposition 8.4, where we show that

$$\lim_{t\to\infty} g_{\alpha_2}(t) < 0 \quad \text{and} \quad \int_0^\infty g_{\alpha_2}(t) Q t^4 \, dt > 0.$$

This property will play a key role in the proof of Proposition 9.1. To carry out a careful estimate of g_{α_2}, we employ slightly different comparison principles across three successive intervals: $[0, 1.8]$, $[1.8, 3.5]$, and $[3.5, \infty)$. The selection of these intervals, along with their corresponding upper and lower comparison functions, is not entirely straightforward: they are the result of multiple rounds of ad hoc numerical experimentation, iterative refinement, and optimization. The first breakpoint $t_1 = 1.8$ is chosen since the coefficient $A(t) = 2/t^2 + 3tQ'Q > 0$ for all $t \geq 1.8$ (see Lemma A.2). Note that the solution stays positive whilst the coefficient $A(t)$ changes its sign twice on the interval $[0, t_1]$. The second breakpoint $t_2 = 3.5$ is chosen because the solution changes sign once on $[t_1, t_2]$, a situation where Lemma 7.2 no longer applies. Moreover, the decay of Q on this sub-interval is not negligible in the sense that this segment contributes non-trivially to the integral $\int g_{\alpha_2} Q t^4 dt$. For $t > t_2$, the solution has a fixed sign and its contribution to the integral is comparatively small. For convenience of presentation, we work with $I(t) = g_{\alpha_2}(t)t^2$.

Lemma 8.1 (First comparison, $\alpha_2 = 5.8, 0 \leq t \leq 1.8$) *Consider*

$$\begin{cases} I'' = (\frac{2}{t^2} + 3tQ'Q)I + t^2 Q, \\ I = \alpha_2 t^2 + \mathcal{O}(t^4), \quad t \to 0^+, \end{cases} \tag{8.1}$$

where $\alpha_2 = 5.8$. *It holds that* $0 < I_-(t) \leq I(t) \leq I_+(t)$, $\forall\, 0 < t \leq 1.2$, *where* $I_\pm(t) = t^2(\widetilde{g}(t) \pm \frac{t}{1600})$, *and*

$$\begin{aligned} \widetilde{g} = {} & \frac{2{,}655{,}007t^{22}}{766{,}661} - \frac{61{,}031{,}641t^{21}}{765{,}161} + \frac{1{,}193{,}343{,}423t^{20}}{1{,}392{,}527} - \frac{4{,}889{,}665{,}718t^{19}}{857{,}587} \\ & + \frac{22{,}540{,}646{,}527t^{18}}{857{,}154} - \frac{13{,}934{,}342{,}915t^{17}}{156{,}279} + \frac{88{,}378{,}047{,}540t^{16}}{384{,}383} \\ & - \frac{82{,}501{,}815{,}384t^{15}}{179{,}359} + \frac{59{,}891{,}505{,}057t^{14}}{83{,}009} - \frac{188{,}840{,}245{,}805t^{13}}{212{,}143} \\ & + \frac{272{,}712{,}958{,}583t^{12}}{316{,}809} - \frac{117{,}464{,}741{,}599t^{11}}{181{,}890} + \frac{3{,}127{,}972{,}553t^{10}}{8468} \\ & - \frac{1{,}847{,}962{,}994t^{9}}{11{,}735} + \frac{269{,}766{,}735t^{8}}{5476} - \frac{104{,}060{,}861t^{7}}{8909} \\ & + \frac{11{,}561{,}100t^{6}}{5101} - \frac{1{,}372{,}577t^{5}}{5985} - \frac{453{,}401t^{4}}{8690} - \frac{5242t^{3}}{6305} + \frac{5324t^{2}}{11{,}721} + \frac{29}{5}. \end{aligned}$$

Denote $\widetilde{I}(t) = t^2\widetilde{g}(t)$. *We have* $\int_0^{1.8} I(t)t^2 Q(t)dt \geq 1.07$ *and*

$$|I(1.8) - \widetilde{I}(1.8)| \leq 0.0062, \qquad |I'(1.8) - \widetilde{I}'(1.8)| \leq 0.0096. \tag{8.2}$$

Proof We begin by noting that $I_- > 0, \forall t \in [0,\ 1.8]$. Also $0 < I_-(t) < I(t) < I_+(t)$, $\frac{I_-'(t)}{I_-(t)} < \frac{I'(t)}{I(t)} < \frac{I_+'(t)}{I_+(t)}$ for $t = 0^+$. For $0 < t \le 1.2$, we have the easily-checked inequalities (see Lemma A.2 for W_1 and (B.2) for P_2; below $P_2^+ := (W_1 - \frac{75}{10^6})(W_1' + \frac{42}{10^5})$ and note $P_2 \le Q'Q \le P_2^+$ for $t \le 1.8$)

$$\frac{2}{t^2} + 3tP_2^+ + \frac{t^2(W_1 + \frac{75}{10^6})}{I_-} < \frac{I_+''}{I_+}, \qquad \frac{2}{t^2} + 3tP_2 + \frac{t^2(W_1 - \frac{75}{10^6})}{I_+} > \frac{I_-''}{I_-},$$
$$\forall 0 < t \le 1.2. \tag{8.3}$$

Bootstrapping these with $I_- \le I \le I_+$, we have $\frac{I_-'}{I_-} < \frac{I'}{I} < \frac{I_+'}{I_+}$, $\forall 0 < t \le 1.2$. In particular,

$$|I(t) - \widetilde{I}(t)| \le \frac{t^3}{1600}, \qquad \forall 0 \le t \le 1.2.$$

Note that $I_+'(1.2) < 0$. We have $\frac{I_-'}{I_-} I_+ \le I' \le \frac{I_+'}{I_+} I_-$ at $t = 1.2$, which yields

$$|I'(1.2) - \widetilde{I}'(1.2)| \le \frac{62}{10^4}. \tag{8.4}$$

Next we estimate $\widetilde{I} - I$ on $[1.2,\ 1.8]$. Denote $\mathrm{Er} = \widetilde{I}'' - 2\widetilde{g} - 3tQ'Q\widetilde{I} - t^2 Q$ and note

$$F_- \le \mathrm{Er} \le F_+, \qquad \forall t \in [1.2,\ 1.8];$$

where

$$F_- := \widetilde{I}'' - 2\widetilde{g} - 3tP_2^+\widetilde{I} - t^2(W_1 + \frac{75}{10^6}),$$
$$F_+ := \widetilde{I}'' - 2\widetilde{g} - 3tP_2\widetilde{I} - t^2(W_1 - \frac{75}{10^6}).$$

Easy to check that $F_-(t) > 0, \forall t \in [1.2,\ 1.8]$ and

$$\|\mathrm{Er}\|_{L_t^\infty([1.2,1.8])} \le \|F_\pm\|_{L_t^\infty([1.2,1.8])} \le \frac{39}{10^4}, \qquad \max_{1.2 \le t \le 1.8} |\frac{2}{t^2} + 3tQ'Q| \le \frac{7}{5}. \tag{8.5}$$

Denote $\eta = \widetilde{I} - I$ and $a = \frac{2}{t^2} + 3tQ'Q$. Clearly

$$\eta'' = a\eta + \mathrm{Er}. \tag{8.6}$$

Since $|\eta(1.2)| \le \frac{1.2^3}{1600}$, $|\eta'(1.2)| \le \frac{62}{10^4}$, it follows easily that

$$\max_{1.2 \le t \le 1.8} |\eta| \le 0.0062, \qquad \max_{1.2 \le t \le 1.8} |\eta'| \le 0.0114. \tag{8.7}$$

Now note $\frac{2}{t^2} + 3tP_2(t) \le a(t) \le \frac{2}{t^2} + 3tP_2^+(t) < 0$ for all $t \in [1.2,\ 1.6]$ and $\int_{1.2}^{1.6} (-\frac{2}{t^2} - 3tP_2)dt \le 0.244$. Also $\max_{1.6 \le t \le 1.8} |a(t)| \le 0.15$ and $\int_{1.2}^{1.8} F_+ dt \le \frac{165}{10^5}$. Using these together with the formula $\eta'(t) = \eta'(1.2) + \int_{1.2}^t a\eta ds + \int_{1.2}^t E_r ds$, we can improve the estimate to

$$\max_{1.2 \le t \le 1.8} |\eta'| \le 0.0096. \tag{8.8}$$

Finally

$$\int_0^{1.8} t^2 IQdt \geq \int_0^{1.8} t^2(\widetilde{I} - \tfrac{62}{10^4})(W_1 - \tfrac{75}{10^6})dt > \tfrac{107}{100}. \tag{8.9}$$

□

Next we estimate $I(t)$ in the regime $1.8 \leq t \leq 3.5$. Note that $2 + 3t^3 Q'Q > 0$ for $t \geq 1.8$ and we can resort to much simpler comparison.[34] We introduce the following upper and lower solutions:

$$\begin{aligned}
J_+(t) &= \frac{t^{14}}{3{,}302{,}419} - \frac{t^{13}}{189{,}311} - \frac{t^{12}}{27{,}017} + \frac{11t^{11}}{5279} - \frac{17t^{10}}{556} + \frac{923t^9}{3491} - \frac{5962t^8}{3809} \\
&\quad + \frac{20{,}181t^7}{2999} - \frac{37{,}198t^6}{1725} + \frac{104{,}356t^5}{2011} \\
&\quad - \frac{38{,}323t^4}{412} + \frac{428{,}192t^3}{3543} - \frac{215{,}987t^2}{2017} + \frac{162{,}617t}{2916} - \frac{46{,}355}{4182}; \\
J_-(t) &= \frac{t^{14}}{2{,}629{,}395} - \frac{t^{13}}{124{,}156} + \frac{t^{12}}{121{,}108} + \frac{4t^{11}}{2441} - \frac{187t^{10}}{6760} \\
&\quad + \frac{1218t^9}{4849} - \frac{5023t^8}{3296} + \frac{20{,}281t^7}{3051} - \frac{28{,}489t^6}{1325} \\
&\quad + \frac{540{,}785t^5}{10{,}386} - \frac{307{,}187t^4}{3277} + \frac{47{,}397{,}503t^3}{388{,}000} \\
&\quad - \frac{174{,}823t^2}{1612} + \frac{171{,}747t}{3038} - \frac{12{,}413}{1105}.
\end{aligned} \tag{8.10}$$

Lemma 8.2 (Second comparison, $1.8 \leq t \leq 3.5$) *We have* $\int_{1.8}^{3.5} I(t)Q(t)t^2 dt \geq -0.401$ *and*

$$\begin{aligned}
&J_-(t) \leq I(t) \leq J_+(t), \qquad J'_-(t) \leq I'(t) \leq J'_+(t), \qquad \forall t \in [1.8,\ 3.5]; \\
&0.83 \leq -I(3.5) \leq 1.051, \qquad 0.16 \leq -I'(3.5) \leq 0.57.
\end{aligned} \tag{8.11}$$

Proof Easy to check $J_+(\frac{9}{5}) \geq I(\frac{9}{5}) \geq J_-(\frac{9}{5})$, $J'_+(\frac{9}{5}) > I'(\frac{9}{5}) \geq J'_-(\frac{9}{5})$. Lemma A.2 yields

$$\begin{aligned}
&\underbrace{(W'_1 - \tfrac{5}{10^5})(W_1 + \tfrac{6}{10^6})}_{B_2} \leq Q'Q \leq \underbrace{(W'_1 + \tfrac{5}{10^5})(W_1 - \tfrac{6}{10^6})}_{B_1} \leq 0, \\
&\forall t \in [\tfrac{9}{5},\ \tfrac{5}{2}].
\end{aligned} \tag{8.12}$$

Denote $\mathcal{K}_\pm = J''_\pm - (\frac{2}{t^2} + 3tQ'Q)J_\pm - t^2 Q$. Then $\mathcal{K}_+ \geq 0$, $\mathcal{K}_- \leq 0$ on $[1.8,\ 2.5]$ follow from checking

$$\begin{aligned}
&J''_+ - \tfrac{2}{t^2}J_+ - 3tB_i J_+ - t^2(W_1 + \tfrac{6}{10^6}) \geq 0; \\
&J''_- - \tfrac{2}{t^2}J_- - 3tB_i J_- - t^2(W_1 - \tfrac{6}{10^6}) \leq 0, \qquad i = 1, 2.
\end{aligned} \tag{8.13}$$

[34] We meant the following simple comparison argument: If $J''_+ - (\frac{2}{t^2} + 3tQ'Q)J_+ - t^2 Q \geq 0$ for $t \in [1.8, 3.5]$, then for $Z = J_+ - I$ we have $Z'' \geq (\frac{2}{t^2} + 3tQ'Q)Z$, $\forall t \in [1.8, 3.5]$. Since $\frac{2}{t^2} + 3tQ'Q > 0$ for all $t \geq 1.8$, and $Z(1.8) > 0$, $Z'(1.8) \geq 0$, it follows that $Z(t) \geq 0$ for all $t \in [1.8, 3.5]$, i.e., $J_+(t) \geq I(t)$, $\forall t \in [1.8, 3.5]$. The argument for $I(t) \geq J_-(t)$ is similar.

For $t\in[2.5,\ 3.5]$ we note that $J_\pm<0$ and (see Lemma A.2 for W_2 and we are using a larger constant)

$$\mathcal{K}_+\geq J_+''-\tfrac{2}{t^2}J_++3t(-W_2'+\tfrac{5}{10^5})(W_2+\tfrac{6}{10^6})J_+-t^2(W_2+\tfrac{6}{10^6})\geq 0;\tag{8.14}$$

$$\mathcal{K}_-\leq J_-''-\tfrac{2}{t^2}J_-+3t(-W_2'-\tfrac{5}{10^5})(W_2-\tfrac{6}{10^6})J_--t^2(W_2-\tfrac{6}{10^6})\leq 0.\tag{8.15}$$

Thus $\mathcal{K}_+\geq 0$, $\mathcal{K}_-\leq 0$ for all $t\in[1.8,\ 3.5]$ and $J_\pm$ are the desired super-solution/sub-solutions.

Finally we have (note $\max_{1.8\leq t\leq 3.5}|J_-|\leq 1.06$)

$$\begin{aligned}\int_{1.8}^{3.5}I(t)Q(t)t^2dt&\geq\int_{\frac{9}{5}}^{\frac{7}{2}}J_-(t)t^2Qdt\geq\int_{\frac{9}{5}}^{\frac{5}{2}}J_-(t)t^2W_1dt\\&\quad+\int_{\frac{5}{2}}^{\frac{7}{2}}J_-(t)t^2W_2dt-\frac{6}{10^6}\cdot\frac{106}{100}\int_{\frac{9}{5}}^{\frac{7}{2}}t^2dt\\&\geq -0.401.\end{aligned}\tag{8.16}$$

The estimate of $I(3.5)$ and $I'(3.5)$ follows easily from $J_\pm(\frac{35}{10})$ and $J_\pm'(\frac{35}{10})$. □

Finally we turn to the case $t\geq 3.5$. We shall prove $0<H_-(t)\leq -I(t)\leq H_+(t)$, where

$$\begin{cases}H_-''=\frac{1.8525}{t^2}H_--2.8te^{-t};\\ H_-(3.5)=0.83,\ \ H_-'(3.5)=0.16;\end{cases}$$
$$\begin{cases}H_+''=\frac{2}{t^2}H_+-2.7te^{-t};\\ H_+(3.5)=1.051,\ \ H_+'(3.5)=0.57.\end{cases}\tag{8.17}$$

Lemma 8.3 (Third comparison, $t\geq 3.5$) *We have* $0<H_-(t)\leq -I(t)\leq H_+(t)$, $\forall t\geq 3.5$. *Furthermore*

$$\int_{3.5}^{\infty}I(t)t^2Q(t)dt\geq -0.65.\tag{8.18}$$

Proof Lemmas A.1–A.2 imply

$$2.7\leq tQe^t\leq 2.8,\quad 2+3t^3Q'Q\geq 1.8525,\quad \forall t\geq 3.5.$$

Clearly

$$\begin{aligned}H_-(t)&=\frac{10{,}788\cdot 2^{19/20}\cdot 7^{1/20}-548{,}800\Gamma(\frac{21}{20},\frac{7}{2})+548{,}800\Gamma(\frac{21}{20},t)}{568{,}400}t^{39/20}\\&\quad+\frac{548{,}800\Big(\Gamma(\frac{79}{20},\frac{7}{2})-\Gamma(\frac{79}{20},t)\Big)+103{,}733\cdot\sqrt[20]{2}\cdot 7^{19/20}}{568{,}400t^{19/20}},\end{aligned}$$

where $\Gamma(a,t):=\int_t^\infty e^{-s}s^{a-1}ds$. Note

$$10{,}788\cdot 2^{19/20}\cdot 7^{1/20}-548{,}800\Gamma(\tfrac{21}{20},\tfrac{7}{2})>0\tag{8.19}$$

and $H_-(t)\gtrsim t^{1.95}$, $\forall t\geq 3.5$.

By comparison we obtain $-I(t) > H_-(t), \forall t \geq 3.5$. Comparing H_+ with $-I$ then leads to the bound $H_+(t) > -I(t), \forall t \geq 3.5$. Note that

$$H_+(t) = \frac{1523t^2}{18{,}375} - \frac{9t^2}{10e^{7/2}} - \frac{27e^{-t}t}{10} - \frac{27e^{-t}}{5} - \frac{27e^{-t}}{5t} + \frac{749}{6000t} + \frac{7677}{80e^{7/2}t}.$$

Clearly

$$-\int_{7/2}^{\infty} I(t)t^2 Q dt \leq \int_{7/2}^{\infty} H_+(t)t \cdot \frac{19}{7}e^{-t} dt \leq 0.65. \tag{8.20}$$

In the above we used $Q(t) \leq \frac{19}{7t}e^{-t}$ (see Lemma A.1). ☐

We now state and prove our main proposition. Recall $g_{\alpha_2}(t)$ is defined in (7.29) with $\alpha_2 = 5.8$.

Proposition 8.4 *For $\alpha_2 = 5.8$, we have $\lim_{t\to\infty} g_{\alpha_2}(t) < 0$ and $\int_0^\infty g_{\alpha_2} Q t^4 dt > 0$.*

Proof For $\alpha_2 = 5.8$, we have $\lim_{t\to\infty} g_{\alpha_2}(t) = \lim_{t\to\infty} I(t)/t^2 < 0$ and $\int_0^\infty g_{\alpha_2}(t) \times t^4\, dt = \int_0^\infty I(t)t^2\, dt > 0$. The latter follows from Lemmas 8.1, 8.2 and 8.3. ☐

9 Proof of the case $\ell \geq 1$

In this section we complete the proof for the case $\ell = 1$ and $\ell \geq 2$. The first main result of this section is Theorem 9.4 where we rule out any nonzero embedded eigenvalue for the case $\ell = 1$. To prove Theorem 9.4 we need to prove several technical propositions. The first one employs a "shooting" argument.

Proposition 9.1 *There exists some $f_* \in \mathcal{F} = \left\{ f \in H^2((0,\infty), \mathbb{R}) : \quad f(t) = \text{const} \cdot t^2 + \mathcal{O}(t^4) \text{ as } t \to 0^+ \right\}$ such that $Sf_* = t^2 Q$ and $\int_0^\infty f_* t^2 Q dt < 0$. Here $S = -\partial_{tt} + \frac{2}{t^2} + 3tQ'Q$. Moreover $|f_*(t)| \lesssim t^{-1}$ for $t \geq 1$.*

Proof of Proposition 9.1 We rewrite $g = -t^{-2} f_*$ solving $-g'' - \frac{4}{t}g' + 3tQQ'g = -Q$. Consider

$$\begin{cases} g_\alpha'' = -\frac{4}{t}g_\alpha' + 3tQQ'g_\alpha + Q \\ g_\alpha(0) = \alpha, \quad g_\alpha'(0) = 0. \end{cases}$$

The goal is to find a constant α such that $|g_\alpha(t)| \lesssim t^{-3}$ as $t \to \infty$, and $\int_0^\infty g_\alpha(t) Q \times t^4 dt > 0$.

Clearly for $\alpha_1, \alpha_2 \in \mathbb{R}$, we have $g_{\alpha_1} - g_{\alpha_2} = (\alpha_1 - \alpha_2)\eta_1$, where

$$\eta_1'' = -\frac{4}{t}\eta_1' + 3tQ'Q\eta_1, \qquad \eta_1(0) = 1, \quad \eta_1'(0) = 0. \tag{9.1}$$

By Lemma 7.4 (observe $\varphi(t) = t^2\eta_1(t)$), we know $|\eta_1(t) + \gamma_1| = \mathcal{O}(t^{-3})$ for $t \geq 10$. Since

$$\|g_{\alpha_1} - g_{\alpha_2}\|_\infty \leq |\alpha_1 - \alpha_2| \cdot \|\eta_1\|_\infty,$$

we deduce that the map $\alpha \to g_\alpha(\infty) = \lim_{t\to\infty} g_\alpha(t)$ is continuous. Note that the existence of $\lim_{t\to\infty} g_\alpha(t)$ follows from Lemma 7.1. In particular, if $\lim_{t\to\infty} g_\alpha(t) = 0$, then $|g_\alpha(t)| \lesssim 1/t^3$ for $t \geq 1$. By linearity, it is easy to see that if $\int_0^\infty g_{\alpha_1} Qt^4 dt > 0$, $\int_0^\infty g_{\alpha_2} Qt^4 dt > 0$, then $\int_0^\infty g_\alpha Qt^4 dt > 0$ for any $\alpha \in (\alpha_1, \alpha_2)$. The proof is completed by the following two facts:

(1) For $\alpha_1 = 0$, we have $g_{\alpha_1}(t) > 0$ for all $t \in [0, \infty)$ (Proposition 7.6).

(2) For $\alpha_2 = 5.8$, we have $\lim_{t\to\infty} g_{\alpha_2}(t) < 0$ and $\int_0^\infty g_{\alpha_2} Qt^4 dt > 0$ (Proposition 8.4). □

Recall $S = -\partial_{tt} + \frac{2}{t^2} + 3tQ'Q$ and denote

$$\mathcal{F} = \left\{ f \in H^2((0,\infty), \mathbb{R}) : \ f(t) = \text{const} \cdot t^2 + \mathcal{O}(t^4) \text{ as } t \to 0^+ \right\}. \tag{9.2}$$

For any $f_1, f_2 \in \mathcal{F}$, it is easy to check

$$\int_0^\infty f_1 S f_2 dt = \int_0^\infty f_2 S f_1 dt = \int_0^\infty \left(f_1' f_2' + (\tfrac{2}{t^2} + 3tQ'Q) f_1 f_2 \right) dt.$$

The following proposition is the key to proving Theorem 9.4.

Proposition 9.2 *For any $f \in \mathcal{F}$ with $\int_0^\infty f t^2 Q dt = 0$, we have $\int_0^\infty f S f dt \geq 0$.*

Proof By Lemma 7.5, we have for some $f_0 \in \mathcal{F}$ with $f_0 > 0, \forall t > 0$ and $\int_0^\infty f_0^2 dt = 1$, it holds that

$$S f_0 = -e_0 f_0, e_0 > 0, \quad \text{and} \quad \int_0^\infty f S f dt \geq 0,$$
$$\text{for any } f \in \mathcal{F} \text{ with } \int_0^\infty f f_0 dt = 0. \tag{9.3}$$

By Proposition 9.1, we have for some $f_* \in \mathcal{F}$, $S f_* = t^2 Q$ and $\int_0^\infty f_* S f_* dt \leq 0$. Now we make the decomposition

$$\begin{cases} f = a_1 f_0 + f_1, & \int_0^\infty f_1 f_0 dt = 0, \quad a_1 = \int_0^\infty f f_0 dt; \\ f_* = a_2 f_0 + f_2, & \int_0^\infty f_2 f_0 dt = 0, \quad a_2 = \int_0^\infty f_* f_0 dt. \end{cases}$$

The fact $a_2 < 0$ follows from

$$-e_0 \int_0^\infty f_0 f_* dt = \int_0^\infty f_0 S f_* dt = \int_0^\infty f_0 t^2 Q dt > 0.$$

Using $\int_0^\infty (a_2 f - a_1 f_*) f_0 dt = 0$ and $\int_0^\infty f t^2 Q dt = 0$, we get

$$0 \leq \int_0^\infty (a_2 f - a_1 f_*) S (a_2 f - a_1 f_*) dt = a_2^2 \int_0^\infty f S f dt + a_1^2 \int_0^\infty f_* S f_* dt$$
$$\implies \quad \int_0^\infty f S f dt \geq -\frac{a_1^2}{a_2^2} \int_0^\infty f_* S f_* dt \geq 0.$$

□

Remark 9.3 In the preceding proof, the function f_* constructed in Proposition 9.1, which exhibits $\mathcal{O}(t^{-1})$ decay, plays an essential role. We wish to emphasize that this spatial decay property is, in fact, quite important. To illustrate, consider the seemingly natural alternative $\widetilde{f}_* = -J_{\alpha_1}$, where J_{α_1} is from Proposition 7.6. Noting that $S = -\partial_{tt} + \frac{2}{t^2} + 3tQ'Q$, we have

$$\begin{aligned} J''_{\alpha_1} &= (\tfrac{2}{t^2} + 3tQ'Q)J_{\alpha_1} + t^2 Q \\ \Rightarrow & S(-J_{\alpha_1}) = t^2 Q. \end{aligned} \tag{9.4}$$

Proposition 7.6 gives $J_{\alpha_1} > 0$ and $J_{\alpha_1} \sim t^2$ for $t \gg 1$, so

$$\int_0^\infty \widetilde{f}_* S \widetilde{f}_* \, dt = \int_0^\infty (-J_{\alpha_1}) \, t^2 Q \, dt < 0.$$

Now suppose $f \in C^\infty((0,\infty),\mathbb{R})$ satisfies $f(t) = \text{const} \cdot t^2 + \mathcal{O}(t^4)$ as $t \to 0^+$, and $|f(t)| \le C_2 e^{-c_3 t}$ for $t \ge 1$, with constants $C_2, c_3 > 0$. Assume also that $\int_0^\infty f\, t^2 Q \, dt = 0$. If one attempted to prove $\int_0^\infty f S f \, dt \ge 0$ using $\widetilde{f}_*$ following the argument in Proposition 9.2, a technical difficulty arises concerning the localization of J_{α_1}.

To clarify, we adopt the notation of Proposition 9.2. Let $N \ge 2$ (which will later be taken to infinity) and fix $\phi \in C_c^\infty(\mathbb{R})$ with $\phi(z) = 1$ for $|z| \le 1$ and $\phi(z) = 0$ for $|z| \ge 2$. We then have

$$\int_0^\infty f S f \, dt \ge -\frac{a_1^2}{a_{2,N}^2} \int_0^\infty f_N S f_N \, dt,$$

where $f_N(t) = \widetilde{f}_*(t)\,\phi\left(\frac{t}{N}\right)$ and $a_{2,N} = \int_0^\infty \widetilde{f}_*(t)\,\phi(\frac{t}{N}) f_0(t)\, dt$. Thanks to the exponential decay of f_0,

$$-e_0 a_{2,N} = \int_0^\infty S f_0 \, f_N \, dt = \int_0^\infty f_0 \, S f_N \, dt \to \int_0^\infty f_0 \, t^2 Q \, dt > 0, \quad \text{as } N \to \infty,$$

so $a_{2,N} \neq 0$ for sufficiently large N. On the other hand,

$$\begin{aligned} \int_0^\infty f_N S f_N \, dt &= \int_0^\infty J_{\alpha_1}\phi(\tfrac{t}{N}) S(J_{\alpha_1}\phi(\tfrac{t}{N}))dt \\ &= -\int_0^\infty J_{\alpha_1}\phi(\tfrac{t}{N})^2 t^2 Q \, dt \; + \int_0^\infty (J_{\alpha_1})^2 (\tfrac{1}{N}\phi'(\tfrac{t}{N}))^2 dt. \end{aligned}$$

While the first term is negative (and finite) for large N, the main issue lies in the second term. Because $J_{\alpha_1} \sim t^2$ as $t \to \infty$, we have $(J_{\alpha_1})^2 \sim t^4$, and consequently $\int_0^\infty f_N S f_N \, dt$ is not necessarily negative.

We are now ready to tackle the case $\ell = 1$. Consider the system for $F, G : [0,\infty) \to \mathbb{R}$:

$$\begin{cases} F'' = (\frac{2}{t^2} + 1 - \tau - 2Q^2)F - Q^2 G \\ G'' = (\frac{2}{t^2} + 1 + \tau - 2Q^2)G - Q^2 F. \end{cases} \tag{9.5}$$

Theorem 9.4 *Let $0 \neq \tau \in \mathbb{R}$. Assume $F, G \in H^2((0,\infty) \to \mathbb{R})$ and solve the system (9.5) on the interval $(0,\infty)$, and satisfy $F(t) = \mathrm{const} \cdot t^2 + \mathcal{O}(t^4)$, $G(t) = \mathrm{const} \cdot t^2 + \mathcal{O}(t^4)$ as $t \to 0^+$. Then $F \equiv 0$ and $G \equiv 0$.*

Remark 9.5 To see that the H^1 integrability of U and V suffices for the computation below, one can check

$$\int_0^\infty t(\tfrac{1}{2}(U')^2)'\chi(\tfrac{t}{N})dt = -\int_0^\infty \tfrac{1}{2}(U')^2\chi(\tfrac{t}{N})dt - \underbrace{\int_0^\infty (U')^2 \tfrac{1}{2}\tfrac{t}{N}\chi'(\tfrac{t}{N})dt}_{\text{Vanishes to zero as } N\to\infty}. \tag{9.6}$$

In the above, χ is a bump function such that $\chi(z) \equiv 1$ for $|z| \le 1$ and $\chi(z) \equiv 0$ for $|z| \ge 2$.

Proof of Theorem 9.4 Denote $U = F + G$ and $V = F - G$. Then

$$\begin{cases} U'' = (\frac{2}{t^2} + 1 - 3Q^2)U - \tau V \\ V'' = (\frac{2}{t^2} + 1 - Q^2)V - \tau U. \end{cases}$$

It is not difficult to check that (recall $t = |x|$)

$$L_-(x_1 Q) = -2\partial_1 Q, \quad L_+(\tfrac{U}{t}\tfrac{x_1}{t}) = \tau \tfrac{V}{t}\tfrac{x_1}{t}, \quad L_-(\tfrac{V}{t}\tfrac{x_1}{t}) = \tau \tfrac{U}{t}\tfrac{x_1}{t}.$$

Thus (note $\tau \neq 0$)

$$\begin{aligned} \int_{\mathbb{R}^3} \tfrac{U}{t}\tfrac{x_1}{t} x_1 Q dx &= \tfrac{1}{\tau}\int_{\mathbb{R}^3} L_-(\tfrac{V}{t}\tfrac{x_1}{t}) x_1 Q dx = -\tfrac{2}{\tau}\int_{\mathbb{R}^3} \tfrac{V}{t}\tfrac{x_1}{t}\partial_1 Q dx \\ &= -\tfrac{2}{\tau^2}\int_{\mathbb{R}^3} \tfrac{U}{t}\tfrac{x_1}{t} L_+(\partial_1 Q) dx = 0. \end{aligned}$$

This implies the orthogonality property $\int_0^\infty U(t)t^2 Q(t)dt = 0$ to be used below. By using the multiplier $\frac{1}{2}U + tU'$ for the U-equation and $\frac{1}{2}V + tV'$ for the V-equation (see Remark 9.5), we get

$$\int_0^\infty \big((U')^2 + (\tfrac{2}{t^2} + 3tQ'Q)U^2\big)dt + \int_0^\infty \big((V')^2 + (\tfrac{2}{t^2} + tQ'Q)V^2\big)dt = 0. \tag{9.7}$$

By Proposition 9.2 and noting $\int_0^\infty U t^2 Q dt = 0$, we have

$$\int_0^\infty \big((U')^2 + (\tfrac{2}{t^2} + 3tQ'Q)U^2\big)dt \ge 0.$$

By Lemma A.2, we have $2 + t^3 Q'Q > 0$, $\forall t > 0$. Thus $V \equiv 0$, $U \equiv 0$. □

Finally we settle the case $\boldsymbol{\ell \ge 2}$. It suffices to consider (below $c = \ell(\ell+1) \ge 6$)

$$\begin{cases} \partial_{tt} F = (\frac{c}{t^2} + 1 - \tau - 2Q^2)F - Q^2 G \\ \partial_{tt} G = (\frac{c}{t^2} + 1 + \tau - 2Q^2)G - Q^2 F. \end{cases}$$

Here $F(t)$, $G(t) = \mathcal{O}(t^3)$ as $t \to 0^+$, and F, $G \in H^k((0,\infty),\mathbb{R})$ for any integer $k \ge 1$.

Theorem 9.6 *Let $\tau \in \mathbb{R}$. We have $F \equiv 0$, $G \equiv 0$ on the whole interval $[0, \infty)$.*

Proof Denote $U = F + G$ and $V = F - G$. Then

$$U'' = (\tfrac{c}{t^2} + 1 - 3Q^2)U - \tau V, \qquad V'' = (\tfrac{c}{t^2} + 1 - Q^2)V - \tau U.$$

By using the multiplier $\frac{1}{2}U + tU'$ for the U-equation and $\frac{1}{2}V + tV'$ for the V-equation, we obtain

$$\int_0^\infty \Big((U')^2 + (\tfrac{c}{t^2} + 3tQ'Q)U^2\Big)dt + \int_0^\infty \Big((V')^2 + (\tfrac{c}{t^2} + tQ'Q)V^2\Big)dt = 0.$$

Lemma A.2 gives $2 + t^3 Q'Q > 0$, $\forall t > 0$. Since $c \geq 6$, it follows that $U \equiv 0$ and $V \equiv 0$. ☐

Appendix A: The ground state Q and the approximate ground state $\widetilde{Q}$

Denote for $t > 0$:

$$\begin{aligned}
q_1(t) &= -\frac{28t^{11}}{94{,}431} + \frac{127t^{10}}{76{,}892} - \frac{604t^9}{151{,}861} + \frac{293t^8}{59{,}051} - \frac{113t^7}{80{,}657} + \frac{202t^6}{73{,}305} - \frac{3t^5}{44{,}981} + \frac{54{,}169t^4}{401{,}949} \\
&\quad - \frac{t^3}{8{,}885{,}055} + \frac{127{,}023t^2}{185{,}578} + \frac{21{,}539}{93{,}423};\\
q_2(t) &= \frac{t^{11}}{1{,}183{,}575} - \frac{t^{10}}{120{,}831} + \frac{5t^9}{86{,}563} + \frac{2t^8}{74{,}523} - \frac{17t^7}{36{,}388} + \frac{3025t^6}{287{,}391} \\
&\quad - \frac{5162t^5}{329{,}873} + \frac{31{,}027t^4}{204{,}823} - \frac{1415t^3}{123{,}249} + \frac{295{,}367t^2}{428{,}350} - \frac{21t}{15{,}850} + \frac{18{,}176}{78{,}783};\\
p_2(t) &= q_2(t) + q_1(\tfrac{9}{10}) - q_2(\tfrac{9}{10}) + (q_1'(\tfrac{9}{10}) - q_2'(\tfrac{9}{10}))(t - \tfrac{9}{10});\\
g(t) &= t^{-1}\left(2e^t \mathrm{Ei}(-4t) - e^{-t}\mathrm{Ei}(-2t)\right), \quad (\text{for } x < 0,\ \mathrm{Ei}(x) := \int_{-\infty}^x \tfrac{1}{u}e^u du.)
\end{aligned} \tag{A.1}$$

The Costin-Huang-Schlag [20] approximate ground state $\widetilde{Q}$ is defined as

$$\widetilde{Q}(t) = \begin{cases} \frac{1}{q_1(t)}, & \text{if } t \leq \frac{9}{10};\\ \frac{1}{p_2(t)}, & \text{if } \frac{9}{10} < t \leq \frac{5}{2};\\ \frac{1}{t}\big(A_1 e^{-t} + B_1 t g(t)\big), & \text{if } t \geq \frac{5}{2}, \end{cases} \tag{A.2}$$

where $f_1(t) = \frac{1}{t}e^{-t}$,

$$A_1 = \frac{p_2\left(\frac{5}{2}\right)g'\left(\frac{5}{2}\right) + g\left(\frac{5}{2}\right)p_2'\left(\frac{5}{2}\right)}{p_2^2\left(\frac{5}{2}\right)\left(f_1\left(\frac{5}{2}\right)g'\left(\frac{5}{2}\right) - g\left(\frac{5}{2}\right)f_1'\left(\frac{5}{2}\right)\right)}, \qquad B_1 = -\frac{p_2\left(\frac{5}{2}\right)f_1'\left(\frac{5}{2}\right) + f_1(\frac{5}{2})p_2'\left(\frac{5}{2}\right)}{p_2^2\left(\frac{5}{2}\right)\left(f_1\left(\frac{5}{2}\right)g'\left(\frac{5}{2}\right) - g\left(\frac{5}{2}\right)f_1'\left(\frac{5}{2}\right)\right)}.$$

Lemma A.1 (Properties of $\widetilde{Q}$, [20]) *The approximate ground state $\widetilde{Q}$ satisfy the following*:

(1) *The function* $\widetilde{Q} \in C^1([0,\infty) \cap C_p^2([0,\infty))$, *and is decreasing for* $t \in [0, 5/2]$.
(2) $\forall t \geq 0$, *we have* $|Q(t) - \widetilde{Q}(t)| \leq \eta_0(t) := \frac{7}{10^5} \cdot \frac{1}{1+t} e^{-t}$, $|Q'(t) - \widetilde{Q}'(t)| \leq 5\eta_0(t)$.
(3) *We have* $\frac{187}{69} \cdot \frac{e^{-t}}{t} < \widetilde{Q}(t) < \frac{350}{129} \cdot \frac{e^{-t}}{t}$, $\forall t \geq \frac{5}{2}$. *Thus* $\frac{271}{100} \cdot \frac{e^{-t}}{t} < Q(t) < \frac{2714}{1000} \cdot \frac{e^{-t}}{t}$, $\forall t \geq \frac{5}{2}$.

Proof (1) and (3) are proved in Lemma 2.4 of [20]. The bound (2) is derived in Lemma 3.6 of [20]. □

Note that the polynomial function p_2 used in $\widetilde{Q}$ has rational coefficients. As a result, $\widetilde{Q}$ itself is a piecewise rational polynomial over the entire interval $[0, \frac{5}{2}]$. In particular $\widetilde{Q}(\frac{5}{2})$ and $\widetilde{Q}'(\frac{5}{2})$ are explicit rational numbers.

For our exact computations, we require a highly accurate polynomial approximation of Q on $[0, \frac{5}{2}]$. While such an approximation could be constructed from scratch, the existence of $\widetilde{Q}$ allows us to simply approximate $\widetilde{Q}$ instead. The polynomials below were obtained by approximating $\widetilde{Q}$ on various subintervals. The first one is Q_*^2, a degree-14 polynomial that approximates Q^2 remarkably well on $[0, \frac{8}{5}]$. Considerable care was taken to select a polynomial of relatively low degree, balancing accuracy with ease of presentation.

$$\begin{aligned} Q_*^2(t) = &-\frac{32{,}389t^{14}}{844} + \frac{692{,}573t^{13}}{1409} - \frac{14{,}647{,}839t^{12}}{5182} + \frac{13{,}634{,}891t^{11}}{1418} \\ &- \frac{31{,}307{,}644t^{10}}{1461} + \frac{22{,}954{,}676t^9}{703} - \frac{1{,}268{,}792t^8}{37} \\ &+ \frac{34{,}165{,}996t^7}{1415} - \frac{12{,}679{,}168t^6}{1223} + \frac{2{,}124{,}876t^5}{1117} + \frac{78{,}128t^4}{353} \\ &+ \frac{15{,}317t^3}{829} - \frac{138{,}192t^2}{1231} + \frac{15{,}295}{813}. \end{aligned} \tag{A.3}$$

The next polynomial W_1 is used[35] over the entire interval $[0, \frac{5}{2}]$.

$$\begin{aligned} W_1(t) = &-\frac{126{,}751t^{22}}{80{,}621{,}266} + \frac{998{,}813t^{21}}{24{,}102{,}701} - \frac{44{,}784{,}703t^{20}}{90{,}944{,}085} + \frac{532{,}787{,}760t^{19}}{154{,}914{,}809} - \frac{1{,}478{,}230{,}681t^{18}}{97{,}178{,}472} \\ &+ \frac{1{,}104{,}148{,}971t^{17}}{27{,}114{,}439} - \frac{6{,}386{,}998{,}649t^{16}}{165{,}889{,}120} \\ &- \frac{4{,}563{,}349{,}408t^{15}}{24{,}620{,}175} + \frac{12{,}295{,}664{,}789t^{14}}{11{,}848{,}354} - \frac{25{,}742{,}069{,}395t^{13}}{8{,}929{,}817} + \frac{22{,}711{,}315{,}187t^{12}}{4{,}190{,}364} \\ &- \frac{21{,}505{,}639{,}879t^{11}}{2{,}916{,}474} + \frac{32{,}713{,}853{,}029t^{10}}{4{,}462{,}850} \\ &- \frac{12{,}946{,}708{,}519t^9}{2{,}496{,}198} + \frac{3{,}403{,}022{,}733t^8}{1{,}411{,}424} - \frac{6{,}729{,}001{,}962t^7}{11{,}651{,}297} - \frac{626{,}483{,}074t^5}{65{,}670{,}795} \\ &+ \frac{752{,}248{,}559t^4}{20{,}846{,}520} - \frac{89{,}936{,}485t^2}{6{,}984{,}383} + \frac{93{,}423}{21{,}539}. \end{aligned} \tag{A.4}$$

[35] In principle, W_1^2 could serve as a replacement for Q_*^2 on $[0, \frac{8}{5}]$. However, its relatively high degree makes it computationally less favorable. In contrast, Q_*^2 benefits from a modest degree, which significantly accelerates subsequent calculations.

Finally, a low-degree polynomial W_2 is introduced to approximate Q on $[\frac{5}{2}, \frac{7}{2}]$ for use in the $l = 1$ analysis.

$$W_2(t) = \frac{157t^8}{1{,}255{,}706} - \frac{5215t^7}{1{,}535{,}379} + \frac{25{,}008t^6}{610{,}441} - \frac{1{,}118{,}777t^5}{3{,}886{,}821} + \frac{1{,}230{,}788t^4}{949{,}047} - \frac{784{,}873t^3}{202{,}656} + \frac{10{,}232{,}110t^2}{1{,}348{,}117} - \frac{2{,}084{,}185t}{228{,}972} + \frac{43{,}251}{8233}. \tag{A.5}$$

The following lemma assembles the key quantitative estimates for Q and its polynomial approximations. The analysis relies primarily on Lemma 1.4 for bounding polynomials and Corollary 1.5 for bounding rational polynomials.

Lemma A.2 (Polynomial approximation of Q) *The following hold.*

(1) $Q_*^2 > 0$, $\frac{d}{dt}(Q_*^2) \le 0$, $\forall t \in [0, \frac{8}{5}]$. *Also* $\max_{0\le t\le 8/5} |Q^2 - Q_*^2| \le \frac{75}{10^5}$.

(2) $\max_{0\le t\le 2.5} |Q - W_1| \le \frac{75}{10^6}$, $\max_{0\le t\le 2.5} |Q' - W_1'| \le \frac{42}{10^5}$. *Furthermore*

$$\max_{1.8\le t\le 2.5} |Q - W_1| \le \frac{6}{10^6}, \qquad \max_{1.8\le t\le 2.5} |Q' - W_1'| \le \frac{5}{10^5}.$$

(3) $\max_{2.5\le t\le 3.5} |Q - W_2| \le \frac{5}{10^6}$, $\max_{2.5\le t\le 3.5} |Q' - W_2'| \le \frac{156}{10^7}$.

(4) $2 + 3t^3 Q'Q > 0$, $\forall t \ge 1.7$ *and* $2 + 3t^3 Q'Q \ge 1.8525$, $\forall t \ge 3.5$. *Also* $2 + t^3 QQ' > 0, \forall t > 0$.

(5) $-Q'/Q < \frac{2714}{2710} + \frac{1}{t}$ *for* $t \ge 5/2$.

Proof (1) The estimates $Q_*^2 > 0$, $\frac{d}{dt}(Q_*^2) \le 0$, $\forall t \in [0, 8/5]$ follow from Lemma 1.4:

$$\mathcal{M}(Q_*^2, [0, \tfrac{8}{5}], \tfrac{1}{10^4}) > 0, \quad \mathcal{M}(\tfrac{-\partial_t(Q_*^2)}{t}, [0, \tfrac{8}{5}], \tfrac{1}{10^4}) > 0. \tag{A.6}$$

By Lemma A.1, we have $|Q - \widetilde{Q}| \le \frac{7}{10^5}$. The desired inequality $\max_{0\le t\le 8/5} |Q^2 - Q_*^2| \le \frac{75}{10^5}$ follows easily from the inequalities (note $\min_{0\le t\le 8/5} \widetilde{Q} = \widetilde{Q}(\frac{8}{5}) > \frac{7}{10^5}$)

$$\sup_{0\le t\le 8/5} |Q_*^2 - (\widetilde{Q} - \tfrac{7}{10^5})^2| \le \tfrac{75}{10^5}, \qquad \sup_{0\le t\le 8/5} |Q_*^2 - (\widetilde{Q} + \tfrac{7}{10^5})^2| \le \tfrac{75}{10^5}. \tag{A.7}$$

Recall $1/\widetilde{Q} = q_1$ on $[0, \frac{9}{10}]$ and $1/\widetilde{Q} = p_2$ on $[\frac{9}{10}, \frac{5}{2}]$. To show (A.7), it suffices to prove

$$\Big\| \frac{q_1^2 Q_*^2 - (1 - 7\cdot 10^{-5} q_1)^2}{q_1^2} \Big\|_{L^\infty([0, \frac{9}{10}])} \le \frac{75}{10^5},$$
$$\Big\| \frac{p_2^2 Q_*^2 - (1 - 7\cdot 10^{-5} p_2)^2}{p_2^2} \Big\|_{L^\infty([\frac{9}{10}, \frac{8}{5}])} \le \frac{75}{10^5}; \tag{A.8}$$

$$\Big\| \frac{q_1^2 Q_*^2 - (1 + 7\cdot 10^{-5} q_1)^2}{q_1^2} \Big\|_{L^\infty([0, \frac{9}{10}])} \le \frac{75}{10^5},$$
$$\Big\| \frac{p_2^2 Q_*^2 - (1 + 7\cdot 10^{-5} p_2)^2}{p_2^2} \Big\|_{L^\infty([\frac{9}{10}, \frac{8}{5}])} \le \frac{75}{10^5}. \tag{A.9}$$

(A.8)–(A.9) follow from applying Corollary 1.5 with $\mathsf{h} = \frac{1}{10^4}$.

(2) The desired inequalities follow from $|Q-\widetilde{Q}|\le\frac{7}{10^5}$, $|Q'-\widetilde{Q}'|\le\frac{35}{10^5}$ in Lemma A.1, and

$$\max_{0\le t\le 2.5}|\widetilde{Q}-W_1|\le\frac{32}{10^7},\qquad \max_{0\le t\le 2.5}|\widetilde{Q}'-W_1'|\le\frac{57}{10^6}.$$

Since $1/\widetilde{Q}=q_1$ on $[0,\frac{9}{10}]$ and $1/\widetilde{Q}=p_2$ on $[\frac{9}{10},\frac{5}{2}]$, it suffices to show

$$\left\|\frac{1-q_1W_1}{q_1}\right\|_{L^\infty([0,\frac{9}{10}])}\le\frac{32}{10^7},\qquad \left\|\frac{1-p_2W_1}{p_2}\right\|_{L^\infty([\frac{9}{10},\frac{5}{2}])}\le\frac{32}{10^7};\tag{A.10}$$

$$\left\|\frac{q_1'+q_1^2W_1'}{q_1^2}\right\|_{L^\infty([0,\frac{9}{10}])}\le\frac{57}{10^6},\qquad \left\|\frac{p_2'+p_2^2W_1'}{p_2^2}\right\|_{L^\infty([\frac{9}{10},\frac{5}{2}])}\le\frac{57}{10^6}.\tag{A.11}$$

All four inequalities follow from Corollary 1.5: the first inequality in (A.11) uses $\mathsf{h}=10^{-5}$, the rest $\frac{1}{10^4}$.

Refined bounds for W_1-Q on $[1.8,2.5]$ follow from[36] $|Q-\widetilde{Q}|<\frac{42}{10^7}$ and $|Q'-\widetilde{Q}'|<\frac{21}{10^6}$ (Lemma A.1), and

$$\max_{1.8\le t\le 2.5}|\widetilde{Q}-W_1|\le\frac{18}{10^7},\qquad \max_{1.8\le t\le 2.5}|\widetilde{Q}'-W_1'|\le\frac{29}{10^6}.\tag{A.12}$$

Since $1/\widetilde{Q}=p_2$ on $[\frac{9}{10},\frac{5}{2}]$, it suffices to show

$$\left\|\frac{1-p_2W_1}{p_2}\right\|_{L^\infty([\frac{9}{5},\frac{5}{2}])}\le\frac{18}{10^7},\qquad \left\|\frac{p_2'+p_2^2W_1'}{p_2^2}\right\|_{L^\infty([\frac{9}{5},\frac{5}{2}])}\le\frac{29}{10^6}.\tag{A.13}$$

Both follow from Corollary 1.5 with $\mathsf{h}=\frac{1}{10^4}$.

(3) We first observe[37] that $|Q(\frac{5}{2})-\widetilde{Q}(\frac{5}{2})|\le\frac{165}{10^8}$, $|Q(\frac{7}{2})-\widetilde{Q}(\frac{7}{2})|\le\frac{47}{10^8}$, so that

$$\max\{|Q(\tfrac{5}{2})-W_2(\tfrac{5}{2})|,\ |Q(\tfrac{7}{2})-W_2(\tfrac{7}{2})|\}\le\frac{18}{10^7}.$$

Denote $F:=W_2''+\frac{2}{t}W_2'-W_2+W_2^3$. Corollary 1.5 with $f=tF$, $g=t$, $\alpha=\frac{48}{10^7}$, $\mathsf{h}=\frac{1}{10^4}$ yields

$$\max_{5/2\le t\le 7/2}|F(t)|\le\frac{48}{10^7}.\tag{A.14}$$

To show $\max_{5/2\le t\le 7/2}|Q-W_2|\le\frac{5}{10^6}$, we run a maximum principle argument as follows. Firstly, we note that $0<W_2(t)<\frac{9}{100}$, $\forall t\in[5/2,7/2]$. This follows from Lemma 1.4 via checking:

$$\mathcal{M}(W_2,[\tfrac{5}{2},\tfrac{7}{2}],\tfrac{1}{10^4})>0,\qquad \mathcal{M}(\tfrac{9}{100}-W_2,[\tfrac{5}{2},\tfrac{7}{2}],\tfrac{1}{10^4})>0.\tag{A.15}$$

Note that $Q(t)\le Q(5/2)\le 9/100$ (Lemma A.1), $\forall t\in[5/2,7/2]$. Thus

$$\max\{3W_2(t)^2,\ 3Q(t)^2\}\le\frac{243}{10{,}000},\qquad \forall t\in[\tfrac{5}{2},\tfrac{7}{2}].$$

[36] Recall $\eta_0=7\cdot 10^{-5}\frac{1}{1+t}e^{-t}$ in Lemma A.1 and note that $\frac{1}{1+1.8}e^{-1.8}<0.06$.

[37] Note that in this part we only appeal to the estimate of $\widetilde{Q}(5/2)$, $\widetilde{Q}'(5/2)$ and $\widetilde{Q}(7/2)$.

Observe that

$$W_2'' + \frac{2}{t}W_2' - W_2 + W_2^3 = F, \qquad \max_{5/2\le t\le 7/2} |F(t)| \le \frac{48}{10^7};$$

$$Q'' + \frac{2}{t}Q' - Q + Q^3 = 0.$$

Let $\eta = Q - W_2$. Then $\max\{|\eta(5/2)|, |\eta(7/2)|\} \le \frac{18}{10^7}$ and

$$-\eta'' - \frac{2}{t}\eta' + \eta - 3W_2^2\eta - 3W_2\eta^2 - \eta^3 = F. \tag{A.16}$$

Now suppose for some $t_* \in (\frac{5}{2}, \frac{7}{2})$, $|\eta(t_*)|$ attains the maximum value $\max_{\frac{5}{2}\le t\le\frac{7}{2}} |\eta(t)|$. We discuss two cases.

Case 1: $\eta(t_*) = \max_{\frac{5}{2}\le t\le\frac{7}{2}} |\eta(t)|$. Then

$$\begin{aligned}(1 - 3W_2(t_*)^2)\eta(t_*) &\le 3W_2(t_*)\eta(t_*)\eta(t_*) + \eta(t_*)^3 + F(t_*)\\ &\le 3W_2(t_*)\frac{9}{100}\eta(t_*) + (\frac{9}{100})^2\eta(t_*) + F(t_*),\end{aligned} \tag{A.17}$$

where we used $\eta(t_*) = Q(t_*) - W_2(t_*) \le Q(t_*) \le 9/100$. This clearly yields a (weak) bound

$$\max_{\frac{5}{2}\le t\le\frac{7}{2}} |\eta(t)| \le \frac{48\cdot 10^{-7}}{1-2\cdot\frac{243}{10{,}000} - \frac{81}{10{,}000}} < \frac{51}{10^7}. \tag{A.18}$$

By using this bound, we estimate again

$$(1 - 3W_2(t_*)^2)\eta(t_*) \le 3\cdot\frac{9}{100}\cdot(\frac{51}{10^7})^2 + (\frac{51}{10^7})^3 + \frac{48}{10^7}.$$

This implies $\max_{\frac{5}{2}\le t\le\frac{7}{2}} |\eta(t)| \le \frac{5}{10^6}$.

Case 2: $\eta(t_*) = -\max_{\frac{5}{2}\le t\le\frac{7}{2}} |\eta(t)|$. Let $\widetilde{\eta} = -\eta$. Clearly

$$-\widetilde{\eta}'' - \frac{2}{t}\widetilde{\eta}' + \widetilde{\eta} - 3W_2^2\widetilde{\eta} + 3W_2\widetilde{\eta}^2 - \widetilde{\eta}^3 = -F.$$

It follows that

$$\widetilde{\eta}(t_*) \le \frac{48\cdot 10^{-7}}{1 - \frac{243}{10{,}000} - \frac{81}{10{,}000}} < \frac{5}{10^6}. \tag{A.19}$$

Thus $\max_{\frac{5}{2}\le t\le\frac{7}{2}} |\eta(t)| \le \frac{5}{10^6}$.

Next we estimate $|Q' - W_2'| = |\eta'|$ on $[5/2, 7/2]$.

Note $|\eta'(\frac{5}{2})| \le |Q'(\frac{5}{2}) - \widetilde{Q}'(\frac{5}{2})| + |\widetilde{Q}'(\frac{5}{2}) - W_2'(\frac{5}{2})| \le \frac{83}{10^7}$ (Lemma A.1). Observe that

$$(t^2\eta')' = t^2(h(Q) - h(W_2)) - t^2F(t), \qquad h(z) := z - z^3;$$

$$|\eta'(t)| \le |\eta'(\tfrac{5}{2})| + t^{-2}\int_{\frac{5}{2}}^t s^2 ds \max_{\frac{5}{2}\le t\le\frac{7}{2}} |h(Q) - h(W_2)| + t^{-2}\int_{\frac{5}{2}}^t s^2 ds \max_{\frac{5}{2}\le t\le\frac{7}{2}} |F(t)|$$

$$\leq \frac{83}{10^7} + (\tfrac{t}{3} - \tfrac{125}{24t^2})\Big|_{t=\frac{7}{2}} \Big(\max_{\frac{5}{2}\leq t\leq\frac{7}{2}} |\eta(t)| + \max_{\frac{5}{2}\leq t\leq\frac{7}{2}} |F(t)| \Big)$$

$$\leq \frac{83}{10^7} + \frac{109}{147}(\tfrac{5}{10^6} + \tfrac{48}{10^7}) \leq \frac{156}{10^7}, \tag{A.20}$$

where in the second inequality we used that $t^{-2}\int_{5/2}^t s^2 ds = \frac{t}{3} - \frac{125}{24t^2}$ increases for $t \geq \frac{5}{2}$, and $|h(Q) - h(W_2)| \leq |Q - W_2| = |\eta|$ by the mean value theorem ($h'(z) = 1 - 3z^2 \in (0, 1)$ for z between W_2 and Q).

(4) Using $(t^3 Q')' = t^2 Q' + t^3(1 - Q^2)Q$, we write for $t \geq 1.7$:

$$t^3 Q' = \underbrace{1.7^3 Q'(1.7) - 1.7^2 Q(1.7)}_{\geq -3.1} + t^2 Q + \int_{1.7}^t \Big(\underbrace{s^3(1 - Q^2) - 2s}_{>0}\Big) Q ds$$

$$\implies \quad 2 + 3t^3 Q' Q \geq 2 + (-3.1 + t^2 Q) \cdot 3Q \geq 2 - 9.3Q + 8.67Q^2 > 0,$$
$$\forall t \geq 1.7. \tag{A.21}$$

The inequality for $t \geq 3.5$ follows similarly:[38]

$$t^3 Q' = \underbrace{3.5^3 Q'(3.5) - 3.5^2 Q(3.5)}_{>-2} + t^2 Q + \int_{3.5}^t \Big(\underbrace{s^3(1 - Q^2) - 2s}_{>0}\Big) Q ds$$

$$\implies 2 + 3t^3 Q' Q > 2 - 6Q(\tfrac{7}{2}) \geq 2 - 6(W_2(\tfrac{7}{2}) + 156 \cdot 10^{-7}) > 1.8525. \tag{A.22}$$

The inequality $2 + t^3 Q Q' \geq 2 + t^3(W_1 + \frac{75}{10^6})(W_1' - \frac{42}{10^5}) > 0$ follows from Lemma 1.4:

$$\mathcal{M}(2 + t^3(W_1 + \tfrac{75}{10^6})(W_1' - \tfrac{42}{10^5}), [0, \tfrac{17}{10}], \tfrac{1}{10^4}) > 0. \tag{A.23}$$

(5) Using $-tQ' = Q + \int_t^\infty s(Q - Q^3) ds < Q + \int_t^\infty sQ ds$, we get
$-t\frac{Q'(t)}{Q(t)} < 1 + Q(t)^{-1}\int_t^\infty sQ ds < 1 + \frac{2714}{2710}t.$ (By Lemma A.1).

□

Appendix B: Verification of condition (4.15) in Theorem 4.5

In this section, we take the parameters A, T, $\beta(t)$ according to Table 1 and prove the inequalities:

$$f_1(t) > 0, \quad f_1(t) f_2(t) - \tfrac{1}{2} A Q(t)^4 > 0, \qquad \forall t \geq T, \tag{B.1}$$

where

$$f_1 = \tau - 1 + 2Q^2 + 4\tfrac{\beta}{\alpha} Q' Q, \qquad f_2 = \tau + 1 - 2Q^2 - \tfrac{1}{4}(\tfrac{\alpha'}{\alpha})^2 - \tfrac{1}{2A}(\tfrac{\beta}{\alpha})^2 Q^4.$$

[38] By (3) of this lemma, $(\frac{7}{2})^3(W_2'(\frac{7}{2}) - \frac{156}{10^7}) - (\frac{7}{2})^2(W_2(\frac{7}{2}) + \frac{5}{10^6}) > -2$.

We discuss 5 cases. For convenience of notation, we shall denote (see Lemma A.2)

$$P_1 = 2(W_1 - \tfrac{75}{10^6})^2, \qquad P_2 = (W_1 + \tfrac{75}{10^6})(W_1' - \tfrac{42}{10^5}); \tag{B.2}$$

$$P_3 = 2(W_1 + \tfrac{75}{10^6})^2, \qquad P_4 = (W_1 + \tfrac{75}{10^6})^4. \tag{B.3}$$

It is clear that

$$0 < P_1 \le 2Q^2 \le P_3, \qquad P_2 \le Q'Q \le 0, \qquad \forall t \in [0,\ \tfrac{5}{2}].$$

1) $\mathbf{1 \le \tau \le 1.2}$. Note $T = \frac{16}{10}$, $A = \frac{3}{10^4}$, $\beta = e^{2.6t} - 63$ and $2 - \frac{1}{4}(\frac{\alpha'}{\alpha})^2 = \frac{31}{10^2}$. Consider first the regime $t \ge 3.5$. Lemma A.2 yields $Q'/Q > -1.2872$ and

$$\begin{aligned}
&\tfrac{1}{2Q^2} f_1 > 1 + (\tfrac{10}{13} - \tfrac{630}{13}e^{-2.6t})(-\tfrac{12{,}872}{10^4}) > \tfrac{98}{10^4}, \quad \forall t \ge \tfrac{7}{2};\\
&\tfrac{1}{2Q^2} f_1 f_2 - \tfrac{1}{4}AQ^2 > \tfrac{98}{10^4}\cdot(\tfrac{31}{10^2} - 2Q(\tfrac{7}{2})^2 - \tfrac{1}{2A\cdot 2.6^2}Q(\tfrac{7}{2})^4) - \tfrac{1}{4}AQ(\tfrac{7}{2})^2 > 0,\\
&\forall t \ge \tfrac{7}{2}.
\end{aligned} \tag{B.4}$$

Next for $t \in [\mathbf{2.5},\ \mathbf{3.5}]$ we have

$$f_2 \ge \tfrac{31}{10^2} - 2Q(\tfrac{5}{2})^2 - \tfrac{1}{2A}(\tfrac{1}{2.6})^2 Q(\tfrac{5}{2})^4 > \tfrac{27}{10^2} \tag{B.5}$$

and (see Lemma A.2 for W_2)

$$\begin{aligned}
\tfrac{1}{2Q} f_1 - \tfrac{A}{4f_2}Q^3 &> Q + 2\cdot(\tfrac{10}{26} - \tfrac{630}{26}e^{-2.6t})Q' - \tfrac{AQ(2.5)^3}{4\cdot 0.27}\\
&> \underbrace{W_2 - \tfrac{5}{10^6} + 2(\tfrac{10}{26} - \tfrac{630}{26}a)(W_2' - \tfrac{156}{10^7}) - \tfrac{2}{10^7}}_{p}\\
&\ge \mathcal{M}(p, [\tfrac{5}{2},\ \tfrac{7}{2}],\ \tfrac{1}{10^3}) > 0,
\end{aligned} \tag{B.6}$$

where in the last step we used Lemma 1.4. In (B.6), we used

$$e^{-2.6t} = e^{-7.8}e^{-2.6(t-3)} > a(t) := \tfrac{1}{2441}\textstyle\sum_{k=0}^{3}\tfrac{1}{k!}(-\tfrac{13}{5}(t-3))^k, \qquad \forall t \in [\tfrac{5}{2},\ \tfrac{7}{2}].$$

Now for $t \in [\mathbf{T},\ \mathbf{2.5}]$, using $e^{-2.6t} > a_1(t) := \frac{73}{15{,}070}\sum_{k=0}^{5}\frac{1}{k!}(-\frac{13}{5}(t - \frac{41}{20}))^k$ and Lemma A.2, we have

$$f_1 > P_1 + (\tfrac{10}{26} - \tfrac{630}{26}a_1)4P_2 =: Z_1; \tag{B.7}$$

$$f_2 > \tfrac{31}{100} - P_3 - \tfrac{1}{2A}(\tfrac{10}{26} - \tfrac{630}{26}a_1)^2 P_4 =: Z_2. \tag{B.8}$$

Since $\frac{1}{2}AQ(\frac{8}{5})^4 < \frac{3}{10^6}$, clearly (B.1) follows from

$$\mathcal{M}(Z_1, [\tfrac{8}{5},\ \tfrac{5}{2}],\ \tfrac{15}{10^6}) > \tfrac{5}{2}\cdot\tfrac{1}{10^4}, \quad \mathcal{M}(Z_2, [\tfrac{8}{5},\ \tfrac{5}{2}],\ \tfrac{2}{10^5}) > \tfrac{2}{10^2}. \tag{B.9}$$

2) $\mathbf{1.2 \le \tau \le 2.1}$. Note $T = \frac{11}{10}$, $A = \frac{2}{10^3}$, $\beta = e^{1.4t} - \frac{461}{10^2}$ and $\frac{22}{10} - \frac{1}{4}(\frac{\alpha'}{\alpha})^2 = \frac{171}{10^2}$.

The regime $t \geq \frac{5}{2}$ is OK since Lemma A.2 yields $Q'/Q > -\frac{380}{271}$ and (observe $\frac{1}{2}AQ(\frac{5}{2})^4 < \frac{1}{10^4}$)

$$f_1 > \frac{2}{10} + 2Q(\tfrac{5}{2})^2 + \frac{40}{14} \cdot (-\tfrac{380}{271}) \cdot Q(\tfrac{5}{2})^2 > \frac{1}{10}, \qquad \forall t \geq \tfrac{5}{2};$$
$$f_2 > \frac{171}{10^2} - 2Q(\tfrac{5}{2})^2 - \frac{1}{2A} \cdot (\tfrac{10}{14})^2 \cdot Q(\tfrac{5}{2})^4 > 1, \qquad \forall t \geq \tfrac{5}{2}. \tag{B.10}$$

Now for $t \in [\mathbf{T}, \mathbf{2.5}]$, using $e^{-1.4t} > a_2(t) := \frac{8}{100}\sum_{k=0}^{3}\frac{1}{k!}(-\frac{7}{5}(t - \frac{9}{5}))^k$ and Lemma A.2, we have

$$f_1 > \frac{1}{5} + P_1 + (\tfrac{10}{14} - \tfrac{461}{140}a_2)4P_2 =: Z_3;$$
$$f_2 > \frac{171}{100} - P_3 - \frac{1}{2A}(\tfrac{10}{14} - \tfrac{461}{140}a_2)^2 P_4 =: Z_4.$$

Since $\frac{1}{2}AQ(\frac{11}{10})^4 < \frac{1}{10^3}$, clearly (B.1) follows from

$$\mathcal{M}(Z_3, [\tfrac{11}{10}, \tfrac{5}{2}], \tfrac{2}{10^4}) > \tfrac{1}{10}, \quad \mathcal{M}(Z_4, [\tfrac{11}{10}, \tfrac{5}{2}], \tfrac{1}{10^4}) > \tfrac{2}{100}. \tag{B.11}$$

3) $\mathbf{2.1 \leq \tau \leq 5.4}$. Note $T = \frac{856}{10^3}$, $A = \frac{1}{10^2}$, $\beta = t - \frac{845}{10^3}$. The regime $t \geq \frac{5}{2}$ is OK since Lemma A.2 yields $Q'/Q > -\frac{380}{271}$ and (observe $\frac{1}{2}AQ(\frac{5}{2})^4 < \frac{1}{10^4}$ and $(tQ)' < 0$ for $t \geq \frac{5}{2}$)

$$f_1 > \frac{11}{10} + 2Q(\tfrac{5}{2})^2 + 4 \cdot (-\tfrac{380}{271}) \cdot \tfrac{5}{2} \cdot Q(\tfrac{5}{2})^2 > 1, \qquad \forall t \geq \tfrac{5}{2};$$
$$f_2 > \frac{31}{10} - 2Q(\tfrac{5}{2})^2 - \frac{1}{2A} \cdot (\tfrac{5}{2})^2 \cdot Q(\tfrac{5}{2})^4 > 1, \qquad \forall t \geq \tfrac{5}{2}. \tag{B.12}$$

Now for $t \in [\mathbf{T}, \mathbf{2.5}]$, using Lemma A.2, we have

$$f_1 > \frac{11}{10} + P_1 + 4\beta \cdot P_2 =: Z_5; \tag{B.13}$$
$$f_2 > \frac{31}{10} - P_3 - \frac{1}{2A}\beta^2 \cdot P_4 =: Z_6.$$

Note that $\frac{1}{2}AQ(\frac{91}{100})^4 < \frac{9}{10^3}$ and $\frac{1}{2}AQ(T)^4 < \frac{12}{10^3}$. Then (B.1) follows from:

$$\mathcal{M}(Z_5, [\tfrac{91}{100}, \tfrac{5}{2}], \tfrac{15}{10^5}) > \tfrac{7}{10}, \qquad \mathcal{M}(Z_6, [\tfrac{91}{100}, \tfrac{5}{2}], \tfrac{8}{10^5}) > \tfrac{16}{10^2}; \tag{B.14}$$
$$\mathcal{M}(Z_5, [T, \tfrac{91}{100}], \tfrac{2}{10^6}) > 3, \qquad \mathcal{M}(Z_6, [T, \tfrac{91}{100}], \tfrac{9}{10^7}) > \tfrac{41}{10^4}. \tag{B.15}$$

4) $\mathbf{5.4 \leq \tau \leq 12.4}$. Note $T = \frac{2}{3}$, $A = \frac{2}{10^2}$, $\beta = t - \frac{655}{10^3}$. The regime $t \geq \frac{5}{2}$ is OK similar to (B.12). On the other hand for $t \in [\mathbf{T}, \mathbf{2.5}]$, by Lemma A.2 we have

$$f_1 > \frac{44}{10} + P_1 + 4\beta \cdot P_2 =: Z_8;$$
$$f_2 > \frac{64}{10} - P_3 - \frac{1}{2A}\beta^2 \cdot P_4 =: Z_9.$$

Note that $\frac{1}{2}AQ(\frac{68}{100})^4 < \frac{1}{10}$ and $\frac{1}{2}AQ(T)^4 < \frac{12}{100}$. Then (B.1) follows from:

$$\mathcal{M}(Z_8, [\tfrac{68}{100}, \tfrac{5}{2}], \tfrac{52}{10^5}) > 3, \qquad \mathcal{M}(Z_9, [\tfrac{68}{100}, \tfrac{5}{2}], \tfrac{8}{10^5}) > \tfrac{8}{10^2}; \tag{B.16}$$
$$\mathcal{M}(Z_8, [T, \tfrac{68}{100}], \tfrac{1}{9 \cdot 10^5}) > \tfrac{98}{10}, \qquad \mathcal{M}(Z_9, [T, \tfrac{68}{100}], \tfrac{1}{12 \cdot 10^5}) > \tfrac{14}{10^3}. \tag{B.17}$$

5) **12.4 ≤ τ ≤ 19.2**. Note $T = \frac{475}{10^3}$, $A = \frac{32}{10^3}$, $\beta = t - \frac{46}{10^2}$. The regime $t \geq 2.5$ is OK similar to (B.12). On the other hand, for $t \in [\mathbf{T},\ \mathbf{2.5}]$, by Lemma A.2 we have

$$f_1 > \frac{114}{10} + P_1 + 4\beta \cdot P_2 =: Z_{11};$$
$$f_2 > \frac{134}{10} - P_3 - \frac{1}{2A}\beta^2 \cdot P_4 =: Z_{12}.$$

Note that $\frac{1}{2}AQ(\frac{68}{100})^4 < \frac{2}{10}$ and $\frac{1}{2}AQ(T)^4 < \frac{68}{10^2}$. Then (B.1) follows from:

$$\mathcal{M}(Z_{11}, [\tfrac{68}{100},\ \tfrac{5}{2}], \tfrac{2}{10^4}) > \tfrac{95}{10}, \qquad \mathcal{M}(Z_{12}, [\tfrac{68}{100},\ \tfrac{5}{2}], \tfrac{1}{10^4}) > \tfrac{5}{10^2}; \tag{B.18}$$
$$\mathcal{M}(Z_{11}, [T,\ \tfrac{68}{100}], \tfrac{1}{10^5}) > \tfrac{123}{10}, \qquad \mathcal{M}(Z_{12}, [T,\ \tfrac{68}{100}], \tfrac{1}{10^5}) > \tfrac{59}{10^3}. \tag{B.19}$$

Appendix C: Auxiliary estimates for the case $\tau \in [1,\ 12.4]$

For $\tau \in [5.4,\ 12.4]$, we define (we adopt the same notation as in (4.9)–(4.11))

$$\begin{aligned}
\widetilde{U}_a(\tau,t) &= \frac{250{,}858t^{10}}{4155} - \frac{42{,}245t^9}{226} + \frac{7428t^8}{35} - \frac{8522t^7}{109} - \frac{9443t^6}{240}\\
&\quad + \frac{11{,}239t^5}{283} - \frac{698t^4}{265} - \frac{1186t^3}{203} - \frac{10t^2}{749} + t\\
&\quad + \Big(\frac{5t^{10}}{402} - \frac{14t^9}{341} + \frac{7t^8}{135} - \frac{5t^7}{214} - \frac{t^6}{96} + \frac{t^5}{81} - \frac{t^4}{1295} + \frac{t^3}{12{,}292}\Big)\tau^2\\
&\quad + \Big(\frac{303t^{10}}{272} - \frac{642t^9}{179} + \frac{843t^8}{199} - \frac{111t^7}{68} - \frac{207t^6}{244}\\
&\quad + \frac{584t^5}{629} - \frac{26t^4}{435} - \frac{48t^3}{299} - \frac{t^2}{3075}\Big)\tau;\\
\widetilde{V}_a(\tau,t) &= -\frac{13{,}012t^{10}}{297} + \frac{50{,}423t^9}{387} - \frac{35{,}597t^8}{261} + \frac{11{,}461t^7}{333} + \frac{13{,}129t^6}{321}\\
&\quad - \frac{20{,}636t^5}{669} + \frac{431t^4}{177} + \frac{255t^3}{88} + \frac{2t^2}{167}\\
&\quad + \Big(\frac{t^{10}}{5317} - \frac{t^9}{336} + \frac{4t^8}{435} - \frac{t^7}{90} + \frac{3t^6}{389} - \frac{t^5}{459} + \frac{t^4}{2762} - \frac{t^3}{28{,}556}\Big)\tau^2\\
&\quad + \Big(\frac{15t^{10}}{256} - \frac{99t^9}{232} + \frac{493t^8}{450} - \frac{534t^7}{389} + \frac{248t^6}{285} - \frac{25t^5}{102} + \frac{7t^4}{172} - \frac{t^3}{255} + \frac{t^2}{4980}\Big)\tau;\\
\widetilde{U}_b(\tau,t) &= -\frac{3355t^{10}}{78} + \frac{51{,}257t^9}{401} - \frac{55{,}000t^8}{413} + \frac{7825t^7}{244} + \frac{12{,}499t^6}{298}\\
&\quad - \frac{11{,}800t^5}{379} + \frac{236t^4}{95} + \frac{1727t^3}{597} + \frac{4t^2}{327}\\
&\quad + \Big(-\frac{t^{10}}{167} + \frac{5t^9}{288} - \frac{5t^8}{249} + \frac{5t^7}{466} - \frac{t^6}{503} + \frac{t^5}{2013} - \frac{t^4}{10{,}812} + \frac{t^3}{94{,}132}\Big)\tau^2\\
&\quad + \Big(\frac{103t^{10}}{334} - \frac{122t^9}{165} + \frac{95t^8}{201} + \frac{35t^7}{172} - \frac{115t^6}{337} + \frac{23t^5}{240} - \frac{6t^4}{409} + \frac{t^3}{805} - \frac{t^2}{18{,}830}\Big)\tau;\\
\widetilde{V}_b(\tau,t) &= \frac{18{,}556t^{10}}{313} - \frac{107{,}695t^9}{587} + \frac{66{,}232t^8}{319} - \frac{23{,}047t^7}{308} - \frac{13{,}337t^6}{328}\\
&\quad + \frac{7012t^5}{175} - \frac{167t^4}{62} - \frac{1249t^3}{214} - \frac{4t^2}{293} + t\\
&\quad + \Big(\frac{14t^{10}}{741} - \frac{18t^9}{301} + \frac{13t^8}{171} - \frac{7t^7}{166} + \frac{2t^6}{375} + \frac{t^5}{132} + \frac{t^4}{12{,}270}\Big)\tau^2
\end{aligned}$$

$$+\Big(-\frac{376t^{10}}{245}+\frac{634t^9}{129}-\frac{2809t^8}{465}+\frac{1457t^7}{491}+\frac{21t^6}{85}-\frac{268t^5}{353}+\frac{t^4}{33}+\frac{35t^3}{214}+\frac{t^2}{6317}\Big)\tau.$$

For $\tau\in[2.1,\ 5.4]$, we define (we adopt the same notation as in (4.9)–(4.11))

$$\widetilde{U}_a=\frac{42{,}673t^{12}}{692}-\frac{1{,}055{,}462t^{11}}{3005}+\frac{424{,}795t^{10}}{488}-\frac{435{,}706t^9}{359}+\frac{419{,}355t^8}{413}-\frac{130{,}882t^7}{269}+\frac{20{,}712t^6}{211}+\frac{3913t^5}{423}+\frac{405t^4}{241}-\frac{4633t^3}{746}+\frac{t^2}{278}+t+\Big(\frac{89t^{12}}{67}-\frac{1981t^{11}}{262}+\frac{4426t^{10}}{235}-\frac{21{,}136t^9}{795}+\frac{5435t^8}{239}-\frac{6325t^7}{559}+\frac{437t^6}{176}+\frac{23t^5}{135}+\frac{15t^4}{296}-\frac{95t^3}{558}+\frac{t^2}{7198}\Big)\tau+\Big(-\frac{2t^{12}}{231}+\frac{13t^{11}}{401}-\frac{t^{10}}{23}+\frac{5t^9}{276}+\frac{4t^8}{451}-\frac{t^7}{383}-\frac{5t^6}{328}+\frac{3t^5}{227}-\frac{t^4}{1153}+\frac{t^3}{11{,}554}\Big)\tau^2;$$

$$\widetilde{V}_a=-\frac{26{,}307t^{12}}{491}+\frac{81{,}288t^{11}}{271}-\frac{257{,}262t^{10}}{353}+\frac{290{,}879t^9}{293}-\frac{214{,}088t^8}{265}+\frac{91{,}013t^7}{243}-\frac{19{,}248t^6}{263}-\frac{1193t^5}{211}-\frac{267t^4}{238}+\frac{1981t^3}{619}-\frac{t^2}{519}+\Big(-\frac{233t^{12}}{354}+\frac{1250t^{11}}{367}-\frac{1958t^{10}}{259}+\frac{3649t^9}{394}-\frac{6277t^8}{944}+\frac{888t^7}{329}-\frac{228t^6}{409}+\frac{21t^5}{244}-\frac{3t^4}{347}+\frac{t^3}{1861}-\frac{t^2}{53{,}354}\Big)\tau+\Big(\frac{3t^{12}}{152}-\frac{20t^{11}}{207}+\frac{58t^{10}}{279}-\frac{20t^9}{77}+\frac{101t^8}{481}-\frac{23t^7}{200}+\frac{9t^6}{206}-\frac{t^5}{96}+\frac{t^4}{636}-\frac{t^3}{7055}\Big)\tau^2;$$

$$\widetilde{U}_b=-\frac{23{,}768t^{12}}{441}+\frac{30{,}452t^{11}}{101}-\frac{91{,}512t^{10}}{125}+\frac{987{,}850t^9}{991}-\frac{236{,}014t^8}{291}+\frac{129{,}403t^7}{344}-\frac{160{,}111t^6}{2171}-\frac{4746t^5}{859}-\frac{397t^4}{348}+\frac{1284t^3}{401}-\frac{t^2}{498}+\Big(\frac{179t^{12}}{278}-\frac{2323t^{11}}{697}+\frac{2951t^{10}}{399}-\frac{1077t^9}{119}+\frac{1509t^8}{233}-\frac{521t^7}{200}+\frac{34t^6}{65}-\frac{15t^5}{193}+\frac{3t^4}{409}-\frac{t^3}{2412}+\frac{t^2}{79{,}649}\Big)\tau+\Big(-\frac{4t^{12}}{423}+\frac{16t^{11}}{343}-\frac{39t^{10}}{392}+\frac{38t^9}{319}-\frac{19t^8}{219}+\frac{27t^7}{703}-\frac{2t^6}{215}+\frac{t^5}{592}-\frac{t^4}{4961}+\frac{t^3}{67{,}049}\Big)\tau^2;$$

$$\widetilde{V}_b=\frac{27{,}683t^{12}}{447}-\frac{214{,}333t^{11}}{608}+\frac{289{,}907t^{10}}{332}-\frac{320{,}070t^9}{263}+\frac{845{,}926t^8}{831}-\frac{18{,}539t^7}{38}+\frac{25{,}544t^6}{259}+\frac{1335t^5}{146}+\frac{765t^4}{451}-\frac{733t^3}{118}+\frac{t^2}{273}+t+\Big(-\frac{369t^{12}}{280}+\frac{4729t^{11}}{630}-\frac{4883t^{10}}{261}+\frac{7160t^9}{271}-\frac{3865t^8}{171}+\frac{3944t^7}{351}-\frac{547t^6}{223}-\frac{148t^5}{831}-\frac{4t^4}{81}+\frac{74t^3}{435}-\frac{t^2}{7602}\Big)\tau+\Big(\frac{t^{12}}{72}-\frac{74t^{11}}{961}+\frac{79t^{10}}{420}-\frac{107t^9}{408}+\frac{74t^8}{331}-\frac{22t^7}{197}+\frac{8t^6}{323}+\frac{t^5}{255}+\frac{t^4}{1959}-\frac{t^3}{27{,}802}\Big)\tau^2.$$

For $\tau\in[1,\ 2.1]$, we define (we adopt the same notation as in (4.9)–(4.11))

$$\widetilde{U}_a=t+\frac{112t^2}{7071}-\frac{84{,}069t^3}{12{,}941}+\frac{19{,}306t^4}{3625}-\frac{156{,}439t^5}{8122}+\frac{4{,}529{,}108t^6}{18{,}517}-\frac{12{,}407{,}469t^7}{12{,}391}$$

$$
\begin{aligned}
&+\frac{13{,}802{,}505t^8}{6041}-\frac{19{,}938{,}272t^9}{5805}+\frac{53{,}183{,}415t^{10}}{14{,}657}\\
&-\frac{42{,}924{,}858t^{11}}{15{,}571}+\frac{8{,}404{,}358t^{12}}{5591}-\frac{3{,}526{,}473t^{13}}{6148}+\frac{1{,}591{,}789t^{14}}{11{,}029}\\
&-\frac{278{,}264t^{15}}{13{,}315}+\frac{3326t^{16}}{3039}+\frac{2680t^{17}}{49{,}349}\\
&+\Big(-\frac{249t^{17}}{5389}+\frac{4374t^{16}}{6893}-\frac{15{,}398t^{15}}{3823}+\frac{219{,}958t^{14}}{14{,}015}-\frac{212{,}295t^{13}}{5051}\\
&+\frac{688{,}472t^{12}}{8401}-\frac{1{,}117{,}303t^{11}}{9315}+\frac{1{,}588{,}706t^{10}}{11{,}909}\\
&-\frac{4{,}482{,}105t^9}{39{,}941}+\frac{501{,}013t^8}{7201}-\frac{359{,}491t^7}{12{,}021}+\frac{30{,}645t^6}{3956}-\frac{5614t^5}{6493}\\
&+\frac{1923t^4}{10{,}346}-\frac{1271t^3}{7015}+\frac{3t^2}{4790}\Big)\tau\\
&+\Big(\frac{54t^{17}}{7553}-\frac{1012t^{16}}{11{,}019}+\frac{6161t^{15}}{11{,}484}-\frac{39{,}423t^{14}}{20{,}906}+\frac{260{,}558t^{13}}{58{,}587}\\
&-\frac{121{,}265t^{12}}{16{,}324}+\frac{57{,}598t^{11}}{6371}-\frac{64{,}982t^{10}}{7999}\\
&+\frac{68{,}601t^9}{12{,}683}-\frac{17{,}790t^8}{6677}+\frac{7547t^7}{7746}-\frac{2944t^6}{10{,}785}+\frac{2152t^5}{35{,}229}-\frac{33t^4}{4823}+\frac{7t^3}{12{,}667}-\frac{t^2}{40{,}643}\Big)\tau^2;
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{V}_a=&\frac{3681t^{17}}{15{,}061}-\frac{57{,}636t^{16}}{11{,}473}+\frac{825{,}829t^{15}}{18{,}764}-\frac{57{,}112{,}156t^{14}}{255{,}125}+\frac{11{,}597{,}999t^{13}}{15{,}502}\\
&-\frac{20{,}590{,}648t^{12}}{11{,}777}+\frac{25{,}621{,}273t^{11}}{8685}-\frac{51{,}191{,}939t^{10}}{14{,}066}\\
&+\frac{39{,}140{,}407t^9}{11{,}979}-\frac{5{,}210{,}419t^8}{2505}+\frac{13{,}541{,}181t^7}{15{,}350}-\frac{352{,}048t^6}{1631}+\frac{183{,}459t^5}{8386}\\
&-\frac{43{,}947t^4}{9515}+\frac{43{,}703t^3}{12{,}583}-\frac{205t^2}{15{,}119}\\
&+\Big(\frac{271t^{17}}{5454}-\frac{6838t^{16}}{10{,}085}+\frac{52{,}723t^{15}}{12{,}435}-\frac{43{,}681t^{14}}{2714}+\frac{243{,}056t^{13}}{5871}\\
&-\frac{2{,}308{,}060t^{12}}{30{,}281}+\frac{771{,}033t^{11}}{7463}-\frac{949{,}681t^{10}}{9110}\\
&+\frac{1{,}067{,}194t^9}{13{,}673}-\frac{126{,}229t^8}{2964}+\frac{127{,}476t^7}{7783}-\frac{33{,}624t^6}{7837}\\
&+\frac{12{,}196t^5}{15{,}311}-\frac{531t^4}{5368}+\frac{89t^3}{11{,}587}-\frac{4t^2}{12{,}041}\Big)\tau\\
&+\Big(-\frac{80t^{17}}{14{,}649}+\frac{945t^{16}}{13{,}466}-\frac{221{,}639t^{15}}{540{,}245}+\frac{12{,}380t^{14}}{8571}-\frac{232{,}353t^{13}}{67{,}993}\\
&+\frac{74{,}064t^{12}}{12{,}901}-\frac{58{,}461t^{11}}{8284}+\frac{65{,}299t^{10}}{10{,}126}\\
&-\frac{41{,}500t^9}{9413}+\frac{4898t^8}{2173}-\frac{13{,}581t^7}{15{,}874}+\frac{2569t^6}{10{,}804}-\frac{305t^5}{6601}+\frac{56t^4}{9319}-\frac{2t^3}{4113}+\frac{t^2}{46{,}045}\Big)\tau^2;
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{U}_b=&\frac{2463t^{17}}{9715}-\frac{36{,}251t^{16}}{7047}+\frac{285{,}916t^{15}}{6391}-\frac{2{,}011{,}509t^{14}}{8881}+\frac{8{,}515{,}714t^{13}}{11{,}285}\\
&-\frac{23{,}298{,}569t^{12}}{13{,}241}+\frac{119{,}068{,}063t^{11}}{40{,}167}-\frac{44{,}459{,}792t^{10}}{12{,}171}\\
&+\frac{21{,}775{,}652t^9}{6645}-\frac{17{,}028{,}738t^8}{8167}+\frac{8{,}508{,}863t^7}{9624}-\frac{1{,}251{,}442t^6}{5783}
\end{aligned}
$$

$$+\frac{792{,}161t^5}{36{,}032}-\frac{54{,}532t^4}{11{,}771}+\frac{34{,}531t^3}{9939}-\frac{156t^2}{11{,}465}$$
$$+\Big(\frac{68t^{17}}{6929}-\frac{745t^{16}}{7468}+\frac{6177t^{15}}{15{,}824}-\frac{6535t^{14}}{12{,}444}-\frac{4174t^{13}}{3315}+\frac{66{,}375t^{12}}{9109}$$
$$-\frac{146{,}303t^{11}}{8824}+\frac{212{,}006t^{10}}{9193}$$
$$-\frac{201{,}460t^9}{9491}+\frac{43{,}291t^8}{3347}-\frac{88{,}537t^7}{17{,}825}+\frac{19{,}981t^6}{17{,}933}-\frac{1831t^5}{10{,}383}$$
$$+\frac{147t^4}{8275}-\frac{35t^3}{33{,}337}+\frac{t^2}{32{,}146}\Big)\tau$$
$$+\Big(-\frac{28t^{17}}{9343}+\frac{365t^{16}}{9599}-\frac{2944t^{15}}{13{,}441}+\frac{12{,}991t^{14}}{17{,}140}-\frac{32{,}497t^{13}}{18{,}494}$$
$$+\frac{24{,}623t^{12}}{8543}-\frac{16{,}321t^{11}}{4738}+\frac{17{,}317t^{10}}{5685}$$
$$-\frac{8489t^9}{4231}+\frac{90{,}568t^8}{91{,}895}-\frac{2930t^7}{8137}+\frac{1761t^6}{18{,}083}-\frac{143t^5}{7848}+\frac{162t^4}{71{,}441}-\frac{t^3}{5726}+\frac{t^2}{135{,}052}\Big)\tau^2;$$
$$\widetilde{V}_b=\frac{281t^{17}}{6789}+\frac{29{,}119t^{16}}{23{,}022}-\frac{134{,}735t^{15}}{6146}+\frac{1{,}565{,}941t^{14}}{10{,}578}-\frac{6{,}744{,}679t^{13}}{11{,}576}+\frac{9{,}325{,}931t^{12}}{6140}$$
$$-\frac{21{,}688{,}406t^{11}}{7811}+\frac{11{,}901{,}372t^{10}}{3263}$$
$$-\frac{36{,}827{,}911t^9}{10{,}681}+\frac{28{,}814{,}971t^8}{12{,}573}-\frac{11{,}522{,}451t^7}{11{,}476}+\frac{609{,}949t^6}{2486}$$
$$-\frac{47{,}380t^5}{2441}+\frac{53{,}649t^4}{10{,}037}-\frac{70{,}099t^3}{10{,}788}+\frac{243t^2}{15{,}278}+t$$
$$+\Big(-\frac{431t^{17}}{12{,}307}+\frac{1891t^{16}}{4423}-\frac{17{,}301t^{15}}{7532}+\frac{79{,}657t^{14}}{11{,}355}-\frac{75{,}969t^{13}}{5918}$$
$$+\frac{128{,}952t^{12}}{10{,}477}+\frac{15{,}653t^{11}}{11{,}987}-\frac{186{,}559t^{10}}{8361}$$
$$+\frac{394{,}344t^9}{11{,}453}-\frac{395{,}025t^8}{13{,}636}+\frac{260{,}073t^7}{18{,}221}-\frac{29{,}944t^6}{8821}+\frac{271t^5}{18{,}282}-\frac{807t^4}{10{,}802}$$
$$+\frac{1133t^3}{6583}-\frac{2t^2}{9289}\Big)\tau$$
$$+\Big(\frac{27t^{17}}{6227}-\frac{1706t^{16}}{30{,}739}+\frac{4549t^{15}}{14{,}101}-\frac{14{,}721t^{14}}{13{,}070}+\frac{52{,}013t^{13}}{19{,}762}$$
$$-\frac{52{,}482t^{12}}{12{,}091}+\frac{30{,}786t^{11}}{5933}-\frac{64{,}433t^{10}}{14{,}173}$$
$$+\frac{24{,}681t^9}{8450}-\frac{10{,}701t^8}{7799}+\frac{2498t^7}{5227}-\frac{973t^6}{7295}+\frac{267t^5}{7960}-\frac{22t^4}{6887}+\frac{3t^3}{11{,}911}-\frac{t^2}{91{,}156}\Big)\tau^2.$$

Proposition C.1 (Case $\tau\in[1,\ 12.4]$) *Let T, τ_L, τ_R, ϵ_s, $\Delta\tau$ be taken from Table 1 in Sect. 5. We use the notation (4.9)–(4.11) with the explicit forms of $\widetilde{U}_I$, $\widetilde{V}_I$ preceding this proposition. The following hold.*

(*i*) *For any τ, $\widetilde{\tau}\in[\tau_L,\ \tau_R]$ with $|\tau-\widetilde{\tau}|\le\frac{1}{2}\Delta\tau$, we have*

$$\max_{I=a,b}\|Y_I(\tau,T)-\widetilde{Y}_I(\widetilde{\tau},T)\|_{l^\infty}\le\epsilon_s. \tag{C.1}$$

(*ii*) *For corresponding p_1 and p_2 from the table, it holds that*

$$\min_{0\le j\le N}\min\{\widetilde{Y_{a,2}}(\tau_j,T),\ \widetilde{Y_{a,4}}(\tau_j,T)\}>\epsilon_s,$$

$$\min_{0\le j\le N} H^{\epsilon_s}_{p_1,p_2}(\widetilde{Y_a}(\tau_j,T),\widetilde{Y_b}(\tau_j,T))>0, \tag{C.2}$$

where $H^{\epsilon_s}_{p_1,p_2}$ *is defined in* (4.13), *and* $\tau_j=\tau_L+j\Delta\tau$ $(0\le j\le N)$ *with* $N=\frac{\tau_R-\tau_L}{\Delta\tau}$.

Proof Using Q^2_* defined in Lemma A.2, it is straightforward to compute $z(\tau,T)$ defined in (4.11). Note that for $\tau\in[1,\ 12.4]$, $z_I(\tau,T)$ is of degree $d=6$ in the variable τ. Using Lemma 1.4, we find

$$\max_{I=a,b}\max_{\tau\in[\tau_L,\tau_R]} z_I(\tau,T)\le\frac{6N^2}{6N^2-d^2(d^2-1)}\cdot\max_{I=a,b}\max_{0\le j\le N} z_I(\tau_j,T)\le R_1,$$

where R_1 is an explicit rational number.

By Lemma 4.3 and Lemma A.2 (below $\epsilon_Q=\frac{75}{10^5}$), we obtain for $k=\sqrt{\tau_R+1}$

$$\sup_{\tau_L\le\tau\le\tau_R}\max_{I=a,b}\|Y_I(\tau,T)-\widetilde{Y}_I(\tau,T)\|_{l^\infty}\le\frac{1}{2}\sqrt{\frac{2kT+\sinh(2kT)}{k}}\sqrt{R_1}+\frac{3T\sinh(kT)}{2k}\epsilon_Q.$$

Since each component of $\widetilde{Y}_I(\tau,T)$ is a quadratic function of τ, $Z(\tau):=\partial_\tau\widetilde{Y}_I(\tau,T)$ is linear in τ. Thus

$$\sup_{\tau_L\le\tau\le\tau_R}\|Z(\tau)\|_{l^\infty}\le\max\{\|Z(\tau_L)\|_{l^\infty},\ \|Z(\tau_R)\|_{l^\infty}\}\le\frac{2}{5}, \tag{C.3}$$

where the bound $2/5$ is obtained via explicit checking for each of the 4 cases listed in the table. It follows that

$$\max_{I=a,b}\sup_{|\tau-\widetilde{\tau}|\le\frac{1}{2}\Delta\tau}\|\widetilde{Y}_I(\tau,T)-\widetilde{Y}_I(\widetilde{\tau},T)\|_{l^\infty}\le\frac{1}{5}\Delta\tau.$$

Hence (C.1) follows from the inequality:

$$\frac{1}{2}\sqrt{\frac{2kT+\sinh(2kT)}{k}}\sqrt{R_1}+\frac{3T\sinh(kT)}{2k}\epsilon_Q+\frac{1}{5}\Delta\tau\le\epsilon_s. \tag{C.4}$$

On the other hand, notice that $\widetilde{Y}_I(\tau,T)$, $I=a,b$ are **explicit** quadratic functions of τ. Thus (C.2) can be verified via a completely rigorous and exact computation. □

Appendix D: Further auxiliary estimates

Let the dimension $n\ge 1$ and define $p_{\max}=\begin{cases}\frac{2n}{n-2}, & n\ge 3;\\ \infty, & n=1,2;\end{cases}$ and $p_c=\frac{4}{n}+1$. For each $1<p<p_{\max}$, we denote by $Q_p:\mathbb{R}^n\to(0,\infty)$ the unique smooth positive exponentially decaying radial solution to $(1-\Delta)Q_p=Q_p^p$. Note that Q_p is an extremizer for the functional $J(u)=\|\nabla u\|^2_{L^2(\mathbb{R}^n,\mathbb{C})}-\frac{2}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^n,\mathbb{C})}+\|u\|^2_{L^2(\mathbb{R}^n,\mathbb{C})}$.

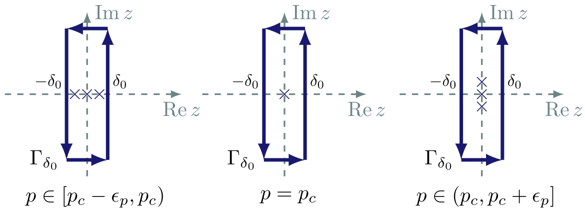


Fig. 3 The case $p_c - \epsilon_p \le p \le p_c + \epsilon_p$

For each $1 < p < p_{\max}$, we denote

$$L_{+,p} = 1 - \Delta - pQ_p^{p-1}, \quad L_{-,p} = 1 - \Delta - Q_p^{p-1}, \quad \widetilde{A}_p = \begin{pmatrix} 0 & L_{-,p} \\ L_{+,p} & 0 \end{pmatrix}. \tag{D.1}$$

Note that for $u = Q_p + \epsilon\eta = Q_p + \epsilon(\eta_1 + i\eta_2)$ ($\eta_1 = \mathrm{Re}(\eta)$, $\eta_2 = \mathrm{Im}(\eta)$), we have

$$J(u) = J(Q_p) + \epsilon^2(\langle L_{+,p}\eta_1, \eta_1\rangle + \langle L_{-,p}\eta_2, \eta_2\rangle) + \mathcal{O}(\epsilon^3). \tag{D.2}$$

Denote $Q_{p,\lambda}(x) = \lambda^{\frac{n}{2}} Q_p(\lambda x)$, $\lambda > 0$. One has $J(Q_{p,\lambda}) = J(Q_p) + \frac{1}{2}\frac{d^2}{d\lambda^2} \times \big(J(Q_{p,\lambda})\big)\big|_{\lambda=1} \cdot (\lambda - 1)^2 + \mathcal{O}(|\lambda - 1|^3)$.

Note that $\|Q_{p,\lambda}\|_{L^2(\mathbb{R}^n)} = \|Q_p\|_{L^2(\mathbb{R}^n)}$ for all $\lambda > 0$. Along the orbit of $Q_{p,\lambda}$, the functional $J(\cdot)$ reduces to the usual energy. It is not difficult to verify that $\frac{d^2}{d\lambda^2}(J(Q_{p,\lambda}))\big|_{\lambda=1} = c_{n,p}(p_c - p)\|Q_p\|^2_{L^2(\mathbb{R}^n)}$, where $c_{n,p} > 0$ depends only on (n, p). In particular this accords with the usual orbital instability (cf. [68]) for $p > p_c$. An infinitesimal version is $\langle L_{+,p}\phi, \phi\rangle < 0$, $p_c < p < p_{\max}$, where $\phi = \frac{n}{2}Q_p + rQ'_p$.

In the next proposition, we collect a few important properties of Q_p, $L_{+,p}$, $L_{-,p}$ and $\widetilde{A}_p$ for $1 < p < p_{\max}$.

Proposition D.1 *The following hold.*

(1) Continuity in p. *For any fixed* $p \in (1, p_{\max})$, *we have* $\|Q_{\widetilde{p}} - Q_p\|_{L^2(\mathbb{R}^n)} + \|Q_{\widetilde{p}} - Q_p\|_{L^\infty(\mathbb{R}^n)} \to 0$ *if* $\widetilde{p} \to p$. *Also we have* $\|Q_{\widetilde{p}}^{\widetilde{p}-1} - Q_p^{p-1}\|_{L^\infty(\mathbb{R}^n)} \to 0$, *as* $\widetilde{p} \to p$.

(2) $\ker(L_{-,p}) = \mathrm{span}\{Q_p\}$, *and* $\ker(L_{+,p}) = \mathrm{span}\{\nabla Q_p\}$.

(3) *If* $1 < p \le p_c = \frac{4}{n} + 1$, *then* $\sigma(\widetilde{A}_p) \cap i\mathbb{R} = \{0\}$.

(4) *For some* $\delta_0 > 0$ *sufficiently small, it holds that* $\Gamma_{\delta_0} \subset \rho(\widetilde{A}_{p_c})$ *and* $\overline{\Omega_{\delta_0}} \cap \sigma(\widetilde{A}_{p_c}) = \{0\}$, *where* Γ_{δ_0} *is depicted in Fig.* 4 *and* Ω_{δ_0} *is the region enclosed by* Γ_{δ_0}.

(5) *If* $|p - p_c| \le \epsilon_p$ *where* $\epsilon_p > 0$ *is sufficiently small, then* $\Gamma_{\delta_0} \subset \rho(\widetilde{A}_p)$.

(6) *It holds that* (*see Fig.* 3)

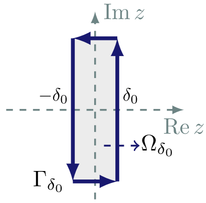
For $p_c - \epsilon_p \le p < p_c$,

$$\sigma(\widetilde{A}_p) \cap \overline{\Omega_{\delta_0}} = \sigma_{\mathrm{dis}}(\widetilde{A}_p) \cap \overline{\Omega_{\delta_0}} = \{0, \lambda_p, -\lambda_p\}; \tag{D.3}$$

For $p = p_c$, $\quad \sigma(\widetilde{A}_{p_c}) \cap \overline{\Omega_{\delta_0}} = \sigma_{\mathrm{dis}}(\widetilde{A}_{p_c}) \cap \overline{\Omega_{\delta_0}} = \{0\}$; $\quad$ (D.4)

For $p_c < p \le p_c + \epsilon_p$,

Fig. 4 Γ_{δ_0} and Ω_{δ_0}



$$\sigma(\widetilde{A}_p)\cap\overline{\Omega_{\delta_0}}=\sigma_{\mathrm{dis}}(\widetilde{A}_p)\cap\overline{\Omega_{\delta_0}}=\{0,\,i\lambda_p,\,-i\lambda_p\}. \tag{D.5}$$

In the above, $\lambda_p>0$ depends on p (here and below we suppress the dependence on the dimension n). The eigenvalues $\pm\lambda_p$, $\pm i\lambda_p$ are simple.

(7) *If $p_c<p_1<p_2<p_{\max}$, then one can find $\delta_1>0$ sufficiently small and $M_1>0$ sufficiently large, such that $\pm\delta_1\in\rho(\widetilde{A}_p)$, $\pm M_1 i\in\rho(\widetilde{A}_p)$ for all $p\in[p_1,p_2]$.*

(8) *If $p_c<p<p_{\max}$, then $\sigma(\widetilde{A}_p)\cap i\mathbb{R}=\{0,i\lambda_p,-i\lambda_p\}$ for some $\lambda_p>0$ depending on p. Furthermore the eigenvalues $\pm i\lambda_p$ are simple.*

Proof (1) We showcase the argument for dimension $n=3$. The proof for general dimensions follows along similar lines. Fix $p\in(1,p_{\max})$. Denote $F(r)=Q_p(r)r$. Clearly $F''=(1-(\frac{F}{r})^{p-1})F$. Multiplying both sides by F' and integrating, we obtain $(F')^2=F^2-\frac{2}{r^{p-1}(p+1)}F^{p+1}+2\frac{p-1}{p+1}\int_r^\infty s^{-p}F^{p+1}(s)ds$.

Note that $F'<0$ for r large. Taking the connected component emanating from $r=\infty$, we obtain

$$\begin{aligned}F'(r)&=-\sqrt{F^2(r)-\frac{2}{r^{p-1}(p+1)}F^{p+1}(r)+2\frac{p-1}{p+1}\int_r^\infty s^{-p}F^{p+1}(s)ds}\\&<-F(r)\sqrt{1-\frac{2}{p+1}Q_p^{p-1}(r)}.\end{aligned} \tag{D.6}$$

Clearly for some $R_*>0$ we have $\frac{2}{p+1}Q_p^{p-1}(r)\le\frac{2}{p+1}Q_p^{p-1}(R_*)\le\frac{1}{3}$, $\forall r\ge R_*$. From this it is evident that one can derive a uniform (in p) exponential decay estimate.

Now consider $\tilde{p}$ with $|\tilde{p}-p|\ll 1$. By using a shooting argument as in [43], one can show $|Q_{\tilde{p}}(0)-Q_p(0)|\to 0$ as $\tilde{p}\to p$. Also by perturbation it is not difficult to check that for any given R_1, $\|Q_{\tilde{p}}(r)-Q_p(r)\|_{L_r^\infty([0,R_1])}+\|Q'_{\tilde{p}}(r)-Q'_p(r)\|_{L_r^\infty([0,R_1])}\to 0$ as $\tilde{p}\to p$. The desired conclusion then easily follows by using the uniform exponential decay estimate established earlier (note that one can achieve uniform decay for all $\tilde{p}$ with $|\tilde{p}-p|\ll 1$).

(2) To show $\ker(L_{-,p})=\mathrm{span}\{Q_p\}$ one can pass to spherical harmonics expansion and work with radial operators $L_{-,p,\ell}=-\partial_{rr}-\frac{n-1}{r}\partial_r+\frac{\ell(\ell+1)}{r^2}+1-Q_p^{p-1}$. For $\ell=0$ one has $L_{-,p,0}Q_p=0$. Since $Q_p>0$, it follows that $L_{-,p,0}\ge 0$. Clearly then $L_{-,p,\ell}>0$ for $\ell\ge 1$. Thus any other possible nontrivial solution must satisfy $L_{-,p,0}f=0$. Clearly the other independent solution must be exponentially growing. Thus one can conclude $\ker(L_{-,p})=\mathrm{span}\{Q_p\}$. A concise proof for the statement $\ker(L_{+,p})=\mathrm{span}\{\nabla Q_p\}$ can be found in Lemma 2.1 of [15] (see Weinstein [73] for the proof for cases $n=1$ and $n=3$, and see Kwong [43] for a general result).

(3) By a computation similar to that around (3.3), we have for any $f \perp Q_p$, it holds that (note here $1 < p \le p_c = 1 + \frac{4}{n}$) $\langle L_{+,p} f, f\rangle \ge 0$. If $L_{+,p} f = i\tau g$, $L_{-,p} g = i\tau f$ for some $0 \ne \tau \in \mathbb{R}$, then

$$\langle L_{+,p} f, f\rangle = \langle i\tau g, \tfrac{1}{i\tau} L_{-,p} g\rangle = -\langle L_{-,p} g, g\rangle. \tag{D.7}$$

Thus $\langle L_{+,p} f, f\rangle + \langle L_{-,p} g, g\rangle = 0 \Rightarrow g = \text{const} \cdot Q_p \Rightarrow f \equiv 0, \ g \equiv 0$. Thus $\sigma(\widetilde{A_p}) \cap i\mathbb{R} = \{0\}$.

(4) By taking $\delta_0 > 0$ sufficiently small, we have $\pm\delta_0 \in \rho(\widetilde{A}_{p_c})$. Since $\sigma(\widetilde{A}_{p_c}) \cap i\mathbb{R} = \{0\}$, it follows that $\Gamma_{\delta_0} \subset \rho(\widetilde{A}_{p_c})$.

(5) Since $|p - p_c| \ll 1$, the statement $\Gamma_{\delta_0} \subset \rho(\widetilde{A}_p)$ follows from the "continuity in p" proved in statement (1) (note that it implies the stability of the resolvent by using the Neumann series).

(6) We first consider $p \in [p_c - \widetilde{\epsilon}, p_c + \widetilde{\epsilon}]$, $\widetilde{\epsilon} = \min\{\epsilon_p, \epsilon_*\}$, where $\epsilon_* > 0$ is the same as in Remark D.4. By using the results from Remark D.3 and Remark D.4 and examining the Riesz projection[39] $\frac{1}{2\pi i}\int_{\Gamma_{\delta_0}} (z - \widetilde{A}_p)^{-1} dz$ for $p \in [p_c - \widetilde{\epsilon}, p_c + \widetilde{\epsilon}]$, we see that there are no other additional eigenvalues in $\overline{\Omega_{\delta_0}}$. Since the dimension of the root space of $\widetilde{A}_p$ at zero is $2n + 2$ for all $p \ne p_c$ and the nonzero simple eigenvalues are stable under perturbation, we can extend the conclusion to the whole regime $p \in [p_c - \epsilon_p, p_c + \epsilon_p]$.

(7) The existence of M_1 is clear. We focus on the existence of δ_1. For each $p \in [p_1, p_2]$, there are only two possibilities:

case a) $[-\frac{1}{2}, \frac{1}{2}] \setminus \{0\} \subset \rho(\widetilde{A}_p)$. In this case we can find $\epsilon_p > 0$ sufficiently small such that $[-\frac{1}{2}, \frac{1}{2}] \setminus \{0\} \subset \rho(\widetilde{A}_q)$ for any $|q - p| \le \epsilon_p$. Here the perturbation argument is done as follows. Firstly we consider a sufficiently small circle around the origin (the circle intersects with the real axis at $\pm\delta_3$ for some $\delta_3 > 0$). The circle is stable under perturbation (i.e., still in the resolvent). Then the compact intervals $[-\frac{1}{2}, -\delta_3]$ and $[\delta_3, \frac{1}{2}]$ are also stable under perturbation.

case b) $\pm\lambda_p \in \sigma_{\text{dis}}(\widetilde{A}_p)$ for some closest $\lambda_p \in (0, \frac{1}{2}]$. Clearly we can find $\epsilon_p > 0$ sufficiently small, such that for $|q - p| \le \epsilon_p$ the closest nonzero eigenvalue $\lambda_q \in \sigma_{\text{dis}}(\widetilde{A}_q)$ satisfies $|\lambda_q - \lambda_p| < \frac{1}{2}\lambda_p$.

Since the interval $[p_1, p_2]$ is compact, by using a covering argument, we can easily deduce the existence of δ_1.

(8) The conclusion clearly holds for $p \in (p_c, p_c + \epsilon_p]$. For $p \in [p_c + \epsilon_p, p_2]$ (here $p_2 < p_{\max}$ is taken to be fixed but arbitrarily close to $p_{\max}$), we can choose a rectangular contour Γ_{δ_1, M_1} passing the points $\pm\delta_1$, $\pm M_1 i$ as specified in statement (7). Clearly the desired conclusion follows by using Riesz projection. □

Remark D.2 One can prove statements (6) and (8) in Proposition D.1 without appealing to the results in Remark D.3 and Remark D.4. The idea is as follows. First consider $p \in [p_c - \epsilon_p, p_c)$. Since $\sigma(\widetilde{A}_p) \cap i\mathbb{R} = \{0\}$ for subcritical p and the root space at 0 has dimension $2n + 2$, we conclude that there must be exactly one pair of real eigenvalues (by using the continuity of $\frac{1}{2\pi i}\int_{\Gamma_{\delta_0}} (z - \widetilde{A}_p)^{-1} dz$). Next consider

[39] Here we use the continuity in p which follows from statement (1) proved earlier.

supercritical $p > p_c$. It suffices to check the existence of a pair of imaginary eigenvalues (this would saturate the missing "2" going from $2n+4$ at $p = p_c$ to $2n+2$ at $p > p_c$). For this we only need to work with the operator $L_{-,p}^{\frac{1}{2}} A L_{-,p}^{\frac{1}{2}}$, $A = PL_{+,p}P$, where P is the projection to the space $Q_p^{\perp} = \{f \in L^2(\mathbb{R}^n, \mathbb{C}) : \langle f, Q_p \rangle = 0\}$. It suffices to show the self-adjoint operator $L_{-,p}^{\frac{1}{2}} A L_{-,p}^{\frac{1}{2}}$ (on $Q_p^{\perp}$) has a negative eigenvalue. This follows from showing $\langle A\phi, \phi \rangle < 0$ for some nontrivial $\phi \in Q_p^{\perp}$. The argument is as follows. Define $Q_1^{(p)} = \frac{2}{p-1} Q_p + r Q_p'$ and recall $L_{+,p} Q_1^{(p)} = -2Q_p$. Let $\phi = r Q_p' + \frac{n}{2} Q_p = Q_1^{(p)} + (\frac{n}{2} - \frac{2}{p-1}) Q_p$. Clearly[40] $\phi \perp Q_p$ and

$$
\begin{aligned}
\langle PL_{+,p}P\phi, \phi \rangle = \langle L_{+,p}\phi, \phi \rangle &= \langle -2Q_p + (\tfrac{n}{2} - \tfrac{2}{p-1})(1-p) Q_p^p, \phi \rangle \\
&= (\tfrac{n}{2} - \tfrac{2}{p-1})(1-p) \langle Q_p^p, r Q_p' + \tfrac{n}{2} Q_p \rangle \\
&= (\tfrac{n}{2} - \tfrac{2}{p-1})(1-p)(\tfrac{n}{2} - \tfrac{n}{p+1}) \| Q_p \|_{L^{p+1}(\mathbb{R}^n)}^{p+1} < 0.
\end{aligned}
$$

Remark D.3 (Kernel of $\widetilde{A}$ for $p = p_c = \frac{4}{n} + 1$) Consider the general NLS $i\partial_t \psi + \Delta\psi + |\psi|^{p-1}\psi = 0$, where $1 < p < \infty$ and $\psi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C}$. The corresponding linearization around the standing wave $Q(x)e^{it}$ (Q is the ground state solving $\Delta Q - Q + Q^p = 0$) leads to the linearized operator $\widetilde{A} = \begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix}$. Here and below we shall suppress the notational dependence of Q, L_+, L_- and $\widetilde{A}$ on p for simplicity of notation. Note that

$$
L_-(xQ) = -2\nabla Q, \qquad L_+ Q_1 = -2Q, \qquad Q_1 = \frac{2}{p-1} Q + x \cdot \nabla Q. \tag{D.8}
$$

Denote $p_c = \frac{4}{n} + 1$ which is the mass-critical power. For $p = p_c$, it is not difficult to check that

$$
\begin{aligned}
&L_-(|x|^2 Q) = -4Q_1, \\
&\qquad \text{(for general } p\text{, one has } L_-(|x|^2 Q) = -4Q_1 + \tfrac{2n}{p-1}(p_c - p)Q); \\
&L_+\rho = |x|^2 Q, \quad \text{(here } \rho \text{ is a radial function).}
\end{aligned} \tag{D.9}
$$

Thus[41] for $p = p_c$, the operator $\widetilde{A}$ has $2n+4$ generalized eigenvectors at the zero eigenvalue. More precisely[42]

$$
\begin{aligned}
\ker(\widetilde{A}) &= \operatorname{span}\{\begin{pmatrix} \nabla Q \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Q \end{pmatrix}\}, \\
\ker(\widetilde{A}^2) &= \ker(\widetilde{A}) \oplus \operatorname{span}\{\begin{pmatrix} Q_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ xQ \end{pmatrix}\};
\end{aligned} \tag{D.10}
$$

[40] This also follows from $\|\lambda^{\frac{n}{2}} Q_p(\lambda \cdot)\|_2 = \|Q_p\|_2$.

[41] One can compute it directly or see Appendix B of [73].

[42] Here we abbreviate $\begin{pmatrix} \partial_j Q \\ 0 \end{pmatrix}$, $j = 1, \ldots, n$ as $\begin{pmatrix} \nabla Q \\ 0 \end{pmatrix}$. Also $\begin{pmatrix} 0 \\ x_j Q \end{pmatrix}$ is written as $\begin{pmatrix} 0 \\ xQ \end{pmatrix}$.

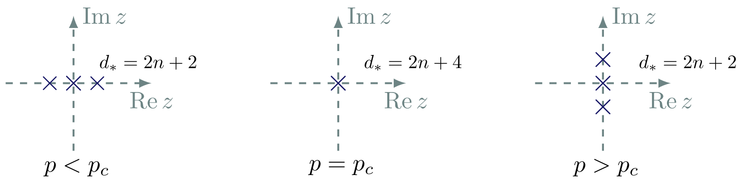


Fig. 5 The bifurcation at $p = p_c$ ($d_* := \dim N_g(\widetilde{A})$)

$$\ker(\widetilde{A}^3) = \ker(\widetilde{A}^2) \oplus \mathrm{span}\{\begin{pmatrix} 0 \\ |x|^2 Q \end{pmatrix}\},$$
$$\ker(\widetilde{A}^4) = \ker(\widetilde{A}^3) \oplus \mathrm{span}\{\begin{pmatrix} \rho \\ 0 \end{pmatrix}\}. \tag{D.11}$$

In deriving the above generalized eigen-spaces, we used the following observations:

$$L_- Q = 0, \qquad L_+ \nabla Q = 0, \qquad L_+ Q_1 = -2Q, \qquad L_-(xQ) = -2\nabla Q;$$
$$\text{(no solutions to } L_+ f = x_j Q \text{ since } \langle x_j Q, \partial_j Q\rangle \neq 0);$$
$$L_-(|x|^2 Q) = -4Q_1 \quad (\text{note that } Q_1 \perp Q \Leftrightarrow p = p_c); \qquad L_+\rho = |x|^2 Q.$$
$$\text{(no solutions to } L_- f = \rho \text{ since } \rho \perp Q \Rightarrow Q_1 \perp |x|^2 Q \Rightarrow$$
$$\langle L_-(|x|^2 Q), |x|^2 Q\rangle = 0 \Rightarrow \text{contradiction.)}$$

Remark D.4 (Transition at $p = p_c$) The eigenvalues of the linearized operator $\widetilde{A} = \begin{pmatrix} 0 & L_- \\ L_+ & 0 \end{pmatrix}$ near zero exhibit a very interesting bifurcation as p gets close to $p_c = 4/n + 1$. As was suggested by Weinstein and rigorously proved[43] in Theorem 2.6 of [15], a pair of real eigenvalues (for $p < p_c$) will first merge at the origin (for $p = p_c$) and then split into a pair of purely imaginary eigenvalues as soon as $p > p_c$. See Fig. 5 for a schematic illustration (note that $N_g(\widetilde{A})$ denotes the generalized eigenspace of $\widetilde{A}$ at zero). To check this, it suffices for us to consider the reduced equation $L_- b = \mu L_+^{-1} b$ where b is radial. Theorem 2.6 of [15] can be reformulated as the following (below $\epsilon_* > 0$ is sufficiently small):

Claim: For $p \in [p_c - \epsilon_*, p_c + \epsilon_*]$, there exists[44] a solution of $L_- b = \mu L_+^{-1} b$ of the form

$$\mu = c_1(p_c - p) + \mathcal{O}(1) \cdot (p_c - p)^2,$$

[43] The proof therein deals with the operator $\mathcal{L} = \begin{pmatrix} 0 & L_- \\ -L_+ & 0 \end{pmatrix}$ which is related to $\widetilde{A}$ by the relation $i\mathcal{L} = P_2^{-1} \widetilde{A} P_2$, $P_2 = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. In yet other words, real nonzero eigenvalues of $\mathcal{L}$ correspond to purely imaginary values of $\widetilde{A}$ and vice versa. Note that there is a typo in the statement of Theorem 2.6 in [15]: it should be $g \perp Q_1$ therein.

[44] Uniqueness and simplicity of the eigenvalue and eigenfunction hold in a more general setting. See Proposition D.1.

$$b = Q + \mu \cdot \tfrac{1}{8}|x|^2 Q + (p_c - p)^2 \widetilde{g}, \tag{D.12}$$

where $c_1 > 0$ is a constant, $\widetilde{g} \perp Q$ and $\|\widetilde{g}\|_2 = \mathcal{O}(1)$ uniformly in p.

We sketch the proof of the above claim as follows. Setting $b = Q + \mu b_1$, we get $L_- b_1 = -\frac{1}{2} Q_1 + \mu L_+^{-1} b_1$.

Set $b_1 = w + \widetilde{\eta}$, where $w = \frac{1}{8}|x|^2 Q$ (note $L_- w = -\frac{1}{2} Q_1 + \frac{n}{4(p-1)}(p_c - p)Q$). Then for $\widetilde{\eta}$ we have the equation

$$L_- \widetilde{\eta} = \tfrac{n}{4(p-1)}(p - p_c)Q + \mu \rho_1 + \mu L_+^{-1} \widetilde{\eta}, \qquad (\rho_1 := L_+^{-1} w). \tag{D.13}$$

We write $\widetilde{\eta} = \eta + cQ$ with $\eta \perp Q$. Then $L_- \eta = \frac{n}{4(p-1)}(p - p_c)Q + \mu \rho_1 + \mu L_+^{-1} \eta - \frac{\mu}{2} c Q_1$. We shall enforce[45] $c = 0$. A contraction argument then yields the result. In more detail, one can define

$$\begin{cases} L_- \eta_{k+1} = \frac{n}{4(p-1)}(p - p_c)Q + \mu_{k+1}(\rho_1 + L_+^{-1} \eta_k); \\ \mu_{k+1} \langle \rho_1 + L_+^{-1} \eta_k, Q \rangle = \frac{n}{4(p-1)}(p_c - p)\|Q\|_2^2. \end{cases} \tag{D.14}$$

Note that $\langle \rho_1, Q \rangle = -\frac{1}{2}\langle w, Q_1 \rangle = (1 + \frac{(p_c - p)n}{2(p-1)})\langle Q, |x|^2 Q \rangle$ which is positive and bounded from above and below. By using the smallness of $|p - p_c|$, one can derive a contraction.

Remark D.5 Yet another way to see the transition at $p = p_c$ is through the self-adjoint operator $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$. By using the minmax formulation and restricting to $Q^\perp$, one has

$$\mu_j = \inf_{g \perp Q, g_1, \ldots, g_{j-1}} \frac{\langle L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} g, g \rangle}{\langle g, g \rangle}, \qquad j \geq 1, \tag{D.15}$$

where for $j \geq 2$, $g_1, \cdots, g_{j-1}$ denote the eigenfunctions corresponding to $\mu_1, \cdots, \mu_{j-1}$ respectively. It is not difficult to check that $\sigma_{\mathrm{ess}}(L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}) = [1, \infty)$. Thus by the minmax principle μ_j will stop at some m_0 with $\mu_{m_0} = \mu_{m_0+1} = \cdots = 1$. As is already realized in [15], μ_j can be equivalently characterized by

$$\mu_j = \inf_{\substack{h \perp Q \\ \langle h, L_-^{-1} h_l \rangle = 0,\, l \leq j-1}} \frac{\langle L_+ h, h \rangle}{\langle L_-^{-1} h, h \rangle}. \tag{D.16}$$

Consider first $p < p_c$. Clearly $\mu_1 = \cdots = \mu_n = 0$ since $L_+(\nabla Q) = 0$. Furthermore $\mu_{n+1} > 0$ (by using the fact that $L_+ h = cQ \Rightarrow \langle L_+ h, Q_1 \rangle = c\langle Q, Q_1 \rangle \Rightarrow c = 0$). For $p < p_c$, we bound μ_{n+1} as

$$\mu_{n+1} \leq \inf_{\substack{g \perp Q \\ g \text{ is radial}}} \frac{\langle L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} g, g \rangle}{\langle g, g \rangle} = \inf_{\substack{h \perp Q \\ h \text{ is radial}}} \frac{\langle L_+ h, h \rangle}{\langle L_-^{-1} h, h \rangle}.$$

[45] The argument in [15] proceeds slightly differently by imposing orthogonality to Q_1.

Note that $L_-(\frac{1}{8}|x|^2 Q) = -\frac{1}{2}Q_1 + \frac{n}{4(p-1)}(p_c - p)Q =: h_1$. Clearly

$$\begin{aligned}
\langle L_-^{-1} h_1, h_1\rangle &= \langle \tfrac{1}{8}|x|^2 Q + \text{const}\cdot Q, h_1\rangle = \langle \tfrac{1}{8}|x|^2 Q, h_1\rangle \\
&= \tfrac{1}{16}\big(1+\tfrac{(p_c-p)n}{p-1}\big)\langle Q, |x|^2 Q\rangle; \\
\langle L_+ h_1, h_1\rangle &= \tfrac{1}{4}\langle L_+ Q_1, Q_1\rangle + 2\langle L_+(-\tfrac{1}{2}Q_1), Q\rangle \cdot \tfrac{n}{4(p-1)}(p_c-p) + \mathcal{O}((p_c-p)^2) \\
&= \tfrac{n}{4(p-1)}(p_c-p)\|Q\|_2^2 + \mathcal{O}((p_c-p)^2).
\end{aligned}$$

It follows easily that for $p < p_c$ (below $c_{n,p} > 0$ and remains $\mathcal{O}(1)$ as $p \to p_c$)

$$0 < \mu_{n+1} \le c_{n,p}\cdot (p_c - p) + \mathcal{O}((p_c-p)^2). \tag{D.17}$$

Next consider $p > p_c$. It is shown in [15] that $\mu_1 < 0$ and $\mu_2 = \cdots = \mu_{n+1} = 0$ with $\mu_{n+2} > 0$ (the key to show $\mu_2 \ge 0$ is to use the property $\langle L_+ Q, Q\rangle < 0$, $L_+\big|_{\{Q^p\}^\perp} \ge 0$ which holds for $1 < p < p_{\max}$). The main point is to show $\mu_1 \to 0$ as $p \to p_c$. To this end, we bound μ_1 as

$$\mu_1 = \inf_{g\perp Q} \frac{\langle L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}} g, g\rangle}{\langle g, g\rangle} = \inf_{h \perp Q} \frac{\langle L_+ h, h\rangle}{\langle L_-^{-1} h, h\rangle}. \tag{D.18}$$

Since μ_1 is an eigenvalue for the self-adjoint operator $L_-^{\frac{1}{2}} L_+ L_-^{\frac{1}{2}}$, the infimum in the above is achieved. We may assume a minimizer $h \perp Q$ with $\|L_-^{-1} h\|_{L^2(\mathbb{R}^n,\mathbb{C})} = 1$ such that

$$L_+ h - \mu_1 L_-^{-1} h = \text{const}\cdot Q. \tag{D.19}$$

Taking inner product with $L_-^{-1}h$, we get $\langle L_+ h, L_-^{-1} h\rangle \le 0$. Using $L_+ = L_- - (p-1)Q^{p-1}$, we get

$$\begin{aligned}
&\|h\|^2_{L^2(\mathbb{R}^n,\mathbb{C})} \le (p-1)\langle Q^{p-1} h, L_-^{-1} h\rangle \lesssim \|h\|_{L^2(\mathbb{R}^n,\mathbb{C})} \\
&\Rightarrow \|h\|_{L^2(\mathbb{R}^n,\mathbb{C})} \lesssim 1.
\end{aligned} \tag{D.20}$$

Similarly by using $\langle L_+ h, L_-^k h\rangle \le 0$, $k \ge 0$, we obtain $\|h\|_{H^l(\mathbb{R}^n,\mathbb{C})} \lesssim 1$, $\forall l \ge 1$.

Since $\|L_-^{-1} h\|_{L^2(\mathbb{R}^n,\mathbb{C})} = 1$, it follows that $\|L_-^{-\frac{1}{2}} h\|_{L^2(\mathbb{R}^n,\mathbb{C})} \sim 1$. Taking inner product with h on both sides of (D.19), we get (below to avoid any confusion we denote by Q_p the ground state corresponding to p)

$$\begin{aligned}
-\mu_1 \|L_-^{-\frac{1}{2}} h\|^2_{L^2(\mathbb{R}^n,\mathbb{C})} &= -\langle L_+ h, h\rangle = -\langle (1 - \Delta - p_c Q_{p_c}^{p_c-1}) h, h\rangle \\
&\quad - \langle (p_c Q_{p_c}^{p_c-1} - p Q_p^{p-1}) h, h\rangle \\
&\le \mathcal{O}(|\langle Q_{p_c}, h\rangle|) \\
&\quad + \|p_c Q_{p_c}^{p_c-1} - p Q_p^{p-1}\|_{L^\infty(\mathbb{R}^n)} \|h\|^2_{L^2(\mathbb{R}^n,\mathbb{C})}.
\end{aligned} \tag{D.21}$$

The argument for the last inequality above is as follows. Firstly we make the decomposition

$$h = \underbrace{\|Q_{p_c}\|_2^{-2}\langle Q_{p_c}, h\rangle}_{=:c_h} Q_{p_c} + h^{\perp}, \tag{D.22}$$

where $h^{\perp} \perp Q_{p_c}$. Denote $L_{+,p_c} = 1 - \Delta - p_c Q_{p_c}^{p_c-1}$. By using $\langle L_{+,p_c} h^{\perp}, h^{\perp}\rangle \geq 0$, we derive

$$\begin{aligned}\langle L_{+,p_c} h, h\rangle &\geq 2\mathrm{Re}(c_h \langle L_{+,p_c} h^{\perp},\ Q_{p_c}\rangle) + |c_h|^2 \\ &= 2\mathrm{Re}(c_h \langle h^{\perp},\ L_{+,p_c} Q_{p_c}\rangle) + |c_h|^2 = \mathcal{O}(|c_h|).\end{aligned} \tag{D.23}$$

Since $h \perp Q_p$, we get $\langle Q_{p_c}, h\rangle = \langle Q_{p_c} - Q_p, h\rangle \xrightarrow{p \to p_c} 0$. Thus for $p \to p_c$ ($p > p_c$), we have $\mu_1 < 0$ and $\mu_1 \to 0$.

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