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Strong Ill-Posedness of the 3D Incompressible Euler Equation in Borderline Spaces

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For the d-dimensional incompressible Euler equation, the usual energy method gives local well-posedness for initial velocity in Sobolev space $H^s(\mathbb{R}^d)$, $s > s_c := d/2 + 1$. The borderline case $s = s_c$ was a folklore conjecture. In the previous paper [2], we introduced a new strategy (large lagrangian deformation and high frequency perturbation) and proved strong ill-posedness in the critical space $H^1(\mathbb{R}^2)$. The main issues in 3D are vorticity stretching, lack of L^p conservation, and control of lifespan. Nevertheless in this work we overcome these difficulties and show strong ill-posedness in 3D. Our results include general borderline Sobolev and Besov spaces.

1 Introduction

The d-dimensional incompressible Euler equation in velocity formulation has the form

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ \nabla \cdot u = 0, \\ u\big|_{t=0} = u_0, \end{cases} \tag{1.1}$$

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where $u = u(t,x) = (u_1(t,x), \cdots, u_d(t,x)) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is the velocity of the fluid. The scalar-valued pressure term $p = p(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ naturally arises as a Lagrange multiplier from the divergence-free constraint $\nabla \cdot u = 0$. Note that there are $(d+1)$ unknowns (u and p), $(d+1)$ equations, and thus the initial value problem (1.1) is formally consistent. The complexity of these equations can be reduced considerably by exploiting the vorticity formulation. For example in 2D, it is convenient to introduce the scalar-valued vorticity function

$$\omega = -\partial_2 u_1 + \partial_1 u_2 = \nabla^{\perp} \cdot u, \quad \nabla^{\perp} := (-\partial_2, \partial_1).$$

Under some suitable regularity and decay assumptions, one can recover u from the vorticity ω through the usual 2D Biot–Savart law:

$$u = \Delta^{-1} \nabla^{\perp} \omega = K_{2\mathrm{D}} * \omega, \quad K_{2\mathrm{D}}(x) = \frac{1}{2\pi} \cdot \frac{x^{\perp}}{|x|^2}, \quad x^{\perp} = (-x_2, x_1).$$

The 2D Euler equation in vorticity form then reads

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = 0, \\ u = \Delta^{-1} \nabla^{\perp} \omega, \\ \omega \Big|_{t=0} = \omega_0. \end{cases} \tag{1.2}$$

A nice feature of (1.2) is that it preserves all L^p, $1 \le p \le \infty$ norm of the vorticity that renders well-posedness (both local and global) theory much more manageable. In the 3D case the situation is more complicated as the vorticity function is genuinely vector valued. Denote

$$\omega = \operatorname{curl} u = \nabla \times u.$$

The 3D Euler equation can be expressed in terms of vorticity as

$$\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u,$$

where u is connected to ω via the 3D Biot–Savart law:

$$u = -\Delta^{-1} \nabla \times \omega = \int_{\mathbb{R}^3} K_{3\mathrm{D}}(x-y) \times \omega(y)\, \mathrm{d}y, \quad K_{3\mathrm{D}}(x) = \frac{1}{4\pi} \cdot \frac{x}{|x|^3}.$$

Compared to 2D, the additional vorticity stretching term $(\omega \cdot \nabla) u$ on the RHS introduces a new arena of mathematical and physical theories in analysis and computation.

Concerning the well-posedness theory, the 1st results date back to Lichtenstein [21] and Gunther [13], which gives local well-posedness in Hölder spaces $C^{k,\alpha}$. The global well-posedness of such classical Hölderian solutions was obtained in Wolibner [31] for 2D Euler. In the Sobolev setting Kato [16] first proved local well-posedness of d-dimensional Euler in the space $C_t^0 H_x^m$ for initial velocity $u_0 \in H^m(\mathbb{R}^d)$ with integer $m > d/2+1$. In the later fundamental work [18] Kato and Ponce removed the restriction that m is an integer and settled well-posedness in the general Sobolev space $W^{s,p}(\mathbb{R}^d)$ with real $s > d/p+1$ and $1 < p < \infty$. These spaces were conjectured to be optimal from a scaling heuristic and energy considerations. We refer to the introduction of [2], [3] and [10] for more extensive references and historical perspectives. A well-known long-standing open problem since the ’88 work of Kato–Ponce [18] was the following:

Conjecture 1.1. The Euler equation (1.1) is ill-posed for a class of initial data in $H^{d/2+1}(\mathbb{R}^d)$.

A synonym for ill-posedness in our context is norm inflation, that is, we are interested in the inflation of critical $H^{d/2+1}$ norms in the time evolution. There also exist a dozen of analogous versions of Conjecture 1.1 in similar Sobolev spaces $W^{d/p+1,p}$ or other Besov or Triebel–Lizorkin-type spaces. In all scenarios a rather delicate and difficult task is to introduce a rigorous mathematical framework and give a precise formulation of the illposedness statement in Conjecture 1.1. Roughly speaking, the main difficulty is that on the one hand one has to construct the solution and show “well-posedness” in some suitable sense, and on the other hand one needs to settle ill-posedness in the aforementioned critical spaces. The critical spaces lie at the threshold for well-posedness/ill-posedness and by their very nature are very sensitive to perturbations. For this it is of utter importance to identify the suitable function spaces in the construction of the solution that involves a subtle interplay of the lifespan and the norm. For 2D Euler this issue is somewhat easier to tackle since (thanks to the a priori control of L^p-norms of vorticity) not-so-rough solutions exist globally in time. However, for 3D and higher dimensions , this is no longer the case, and one has to obtain strong control of the lifespan of local solutions (in certain subcritical function spaces) while examining the persistence (or non-persistence) of various critical or non-critical function norms.

In recent [2], we introduced a new strategy (large Lagrangian deformation and high-frequency perturbation) and proved strong ill-posedness in the critical space $H^1(\mathbb{R}^2)$ for 2D Euler. In this paper we further develop this technique and analyze the 3D incompressible Euler case. The main issues in 3D are vorticity stretching, lack of L^p conservation, and control of lifespan. Nevertheless, in this work, we overcome these difficulties and show strong ill-posedness in 3D. Our results include general borderline Sobolev and Besov spaces. An informal summary of results is the following theorem.

Theorem. Let the dimension $d = 2, 3$. The Euler equation (1.1) is strongly ill-posed in the Sobolev space $W^{d/p+1,p}$ for any $1 < p < \infty$ or the Besov space $B^{d/p+1}_{p,q}$ for any $1 < p < \infty$, $1 < q \le \infty$.

The phrase "strongly illposed" needs some clarification. What we shall show is that in the borderline case, ill-posedness holds in the strongest sense. Namely for *any* given smooth initial data, one can find special perturbations that can be made arbitrarily small in the critical space norm, such that the corresponding perturbed solution is unique in other functional spaces but loses borderline Sobolev regularity instantaneously in time. In yet other words, the ill-posedness happens in a very generic way and is "dense" in the topology of critical norms.

We now state more precisely the main results. The 1st result is for 3D Euler with non-compactly supported data. As is well known, the lifespan of solutions to 3D Euler emanating from smooth initial data is an outstanding open problem. Since we are perturbing smooth initial data using functions with critical Sobolev regularity, we need to make sure the perturbed solution has a positive lifespan in some suitable functional spaces. In the non-compact data case, this issue turns out to be immaterial since we can choose the patches sufficiently far away from each other and the lifespan of each patch is well under control.

Theorem 1.2 (3D non-compact case). Consider the 3D incompressible Euler equation in vorticity form:

$$\begin{cases} \partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla) u, \quad t > 0,\ x = (x_1, x_2, z) \in \mathbb{R}^3; \\ u = -\Delta^{-1} \nabla \times \omega, \\ \omega\Big|_{t=0} = \omega_0. \end{cases} \tag{1.3}$$

For any given $\omega_0^{(g)} \in C_c^\infty(\mathbb{R}^3)$ and any $\epsilon > 0$, we can find a $T_0 = T_0(\omega_0^{(g)}) > 0$ and C^∞ perturbation $\omega_0^{(p)} : \mathbb{R}^3 \to \mathbb{R}^3$ such that the following hold true:

(1) $\|\omega_0^{(p)}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} + \|\omega_0^{(p)}\|_{L^1(\mathbb{R}^3)} + \|\omega_0^{(p)}\|_{L^\infty(\mathbb{R}^3)} < \epsilon$.

(2) Let $\omega_0 = \omega_0^{(g)} + \omega_0^{(p)}$. Let u_0 be the velocity corresponding to the initial vorticity ω_0. We have $u_0 \in H^{\frac{5}{2}}(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.

(3) Corresponding to ω_0, there exists a unique solution $\omega = \omega(t)$ to (1.3) on the whole time interval $[0, T_0]$ such that

$$\sup_{0 \le t \le T_0} (\|\omega(t,\cdot)\|_{L^1} + \|\omega(t,\cdot)\|_{L^\infty}) < \infty.$$

Moreover $\omega \in C^\infty$ and $u \in C^\infty$ so that the solution is actually classical.

(4) For any $0 < t_0 \le T_0$, we have

$$\text{ess-sup}_{0<t\le t_0} \|\omega(t,\cdot)\|_{\dot{H}^{\frac{3}{2}}} = +\infty.$$

Remark 1.3. If the vorticity $\omega_0^{(g)}$ is axisymmetric (see (1.4) below), then we can choose $T_0 > 0$ to be any positive number. This is due to the fact that for 3D Euler smooth axisymmetric flows without swirl exist globally in time.

Remark 1.4. In [32] Yudovich proved the existence and uniqueness of weak solutions to 2D Euler in bounded domains for L^∞ vorticity data. The uniqueness result (for bounded domain in general dimensions $d \ge 2$) was improved in [33] allowing vorticity $\omega \in \cap_{p_0 \le p < \infty} L^p$ and $\|\omega\|_p \le C\theta(p)$ with $\theta(p)$ growing relatively slowly in p (such as $\theta(p) = \log p$). Vishik [30] proved the uniqueness of weak solutions to Euler in $\mathbb{R}^d$, $d \ge 2$, under the following assumptions:

- $\omega \in L^{p_0}$, $1 < p_0 < d$,
- For some $a(k) > 0$ with the property

$$\int_1^\infty \frac{1}{a(k)}\, \mathrm{d}k = +\infty,$$

it holds that

$$\left| \sum_{j=2}^{k} \|P_{2^j}\omega\|_\infty \right| \le \text{const} \cdot a(k), \qquad \forall\, k \ge 4.$$

In other words, uniqueness is guaranteed as long as ω has a little bit integrability and the partial sum of the Besov $\dot{B}^0_{\infty,1}$ norm of ω is allowed to diverge in a controlled fashion. Since we have uniform in time L^∞ control of the vorticity ω (see Theorems 1.2–1.5 below), the uniqueness of the constructed solution is not an issue and we shall not discuss this point further in this work.

The following theorem concerns the 3D Euler case with compactly supported initial vorticity. In this case the situation is more complicated than that in Theorem 1.2. For convenience we will work with a class of axisymmetric vorticity functions ω having the form:

$$\omega(x) = \omega^{\theta}(r,z)e_{\theta}, \quad x = (x_1, x_2, z),\ r = \sqrt{x_1^2 + x_2^2}, \tag{1.4}$$

where ω^{θ} is scalar valued and $e_{\theta} = \frac{1}{r}(-x_2, x_1, 0)$. The corresponding velocity fields are usually called axisymmetric without swirl flows. In this paper we shall call such ω axisymmetric without swirl vorticity or simply axisymmetric vorticity when there is no obvious confusion. The theory of axisymmetric flows on $\mathbb{R}^3$ and some recent developments are reviewed in the beginning of Section 3.

Theorem 1.5 (3D compact case). For any given axisymmetric vorticity $\omega_0^{(g)} \in C_c^{\infty}(\mathbb{R}^3)$ and any $\epsilon > 0$, we can find a perturbation $\omega_0^{(p)} : \mathbb{R}^3 \to \mathbb{R}^3$ such that the following hold true:

(1) $\omega_0^{(p)}$ is compactly supported (in a ball of radius ≤ 1), continuous and

$$\|\omega_0^{(p)}\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} + \|\omega_0^{(p)}\|_{L^{\infty}(\mathbb{R}^3)} < \epsilon.$$

(2) Let $\omega_0 = \omega_0^{(g)} + \omega_0^{(p)}$. Corresponding to ω_0 there exists a unique solution $\omega = \omega(t,x)$ to the Euler equation (1.3) on the time interval $[0,1]$ satisfying

$$\sup_{0\leq t\leq 1} \|\omega(t,\cdot)\|_{L^{\infty}} < \infty,$$
$$\mathrm{supp}(\omega(t,\cdot)) \subset \{x,\ |x| < R\}, \quad \forall\, 0 \leq t \leq 1, \tag{1.5}$$

where $R > 0$ is some constant. Furthermore $\omega \in C_t^0 C_x^0$ and $u \in C_t^0 L_x^2 \cap C_t^0 C_x^{\alpha}$ for any $\alpha < 1$.

(3) $\omega(t)$ has additional local regularity in the following sense: there exists $x_* \in \mathbb{R}^3$ such that for any $x \neq x_*$, there exists a neighborhood $N_x \ni x$, $t_x > 0$ such that $w(t) \in C^{\infty}(N_x)$ for any $0 \leq t \leq t_x$.

(4) For any $0 < t_0 \leq 1$, we have

$$\text{ess-sup}_{0<t\leq t_0} \|\omega(t,\cdot)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} = +\infty. \tag{1.6}$$

More precisely, there exist $0 < t_n^1 < t_n^2 < \frac{1}{n}$, open precompact sets Ω_n^1 and Ω_n^2 with $\Omega_n^1 \subset \overline{\Omega_n^1} \subset \Omega_n^2$, $n = 1, 2, 3, \cdots$ such that

- $\omega(t) \in C^\infty(\Omega_n^2)$ for all $0 \le t \le t_n^2$;
- $\omega(t, x) \equiv 0$ for any $x \in \Omega_n^2 \setminus \Omega_n^1$, $0 \le t \le t_n^2$.
- Define $\omega_n(t, x) = \omega(t, x)$ for $x \in \Omega_n^1$, and $\omega_n(t, x) = 0$ otherwise. Then $\omega_n \in C_c^\infty(\mathbb{R}^3)$,

$$\|\omega_n(t, \cdot)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} > n, \quad \forall\, t_n^1 \le t \le t_n^2. \tag{1.7}$$

and

$$\|(|\nabla|^3 \omega_n)(t, \cdot)\|_{L^2(x \in \mathbb{R}^3 \setminus \Omega_n^2)} \le 1, \quad \forall\, 0 \le t \le t_n^2. \tag{1.8}$$

Remark 1.6. We stress that the situation here in Theorem 1.5 is much more complex than the 2D case in [2]. Due to the nonlocal character of the fractional differentiation operator $|\nabla|^{\frac{3}{2}}$, we have to include the additional constraint (1.8) in our construction in order to derive (1.6) from (1.7). We briefly sketch the argument as follows. Suppose $\|\omega(\tau, \cdot)\|_{\dot{H}^{\frac{3}{2}}} < \infty$, for some $\tau \in [t_n^1, t_n^2]$. Then we write

$$\omega(\tau) = \omega_n(\tau) + g_n(\tau),$$

where $g_n(\tau) = \omega(\tau) - \omega_n(\tau)$ also has finite $\dot{H}^{\frac{3}{2}}$-norm. Clearly

$$\|\omega(\tau)\|^2_{\dot{H}^{\frac{3}{2}}} = \|\omega_n(\tau)\|^2_{\dot{H}^{\frac{3}{2}}} + \|g_n(\tau)\|^2_{\dot{H}^{\frac{3}{2}}} + 2\langle |\nabla|^{\frac{3}{2}} \omega_n(\tau), |\nabla|^{\frac{3}{2}} g_n(\tau)\rangle,$$

where $\langle , \rangle$ denotes the usual L^2 inner product on $L^2(\mathbb{R}^3)$. Now observe that $\mathrm{supp}(g_n(\tau)) \subset \mathbb{R}^3 \setminus \Omega_n^2$ and $\|g_n(\tau)\|_{L^2} \lesssim \|\omega(\tau)\|_2$ ($g_n(\tau)$ and $\omega_n(\tau)$ have disjoint supports; therefore

$$\begin{aligned}
&|\langle |\nabla|^{\frac{3}{2}} \omega_n(\tau), |\nabla|^{\frac{3}{2}} g_n(\tau)\rangle| \\
&\quad = |\langle |\nabla|^3 \omega_n(\tau), g_n(\tau)\rangle| \\
&\quad \le \|(|\nabla|^3 \omega_n)(\tau)\|_{L^2(\mathbb{R}^3 \setminus \Omega_n^2)} \|g_n(\tau)\|_{L^2} \\
&\quad \lesssim \|\omega(\tau)\|_2 \lesssim 1.
\end{aligned}$$

Hence, for n sufficiently large, we have for any $\tau \in [t_n^1, t_n^2]$, either

$$\|\omega(\tau)\|_{\dot{H}^{\frac{3}{2}}} = +\infty$$

or

$$\|\omega(\tau)\|_{\dot{H}^{\frac{3}{2}}} > \frac{n}{2}.$$

This obviously implies (1.6).

Al these previous theorems can be sharpened significantly. We have the following Besov version that essentially includes all previous theorems as special cases. In order not to overburden with notations, we shall state an informal version. The detailed (and more precise) statements can be found in Section 5 and Theorems 5.1–5.4 therein.

Theorem 1.7 (Besov case). Let $d = 2, 3$. For any smooth initial velocity $u_0^{(g)}$, any $\epsilon > 0$, and any $1 < p < \infty$, $1 < q \le \infty$, there exists a nearby initial velocity $u_0 \in B_{p,q}^{\frac{d}{p}+1}$ such that $\|u_0 - u_0^{(g)}\|_{B_{p,q}^{\frac{d}{p}+1}} < \epsilon$, and the corresponding solution satisfies

$$\text{ess-sup}_{0<t<t_0} \|u(t,\cdot)\|_{\dot{B}_{p,\infty}^{\frac{d}{p}+1}} = +\infty$$

for any $t_0 > 0$.

Our last result concerns the ill-posedness in the usual Sobolev $W^{s,p}$ spaces.

Theorem 1.8. Let $d = 2, 3$. For any smooth initial velocity $u_0^{(g)}$, any $\epsilon > 0$, and any $1 < p < \infty$, there exists a nearby initial velocity $u_0 \in W^{d/p+1,p}$ such that $\|u_0 - u_0^{(g)}\|_{W^{d/p+1,p}} < \epsilon$, and the solution corresponding to u_0 satisfies

$$\text{ess-sup}_{0<t<t_0} \|u(t,\cdot)\|_{\dot{W}^{d/p+1,p}} = +\infty$$

for any $t_0 > 0$.

The proof of Theorem 1.8 will be omitted. It can be subsumed under a more general argument that we will address elsewhere.

We now give a brief overview of the proofs of the main theorems. Broadly speaking our approach is taking advantages of both the Lagrangian point of view (in the large Lagrangian deformation step, see below) and the Eulerian approach (in the high frequency perturbation and the "glueing" steps, see below). This hybrid point of view is quite useful in nonlocal fluid-type equations possessing intricate nonlinear structures and can even manifest in global regularity and well-posedness problems (cf. recent [20] for breakthrough result in 2D incompressible elastodynamics and [34] in 2D compressible MHD system). The overall scheme follows the 2D case in [2] and consists of three steps. In the 1st step, we perform a local construction and create large Lagrangian deformation matrix for well-chosen data. In the 2nd step, another high-frequency perturbation argument is used to generate critical norm inflation in a local patch. In the 3rd step, we glue the "patch solutions" obtained from the 1st two steps together and show that they give rise to the desired norm inflation. In this step there are two further subcases: non-compact case in which the patch solutions interact weakly and generate a C^∞ solution, and the compact case in which the patch solutions have strong interactions. In the latter subcase, a very involved analysis is needed to take care of the interactions among the solution patches.

We now point out the new technical ingredients in the proof for the 3D case. Compared with the 2D case, the 1st difficulty in 3D is the lack of L^p conservation of the vorticity. It is deeply connected with the vorticity stretching term $(\omega \cdot \nabla)u$. To simplify the analysis we take the axisymmetric flow without swirl as the basic building block for the whole construction. The vorticity equation in the axisymmetric case (see the beginning of Section 3) takes the form

$$\partial_t \left(\frac{\omega}{r}\right) + (u \cdot \nabla) \left(\frac{\omega}{r}\right) = 0, \quad r = \sqrt{x_1^2 + x_2^2},\, x = (x_1, x_2, z).$$

Owing to the denominator r, the solution formula for ω then acquires an additional metric factor (compared with 2D) that represents the vorticity stretching effect in the axisymmetric setting. A lot of analysis (cf. Proposition 3.11) goes into controlling the metric factor by the large Lagrangian deformation matrix and producing the desired $H^{3/2}$ norm inflation. In our construction the patch solutions that are made of asymmetric without swirl flows typically carry infinite $\|\omega/r\|_{L^{3,1}}$ norm (when summing all the patches together). To glue these solutions together in the 3D compactly supported case, we need to run a new perturbation argument (cf. Lemma 4.1) that allows to add each new patch ω_n with sufficiently small $\|\omega_n\|_\infty$ norm (over the whole lifespan) such that the effect of the large $\|\omega_n/r\|_{L^{3,1}}$ becomes negligible. All in all, the constructed patch solutions converge in the C^0 metric after building several auxiliary lemmas (cf. Lemma 4.5 and Proposition 4.6).

2 Notation and preliminaries

For any two quantities X and Y, we denote $X \lesssim Y$ if $X \le CY$ for some harmless constant $C > 0$. Similarly $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. We shall write $X \lesssim_{Z_1,Z_2,\cdots,Z_k} Y$ if $X \le CY$ and the constant C depends on the quantities $(Z_1,\cdots,Z_k)$. Similarly we define $\gtrsim_{Z_1,\cdots,Z_k}$ and $\sim_{Z_1,\cdots,Z_k}$.

We shall denote by $X+$ any quantity of the form $X+\epsilon$ for any $\epsilon > 0$. For example we shall write

$$Y \lesssim 2^{X+} \tag{2.1}$$

if $Y \lesssim_\epsilon 2^{X+\epsilon}$ for any $\epsilon > 0$. The notation $X-$ is similarly defined.

For any center $x_0 \in \mathbb{R}^d$ and radius $R > 0$, we use $B(x_0,R) := \{x \in \mathbb{R}^d : |x-x_0| < R\}$ to denote the open Euclidean ball. More generally for any set $A \subset \mathbb{R}^d$, we denote

$$B(A,R) := \big\{y \in \mathbb{R}^d : \ |y-x| < R \text{ for some } x \in A\big\}. \tag{2.2}$$

For any two sets $A_1, A_2 \subset \mathbb{R}^d$, we define

$$d(A_1,A_2) = \operatorname{dist}(A_1,A_2) = \inf\big\{|x-y| : \ x \in A_1,\ y \in A_2\big\}.$$

For any f on $\mathbb{R}^d$, we denote the Fourier transform of f has

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x}\, \mathrm{d}x.$$

The inverse Fourier transform of any g is given by

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(\xi) e^{ix\cdot\xi}\, \mathrm{d}\xi.$$

For any $1 \le p \le \infty$ we use $\|f\|_p$, $\|f\|_{L^p(\mathbb{R}^d)}$, or $\|f\|_{L^p_x(\mathbb{R}^d)}$ to denote the usual Lebesgue norm on $\mathbb{R}^d$. The Sobolev space $H^1(\mathbb{R}^d)$ is defined in the usual way as the completion of C_c^∞ functions under the norm $\|f\|_{H^1} = \|f\|_2 + \|\nabla f\|_2$. For any $s \in \mathbb{R}$, we define the homogeneous Sobolev norm of a tempered distribution $f : \mathbb{R}^d \to \mathbb{R}$ as

$$\|f\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2\, \mathrm{d}\xi\right)^{\frac{1}{2}}.$$

We use the Fourier transform to define the fractional differentiation operators $|\nabla|^s$ by the formula

$$\widehat{|\nabla|^s f}(\xi) = |\xi|^s \hat{f}(\xi).$$

For any integer $n \geq 0$ and any open set $U \subset \mathbb{R}^d$, we use the notation $C^n(U)$ to denote functions on U whose n^{th} derivatives are all continuous.

For any $1 \leq p < \infty$, we denote by $L^p_{ul}(\mathbb{R}^d)$ the Banach space endowed with the norm

$$\|u\|_{L^p_{ul}(\mathbb{R}^d)} := \sup_{x\in\mathbb{R}^d} \left(\int_{|y-x|<1} |u(y)|^p \, \mathrm{d}y \right)^{\frac{1}{p}}. \tag{2.3}$$

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be not identically zero. The condition $u \in L^p_{ul}$ is equivalent to

$$\sup_{x\in\mathbb{R}^d} \|\phi(\cdot - x)u(\cdot)\|_{L^p(\mathbb{R}^d)} < \infty.$$

For any $s \in \mathbb{R}$ and any function $u \in H^s_{loc}(\mathbb{R}^d)$, one can define

$$\|u\|_{H^s_{ul}(\mathbb{R}^d)} = \sup_{x\in\mathbb{R}^d} \|\phi(\cdot - x)u(\cdot)\|_{H^s(\mathbb{R}^d)}.$$

In Section 3 and later sections, we need to use Lorentz spaces. We recall the definitions here. For a measurable function f, the nonincreasing rearrangement f^* is defined by

$$f^*(t) = \inf\Big\{ s : \mathrm{Leb}(x : |f(x)| > s) \leq t \Big\}.$$

For $1 \leq p, q < \infty$, the Lorentz space $L^{p,q}$ is the set of functions f that satisfy

$$\|f\|_{L^{p,q}} := \left(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} < \infty.$$

For $q = \infty$, $L^{p,\infty}$ is the set of functions such that

$$\|f\|_{L^{p,\infty}} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty.$$

For $p = \infty$, we set $L^{\infty,q} = L^\infty$ for all $1 \leq q \leq \infty$. Note that $L^{p,p} = L^p$. For $1 < p < \infty$, the space $L^{p,q}$ coincides with the real interpolation from Lebesgue spaces.

We will need to use the Littlewood–Paley frequency projection operators. Let $\varphi(\xi)$ be a smooth bump function supported in the ball $|\xi| \leq 2$ and equal to one on the ball $|\xi| \le 1$. For any real number $N > 0$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, define the frequency localized (LP) projection operators:

$$\widehat{P_{\le N} f}(\xi) := \varphi(\xi/N)\hat{f}(\xi),$$
$$\widehat{P_{> N} f}(\xi) := \big[1 - \varphi(\xi/N)\big]\hat{f}(\xi),$$
$$\widehat{P_N f}(\xi) := \big[\varphi(\xi/N) - \varphi(2\xi/N)\big]\hat{f}(\xi).$$

Similarly we can define $P_{<N}$, $P_{\ge N}$, and $P_{M<\cdot\le N} := P_{\le N} - P_{\le M}$ whenever $N > M > 0$ are real numbers. We will usually use these operators when M and N are dyadic numbers. The summation over N or M are understood to be over dyadic numbers. Occasionally, for convenience of notation, we allow M and N not to be a power of 2.

We recall the following Bernstein estimates: for any $1 \le p \le q \le \infty$, $s \in \mathbb{R}$,

$$\||\nabla|^s P_N f\|_{L_x^p(\mathbb{R}^d)} \sim N^s \|P_N f\|_{L_x^p(\mathbb{R}^d)},$$
$$\|P_{\le N} f\|_{L_x^q(\mathbb{R}^d)} \lesssim_d N^{d(\frac{1}{p}-\frac{1}{q})} \|P_{\le N} f\|_{L_x^p(\mathbb{R}^d)},$$
$$\|P_N f\|_{L_x^q(\mathbb{R}^d)} \lesssim_d N^{d(\frac{1}{p}-\frac{1}{q})} \|P_N f\|_{L_x^p(\mathbb{R}^d)}.$$

For any $s \in \mathbb{R}$, $1 \le p, q \le \infty$, we define the homogeneous Besov seminorm as

$$\|f\|_{\dot{B}^s_{p,q}} := \begin{cases} \Big(\sum_{N>0} N^{sq} \|P_N f\|^q_{L^p(\mathbb{R}^d)}\Big)^{\frac{1}{q}}, & \text{if } 1 \le q < \infty, \\ \sup_{N>0} N^s \|P_N f\|_{L^p(\mathbb{R}^d)}, & \text{if } q = \infty. \end{cases}$$

The inhomogeneous Besov norm $\|f\|_{B^s_{p,q}}$ of $f \in \mathcal{S}'(\mathbb{R}^d)$ is

$$\|f\|_{B^s_{p,q}} = \|f\|_p + \|f\|_{\dot{B}^s_{p,q}}.$$

For any $s \in \mathbb{R}$, $1 < p < \infty$, and $1 \le q \le \infty$, the homogeneous Triebel–Lizorkin seminorm is defined by

$$\|f\|_{\dot{F}^s_{p,q}} := \begin{cases} \Big\| (\sum_{N>0} N^{sq} |P_N f|^q)^{\frac{1}{q}} \Big\|_{L^p}, & \text{if } 1 \le q < \infty, \\ \Big\| \sup_{N>0} N^s |P_N f| \Big\|_{L^p}, & \text{if } q = \infty. \end{cases}$$

The inhomogeneous Triebel–Lizorkin norm is

$$\|f\|_{F^s_{p,q}} = \|f\|_p + \|f\|_{\dot{F}^s_{p,q}}.$$

3 3D Euler with non-compactly supported data

The main building block in our construction for 3D Euler is the axisymmetric flow without swirl on $\mathbb{R}^3$. To unify the notation we first recall the definition and review some useful properties. We shall review some relevant literature along the way.

By an axisymmetric scalar function on $\mathbb{R}^3$, we mean a function of the form $f = f(r,z)$, with $x = (x_1, x_2, z)$, $r = \sqrt{x_1^2 + x_2^2}$. In other words the function is invariant under any rotation about the vertical axis $OZ = \{(0,0,z) : z \in \mathbb{R}\}$. Analogously one can define an axisymmetric vector field on $\mathbb{R}^3$. The velocity field u representing an axisymmetric flow on $\mathbb{R}^3$ generally takes the form

$$u(x_1, x_2, z) = u^r(r,z)e_r + u^\theta(r,z)e_\theta + u^z(r,z)e_z,$$

where (e_r, e_θ, e_z) is the standard orthogonal basis for the cylindrical coordinate system:

$$e_r = \frac{1}{r}(x_1, x_2, 0), \quad e_\theta = \frac{1}{r}(-x_2, x_1, 0),$$
$$e_z = (0,0,1).$$

Here u^r, u^θ, and u^z are called the radial, angular/swirl, and axial velocity respectively. By an axisymmetric flow without swirl, we mean the angular (swirl) component $u^\theta \equiv 0$, that is

$$u(x_1, x_2, z) = u^r(r,z)e_r + u^z(r,z)e_z.$$

The corresponding vorticity $\omega = \nabla \times u$ then reduces to

$$\omega(x_1, x_2, z) = \omega^\theta(r,z)e_\theta = (\partial_z u^r - \partial_r u^z)e_\theta.$$

It follows easily that

$$\begin{aligned}(\omega \cdot \nabla)u &= (\omega^\theta e_\theta \cdot \nabla)u \\ &= \omega^\theta \frac{1}{r} u^r e_\theta = \frac{u^r}{r}\omega.\end{aligned}$$

Therefore, the vorticity equation reads as

$$\partial_t \omega + (u^r \partial_r + u^z \partial_z)\omega = \frac{1}{r} u^r \omega,$$

or simply,

$$\partial_t \left(\frac{\omega}{r}\right) + (u \cdot \nabla) \left(\frac{\omega}{r}\right) = 0.$$

In this way we obtain a transport equation for the quantity ω/r. Since the velocity u is divergence-free, all L^p, $1 \le p \le \infty$ and similar Lorentz space norms of ω/r are preserved in time. These important conservation laws are the key to obtaining global solutions. Indeed Ukhovskii and Yudovich [28] and independently Ladyzhenskaya [19] first proved global well-posedness for initial velocity $u_0 \in H^1$ with initial vorticity satisfying $\omega_0 \in L^2 \cap L^\infty$, $\frac{\omega_0}{r} \in L^2 \cap L^\infty$. In terms of Sobolev regularity, one need the initial velocity $u_0 \in H^s$, $s > 7/2$ to have $\frac{1}{r}\omega_0 \in L^\infty$. In [26], Shirota and Yanagisawa weakened the regularity on velocity to $u_0 \in H^s$, $s > 5/2$, which is the borderline in view of the H^s local well-posedness theory in 3D. Danchin [9] showed global existence and uniqueness of solutions for initial vorticity $\omega_0 \in L^\infty \cap L^{3,1}$ with $\frac{\omega}{r} \in L^{3,1}$ in bounded domains with $C^{2,\alpha}$ boundary or the whole space $\mathbb{R}^3$. In [1] Abidi, Hmidi, and Keraani proved global well-posedness in the space $C_t^0 B_{p,1}^{\frac{3}{p}+1}(\mathbb{R}_+ \times \mathbb{R}^3)$ for initial velocity $u_0 \in B_{p,1}^{\frac{3}{p}+1}(\mathbb{R}^3)$, $1 \le p \le \infty$ with the additional mild assumption that $\frac{\omega_0}{r} \in L^{3,1}(\mathbb{R}^3)$.
The work of Danchin [9] does not address the propagation of critical Besov regularity $B_{p,1}^{\frac{3}{p}+1}$ due to the lack of Beale–Kato–Majda criteria in borderline Besov spaces.
For $p < 3$, the condition $\frac{\omega_0}{r} \in L^{3,1}(\mathbb{R}^3)$ can be derived from $u_0 \in B_{p,1}^{\frac{3}{p}+1}(\mathbb{R}^3)$. See Prop 2.2 in [1].

We now state a few basic lemmas needed for our construction later. Some of these are well-known facts that are already extensively used in the aforementioned works.

Lemma 3.1 ($L^{p,q}$-preservation). Let $1 \le p, q \le \infty$. Suppose u is a given smooth divergence-free vector field on $\mathbb{R}^d$, $d \ge 2$. Let h be the smooth solution to the transport equation

$$\begin{cases} \partial_t h + (u \cdot \nabla) h = f, \\ h\Big|_{t=0} = h_0. \end{cases}$$

Then for any $t > 0$, we have

$$\|h(t)\|_{L^{p,q}(\mathbb{R}^d)} \le \|h_0\|_{L^{p,q}(\mathbb{R}^d)} + \int_0^t \|f(\tau)\|_{L^{p,q}(\mathbb{R}^d)} \, \mathrm{d}\tau.$$

If $f \equiv 0$, then

$$\|h(t)\|_{L^{p,q}(\mathbb{R}^d)} = \|h_0\|_{L^{p,q}(\mathbb{R}^d)}.$$

Proof of Lemma 3.1. See for example Proposition 2 on p. 484 of Danchin [9] or Proposition 2.3 of Abidi–Hmidi–Keraani [1]. In the homogeneous case $f \equiv 0$, one just observes that $h(t) = h_0 \circ \phi(t)$ where the flow map $\phi(t)$ is measure preserving. This obviously implies $\|h_0 \circ \phi(t)\|_{L^{p,q}} = \|h_0\|_{L^{p,q}}$ by using the definition of Lorentz spaces. Alternatively one can use L^p conservation and interpolation, as done in [1]. The general case of nonzero f follows from Duhamel. ■

We shall use Lemma 3.1 often without explicit mentioning. The most useful case for us is the $L^{3,1}$ conservation of vorticity.

The next useful lemma is a Biot–Savart law estimate in the axisymmetric setting. It is the key to the proof of global well-posedness for the axisymmetric (without swirl) flow.

Lemma 3.2. There exists an absolute constant $C > 0$ such that

$$\left\|\frac{u^r}{r}\right\|_{L^\infty(\mathbb{R}^3)} \leq C \left\|\frac{\omega^\theta}{r}\right\|_{L^{3,1}(\mathbb{R}^3)},$$

where $u = u^r e_r + u^z e_z$, $\omega = \nabla \times u = \omega^\theta e_\theta$.

Proof. of Lemma 3.2 See, for example, [1,Proposition 4.1]. The key is to use Lemma 1 from [26], which gives the kernel estimate:

$$|u^r(x)| \lesssim \int_{|y-x|\leq r} \frac{|\omega(y)|}{|x-y|^2}\, dy + r \int_{|y-x|\geq r} \frac{|\omega(y)|}{|x-y|^3}\, dy.$$

■

Occasionally we need the following lemma.

Lemma 3.3. Let $1 \leq p < 3$ and $u \in B_{p,q}^{\frac{3}{p}+1}(\mathbb{R}^3)$ be an axisymmetric divergence-free vector field. Let $\omega = \nabla \times u$ be its vorticity. Then

$$\|\frac{\omega}{r}\|_{L^{3,1}(\mathbb{R}^3)} \lesssim \|u\|_{B_{p,1}^{\frac{3}{p}+1}(\mathbb{R}^3)}.$$

Proof of Lemma 3.3 This is Proposition 2.2 from [1]. The key is to prove $\|\frac{\omega}{r}\|_{L^{3,1}} \lesssim \|\nabla\omega\|_{L^{3,1}}$ that is obtained by using $\omega(0,0,z) \equiv 0$ and the fundamental theorem of calculus. The embedding $B^{\frac{3}{p}-1}_{p,1} \hookrightarrow L^{3,1}$ comes from interpolation

$$(L^p, L^r)_{(\theta,1)} = L^{3,1}$$
$$\left(B^0_{p,1}, B^{\frac{3}{p}-\frac{3}{r}}_{p,1}\right)_{(\theta,1)} = B^{\theta(\frac{3}{p}-\frac{3}{r})}_{p,1} = B^{\frac{3}{p}-1}_{p,1},$$

where $1 \le p < 3 < r < \infty$ and $\frac{1}{3} = \frac{1-\theta}{p} + \frac{\theta}{r}$, followed by the embedding of Besov into Lebesgue. ■

We now state a lemma that sets up some basic properties of the axisymmetric flow and the associated characteristic lines.

Lemma 3.4. Consider the Euler equation (in vorticity form)

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega = (\omega\cdot\nabla)u, \quad (t,x) \in [0,\infty)\times\mathbb{R}^3, \\ u = -\Delta^{-1}\nabla\times\omega, \\ \omega\Big|_{t=0} = \omega_0, \end{cases}$$

where the initial vorticity $\omega_0 = \nabla\times u_0$ and u_0 is a smooth axisymmetric velocity field without swirl.

Write

$$u = u^r e_r + u^z e_z, \tag{3.1}$$

and consider the forward characteristic lines expressed in cylindrical coordinates, that is,

$$\begin{cases} \partial_t\phi^r(t,r,z) = u^r(t,\phi^r(t,r,z),\phi^z(t,r,z)), \\ \partial_t\phi^z(t,r,z) = u^z(t,\phi^r(t,r,z),\phi^z(t,r,z)), \\ (\phi^r,\phi^z)\Big|_{t=0} = (r,z). \end{cases} \tag{3.2}$$

Denote by $\tilde{\phi}(t,r,z) = (\tilde{\phi}^r(t,r,z), \tilde{\phi}^z(t,r,z))$ the inverse map of $\phi = (\phi^r,\phi^z)$. Then the following hold:

- For any $z \in \mathbb{R}$, $t \geq 0$, we have

$$u^r(t,0,z) = 0,$$
$$\phi^r(t,0,z) = 0 = \tilde{\phi}^r(t,0,z). \tag{3.3}$$

Also

$$\phi^r(t,r,z) > 0,\ \tilde{\phi}^r(t,r,z) > 0, \quad \forall\, r > 0, z \in \mathbb{R},\ t \geq 0. \tag{3.4}$$

- For any $t \geq 0$, $r > 0$, $z \in \mathbb{R}$, we have

$$\begin{aligned} &\det\left(\frac{\partial(\phi^r(t),\phi^z(t))}{\partial(r,z)}\right) \\ &= \frac{r}{\phi^r(t,r,z)} \\ &= \exp\left(-\int_0^t \frac{1}{\phi^r(s,r,z)} u^r(s,\phi^r(s,r,z),\phi^z(s,r,z))\,\mathrm{d}s\right). \end{aligned} \tag{3.5}$$

For $t \geq 0$, $r = 0$, $z \in \mathbb{R}$, we have

$$\begin{aligned} &\det\left(\frac{\partial(\phi^r(t),\phi^z(t))}{\partial(r,z)}\right)\Big|_{(r,z)=(0,z)} \\ &= \lim_{r\to 0} \frac{r}{\phi^r(t,r,z)} \\ &= \exp\left(-\int_0^t \lim_{r\to 0} \frac{1}{\phi^r(s,r,z)} u^r(s,\phi^r(s,r,z),\phi^z(s,r,z))\,\mathrm{d}s\right). \end{aligned} \tag{3.6}$$

Here both limits exist and are finite.

- Similarly for any $t \geq 0$, $r \geq 0$, $z \in \mathbb{R}$, we have

$$\det\left(\frac{\partial(\tilde{\phi}^r(t),\tilde{\phi}^z(t))}{\partial(r,z)}\right) = \frac{r}{\tilde{\phi}^r(t,r,z)}. \tag{3.7}$$

Proof of Lemma 3.4 We first show (3.3). Obviously by (3.1), we must have

$$u^r(t,0,z) = 0, \quad \forall\, t \geq 0,\ z \in \mathbb{R},$$

since otherwise u would not be smooth at $x = (0, 0, z)$. By a similar consideration, we can Taylor expand (u_1, u_2) around the point $(0, 0, z)$ to get

$$\begin{aligned}\Big(u_1(t, x_1, x_2, z),\ u_2(t, x_1, x_2, z)\Big) &= c(t, z)(x_1,\ x_2) + O(r^2)\\ &= c(t, z) r e_r + O(r^2)\\ &= u^r(t, r, z) e_r.\end{aligned}$$

Here the coefficient $c(t, z)$ is scalar valued and

$$c(t, z) = (\partial_1 u_1)(t, 0, 0, z) = -(\partial_2 u_2)(t, 0, 0, z).$$

Therefore,

$$u^r(t, r, z) = c(t, z) r + O(r^2)$$

and

$$\begin{aligned}\lim_{r\to 0} \frac{u^r(t, r, z)}{r} &= c(t, z),\\ \lim_{r\to 0} (\partial_r u^r)(t, r, z) &= c(t, z).\end{aligned} \tag{3.8}$$

By (3.2), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi^r &= u^r(t, \phi^r, \phi^z) - u^r(t, 0, \phi^z)\\ &= \left(\int_0^1 (\partial_r u^r)(t, \theta\phi^r, \phi^z)\, \mathrm{d}\theta\right)\phi^r.\end{aligned}$$

Integrating in time then gives

$$\phi^r(t, r, z) = r \exp\left(\int_0^t \int_0^1 (\partial_r u^r)(s, \theta\phi^r(s, r, z), \phi^z(s, r, z))\, \mathrm{d}\theta\, \mathrm{d}s\right). \tag{3.9}$$

Clearly (3.4) follows. Also

$$\phi^r(t, 0, z) = 0, \quad \text{for any } t \geq 0, z \in \mathbb{R}.$$

Next we show (3.5). We shall calculate $\det\Big(\frac{\partial(\phi^r, \phi^z)}{\partial(r, z)}\Big)$ in two ways that in turn would yield the 1st and 2nd identity in (3.5).

Introduce new variables

$$R := \frac{1}{2}\phi^r(t,r,z)^2,$$
$$Z := \phi^z(t,r,z). \tag{3.10}$$

Then

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}R = V^r(t,R,Z) \\ \frac{\mathrm{d}}{\mathrm{d}t}Z = V^z(t,R,Z) \\ (R,Z)\Big|_{t=0} = (R_0,Z_0). \end{cases}$$

Here

$$V^r(t,R,Z) := \sqrt{2R}u^r(t,\sqrt{2R},Z),$$
$$V^z(t,R,Z) := u^z(t,\sqrt{2R},Z).$$

By the incompressibility condition $\nabla \cdot u = 0$, we have

$$\frac{1}{r}\partial_r(ru^r(t,r,z)) + \partial_z u^z(t,r,z) = 0.$$

Therefore, it is easy to check that

$$\partial_R V^r(t,R,Z) + \partial_Z V^z(t,R,Z) = 0.$$

It follows that

$$\det\left(\frac{\partial(R(t),Z(t))}{\partial(R_0,Z_0)}\right) = 1, \quad \forall\, t \geq 0,$$

or

$$\frac{\partial R}{\partial R_0}\frac{\partial Z}{\partial Z_0} - \frac{\partial R}{\partial Z_0}\frac{\partial Z}{\partial R_0} = 1.$$

Letting $R_0 = \frac{1}{2}r^2$, $Z_0 = z$ and using (3.10) then gives

$$\det\left(\frac{\partial(\phi^r,\phi^z)}{\partial(r,z)}\right) = \frac{r}{\phi^r}.$$

For the 2nd way of calculating $\det\left(\frac{\partial(\phi^r,\phi^z)}{\partial(r,z)}\right)$, we need to use the following elementary fact: if $A(t) \in \mathbb{R}^{d\times d}$ and $B(t) \in \mathbb{R}^{d\times d}$ are smooth matrix-valued functions solving the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t) = B(t)A(t)$$

with $A(0) = I_d$ (the identity matrix on $\mathbb{R}^{d\times d}$), then

$$\det(A(t)) = \exp\left(\int_0^t \mathrm{tr}(B(s))\,\mathrm{d}s\right),$$

where $\mathrm{tr}(\cdot)$ denotes the usual matrix trace.

By (3.2), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \frac{\partial\phi^r}{\partial r} & \frac{\partial\phi^r}{\partial z} \\ \frac{\partial\phi^z}{\partial r} & \frac{\partial\phi^z}{\partial z} \end{pmatrix} = \begin{pmatrix} \partial_r u^r & \partial_z u^r \\ \partial_r u^z & \partial_z u^z \end{pmatrix}\begin{pmatrix} \frac{\partial\phi^r}{\partial r} & \frac{\partial\phi^r}{\partial z} \\ \frac{\partial\phi^z}{\partial r} & \frac{\partial\phi^z}{\partial z} \end{pmatrix}.$$

Since $\partial_r u^r + \partial_z u^z = -\frac{1}{r}u^r$, we get

$$(\partial_r u^r)(t,\phi^r,\phi^z) + (\partial_z u^z)(t,\phi^r,\phi^z) = -\frac{1}{\phi^r}u^r(t,\phi^r,\phi^z).$$

Therefore, the 2nd identity in (3.5) follows. It is not difficult to check that this also coincides with (3.9) derived earlier.

The existence of the limits (3.6) is a simple consequence of (3.8) and (3.9).

Finally (3.7) follows easily from (3.5) and the identity

$$D\tilde{\phi} = \Big((D\phi)(\tilde{\phi})\Big)^{-1}.$$

■

We now take a parameter $A \gg 1$ and define (by a slight abuse of notation) a class of axisymmetric functions

$$\begin{aligned} g_A(x_1,x_2,z) &= g_A(r,z) \\ &= \frac{\sqrt{\log A}}{A}\sum_{A\le k\le 2A}\eta_k(r,z), \end{aligned} \tag{3.11}$$

where $r = \sqrt{x_1^2+x_2^2}$ and

$$\eta_k(r,z) = \sum_{\epsilon_1=\pm1}\epsilon_1\eta_0\left(\frac{(r-2^{-k}, z-\epsilon_1 2^{-k})}{2^{-(k+10)}}\right). \tag{3.12}$$

Here we choose $\eta_0 \in C_c^\infty(B(0,1))$ to be a radial compactly supported function such that $0 \le \eta_0 \le 1$. Note that by construction η_k is an odd function of z and so is g_A. Also it is easy to see that

$$\eta_k(2^{-k}r, 2^{-k}z) = \sum_{\epsilon_1=\pm 1} \epsilon_1 \eta_0\left(\frac{(r-1, z-\epsilon_1)}{2^{-10}}\right)$$

$$=: \tilde{\eta}_0(r,z). \tag{3.13}$$

Recall $e_\theta = \frac{1}{r}(-x_2, x_1, 0)$. We first check that

$$\|g_A e_\theta\|_{L^\infty(\mathbb{R}^3)} \lesssim \frac{\sqrt{\log A}}{A}. \tag{3.14}$$

$$\|g_A e_\theta\|_{\dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)} \lesssim \sqrt{\log A}. \tag{3.15}$$

$$\|\frac{g_A}{r} e_\theta\|_{L^{3,1}(\mathbb{R}^3)} \lesssim \sqrt{\log A}. \tag{3.16}$$

$$\|g_A\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} \lesssim \frac{\sqrt{\log A}}{\sqrt{A}}, \tag{3.17}$$

$$\|g_A e_\theta\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} \lesssim \frac{\sqrt{\log A}}{\sqrt{A}}. \tag{3.18}$$

Note that (3.14) is obvious. The property (3.15) follows from the triangle inequality and the fact that each $\eta_k e_\theta$ has the same $\dot{B}^{\frac{3}{2}}_{2,1}$ norm. The inequality (3.16) follows from Lemma 3.3.

For (3.17), albeit standard, we explain how to check it. By definition and direct expansion, we have

$$\|g_A\|^2_{\dot{H}^{\frac{3}{2}}} = \frac{\log A}{A^2} \sum_{A \le k \le 2A} \sum_{A \le l \le 2A} \int_{\mathbb{R}^3} |\nabla|^{\frac{3}{2}} \eta_k \cdot |\nabla|^{\frac{3}{2}} \eta_l \, \mathrm{d}x.$$

Therefore, it suffices to show for each k,

$$\sum_{A \le l \le 2A} \left| \int_{\mathbb{R}^3} |\nabla|^{\frac{3}{2}} \eta_k \cdot |\nabla|^{\frac{3}{2}} \eta_l \, \mathrm{d}x \right| \lesssim 1. \tag{3.19}$$

By scaling $(r,z) \to (2^{-k}r, 2^{-k}z)$ and (3.13), we have

$$\text{LHS of (3.19)} \lesssim \sum_{l \in \mathbb{Z}} \left| \int_{\mathbb{R}^3} |\nabla|^{\frac{3}{2}} \tilde{\eta}_0 \cdot |\nabla|^{\frac{3}{2}} \eta_l \, \mathrm{d}x \right|,$$

where $\tilde{\eta}_0$ was defined in (3.13). Note that $\tilde{\eta}_0 \in C_c^\infty(\mathbb{R}^3)$ since it is supported on $r \sim 1$, $|z| \sim 1$.

Now discuss two cases. If $l \ge 0$, then

$$\left|\int_{\mathbb{R}^3} |\nabla|^{\frac{3}{2}}\tilde{\eta}_0 \cdot |\nabla|^{\frac{3}{2}}\eta_l \,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^3} |\nabla|^3\tilde{\eta}_0 \cdot \eta_l \,\mathrm{d}x\right|$$
$$\lesssim \||\nabla|^3\tilde{\eta}_0\|_{L^\infty} \cdot \|\eta_l\|_{L^1} \lesssim 2^{-3l},$$

which is summable for $l \ge 0$.

On the other hand if $l < 0$, then

$$\left|\int_{\mathbb{R}^3} |\nabla|^{\frac{3}{2}}\tilde{\eta}_0 \cdot |\nabla|^{\frac{3}{2}}\eta_l \,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^3} |\nabla|\tilde{\eta}_0 \cdot \Delta\eta_l \,\mathrm{d}x\right|$$
$$\lesssim \||\nabla|\tilde{\eta}_0\|_{L^2} \cdot \|\Delta\eta_l\|_{L^2}$$
$$\lesssim 2^{\frac{1}{2}l},$$

which is also summable for $l < 0$.

Therefore (3.19) follows and (3.17) is proved.

For (3.18), we note that by (3.13),

$$\eta_k(r,z)e_\theta = \tilde{\eta}_0(2^k r, 2^k z)\frac{1}{2^k r}\big(-2^k x_2,\ 2^k x_1,\ 0\big)$$
$$= \eta_{\mathrm{vec}}(2^k x),$$

where

$$\eta_{\mathrm{vec}}(x) = \eta_{\mathrm{vec}}(x_1, x_2, z) = \tilde{\eta}_0(r,z)\frac{1}{r}(-x_2,\ x_1,\ 0).$$

Since by definition $\tilde{\eta}_0$ is supported on $r \sim 1$, η_{vec} is a smooth function. Therefore, (3.18) can be proved in the same way as (3.17) (note that only scaling is used in the argument).

Lemma 3.5. For any smooth axisymmetric f on $\mathbb{R}^3$ (i.e., $f = f(x) = f(r,z)$, $x = (x_1, x_2, z)$, $r = \sqrt{x_1^2 + x_2^2}$), we have

$$\left(\partial_z\Big(\Delta - \frac{1}{r^2}\Big)^{-1}\partial_z f\right)(r,z) = \left(\partial_{zz}\Big(\Delta - \frac{1}{r^2}\Big)^{-1} f\right)(r,z)$$
$$= C \cdot \int_{\mathbb{R}^5} K(x-y) \cdot \frac{|x'|}{|y'|} f(y)\,\mathrm{d}y, \quad x = (r,0,0,0,z), \tag{3.20}$$

where $C > 0$ is an absolute constant, $x' = (x_1, \cdots, x_4)$, $y' = (y_1, \cdots, y_4)$, and

$$K(\tilde{x}) = \frac{|\tilde{x}'|^2 - 4\tilde{x}_5^2}{|\tilde{x}|^7} + \frac{1}{5C}\delta(\tilde{x}). \tag{3.21}$$

Here $\delta(\cdot)$ is the Dirac delta function on $\mathbb{R}^5$. On the RHS of (3.20) we naturally regard f as an axisymmetric function on $\mathbb{R}^5$ with the identification $f(y) = f(|y'|, y_5)$. Also by axisymmetry the RHS of (3.20) depends only on $(|x'|, x_5)$ so that we can actually choose any x with $|x'| = r$, $x_5 = z$.

Similarly we have

$$\begin{aligned}&\left(\partial_r\left(\Delta - \frac{1}{r^2}\right)^{-1}\partial_z f\right)(r, z)\\ &\quad = C \cdot \int_{\mathbb{R}^5}\left(\frac{1}{|x-y|^5} - 5r \cdot \frac{r - y_1}{|x-y|^7}\right) \cdot \frac{x_5 - y_5}{|y'|} f(y)\, dy, \quad x = (r, 0, 0, 0, z).\end{aligned} \tag{3.22}$$

Proof of Lemma 3.5 We first compute the kernel $(\Delta - \frac{1}{r^2})^{-1}$. To derive the explicit representation, consider the equation

$$\left(\Delta - \frac{1}{r^2}\right)u = f$$

or

$$\left(\partial_{rr} + \frac{1}{r}\partial_r - \frac{1}{r^2} + \partial_{zz}\right)u = f.$$

Set $u = rv$. Then

$$\begin{aligned}\Delta(rv) &= r\Delta v + 2(\partial_r r)\partial_r v + v\Delta(r)\\ &= r\Delta v + 2\partial_r v + \frac{1}{r}v\\ &= r\Delta_5 v + \frac{1}{r}v,\end{aligned}$$

where

$$\Delta_5 = \partial_{rr} + \frac{3}{r}\partial_r + \partial_{zz}$$

is the 5D Laplacian in cylindrical coordinates.

Therefore,

$$\left(\Delta - \frac{1}{r^2}\right)u = \left(\Delta - \frac{1}{r^2}\right)(rv) = r\Delta_5 v$$

and we only need to solve

$$\Delta_5 v = \frac{1}{r} f.$$

Inverting the Laplacian then gives

$$v = C \cdot \int_{\mathbb{R}^5} \frac{-1}{|x-y|^3} \cdot \frac{1}{|y'|} f(y)\,\mathrm{d}y$$

and obviously

$$u(x) = C \cdot \int_{\mathbb{R}^5} \frac{-1}{|x-y|^3} \cdot \frac{|x'|}{|y'|} f(y)\,\mathrm{d}y.$$

Note that in the above expression we have pure convolution in the y_5 variable. Therefore, the 1st equality in (3.20) hold. Differentiating in $z = x_5$ variable twice then gives (3.20). In a similar way we can derive (3.22). ■

Lemma 3.6. Let $\phi = \phi(r,z) = (\phi^r(r,z), \phi^z(r,z))$ be a bi-Lipschitz map on $[0,\infty) \times \mathbb{R}$ such that the following hold:

- for any $r \geq 0$, $z \in \mathbb{R}$,

$$\phi^r(0,z) = 0 \quad \text{and} \quad \phi^z(r,0) = 0; \tag{3.23}$$

- for some integer $n_0 \geq 1$,

$$\|D\phi\|_\infty + \|D\tilde{\phi}\|_\infty \leq 2^{n_0}, \tag{3.24}$$

  where $\tilde{\phi}$ is the inverse map of ϕ.

Define

$$w(r,z) = (T\omega_0)(r,z) = \frac{\omega_0(\phi(r,z))}{\phi^r(r,z)} r, \tag{3.25}$$

where $\omega_0 = g_A$ and g_A is defined in (3.11).

Then

$$\left\| \partial_{zz} (\Delta - \frac{1}{r^2})^{-1} \omega \right\|_\infty \leq C \cdot 2^{2n_0} \cdot \frac{\sqrt{\log A}}{A}, \tag{3.26}$$

where $C > 0$ is an absolute constant.

If in addition the map ϕ preserves the measure $r\,\mathrm{d}r\,\mathrm{d}z$, that is,

$$\frac{\phi^r}{r}\det\left(\frac{\partial(\phi^r,\phi^z)}{\partial(r,z)}\right)\equiv 1;$$

and ϕ is odd in the z-variable, that is,

$$\phi^r(r,-z)=\phi^r(r,z),\quad \forall\, r\geq 0,\, z\in\mathbb{R};$$
$$\phi^z(r,-z)=-\phi^z(r,z),\quad \forall\, r\geq 0,\, z\in\mathbb{R};$$

then for some absolute constant $C_1 > 0$,

$$-\left(\partial_r\left(\Delta-\frac{1}{r^2}\right)^{-1}\partial_z w\right)(0,0)\geq C_1\cdot\sqrt{\log A}\cdot 2^{-8n_0}. \tag{3.27}$$

Remark 3.7. In the proof of (3.26), we do not use the odd symmetry in the z variable of the function g_A. By itself the axisymmetry is enough to control the singular operator $\partial_{zz}(\Delta-\frac{1}{r^2})^{-1}$.

Remark 3.8. For our application later, ω actually corresponds to ω^θ and the expression $-(\Delta-\frac{1}{r^2})^{-1}\partial_z\omega$ in (3.27) will correspond to the radial velocity u^r, see (3.36).

Proof of Lemma 3.6 We shall adopt the same notations as in Lemma 3.5. By (3.23) and (3.24), it is not difficult to check that if $r\sim 2^{-m}$ and $|z|\sim 2^{-m}$ for some integer m, then

$$2^{-m-n_0}\lesssim \phi^r(r,z)\lesssim 2^{-m+n_0},$$
$$2^{-m-n_0}\lesssim |\phi^z(r,z)|\lesssim 2^{-m+n_0}.$$

Similarly if $\phi^r(r,z)\sim 2^{-m}$, $|\phi^z(r,z)|\sim 2^{-m}$, then

$$2^{-m-n_0}\lesssim r\lesssim 2^{-m+n_0},$$
$$2^{-m-n_0}\lesssim |z|\lesssim 2^{-m+n_0}.$$

These facts will be used below.

By (3.25) and (3.11), we have

$$\partial_{zz}\Big(\Delta-\frac{1}{r^2}\Big)^{-1}\omega=\frac{\sqrt{\log A}}{A}\sum_{A\le k\le 2A}\Big(\partial_{zz}\Big(\Delta-\frac{1}{r^2}\Big)^{-1}\Big)(T\eta_k).$$

We shall estimate each piece $\partial_{zz}(\Delta-\frac{1}{r^2})^{-1}(T\eta_k)$.

By (3.20) and (3.21), we write

$$\Big|(\partial_{zz}(\Delta-\frac{1}{r^2})^{-1}(T\eta_k))(x)\Big|\lesssim\Big|\int_{\mathbb{R}^5}K_1(x-y)|x'|\frac{\eta_k(\phi(|y'|,y_5))}{\phi^r(|y'|,y_5)}\,\mathrm{d}y\Big| \tag{3.28}$$

$$+\,r\frac{\eta_k(\phi(r,z))}{\phi^r(r,z)}, \tag{3.29}$$

where

$$K_1(\tilde{x})=\frac{|\tilde{x}'|^2-4\tilde{x}_5^2}{|\tilde{x}|^7}.$$

The contribution of (3.29) is of no problem for us. Indeed on the support of $\eta_k(\phi(r,z))$, we have

$$\phi^r(r,z)\sim 2^{-k},\quad\text{and}\quad|\phi^z(r,z)|\sim 2^{-k}.$$

Therefore, $2^{-k-n_0}\lesssim r\lesssim 2^{-k+n_0}$ and

$$\frac{r}{\phi^r(r,z)}\lesssim 2^{n_0}.$$

Since the supports of the η_k functions are mutually disjoint, it follows that

$$\sum_{A\le k\le 2A} r\frac{\eta_k(\phi(r,z))}{\phi^r(r,z)}\lesssim 2^{n_0}.$$

Hence, we only need to consider the contribution of (3.28) to (3.26). By the same consideration as before we have in (3.28) the integration variable y is localized to the regime:

$$2^{-k-n_0}\lesssim|y'|\lesssim 2^{-k+n_0},$$
$$2^{-k-n_0}\lesssim|y_5|\lesssim 2^{-k+n_0},$$
$$|\phi(|y'|,y_5)|\sim 2^{-k},\quad\text{and}\quad\phi^r(|y'|,y_5)\sim 2^{-k}.$$

Obviously if $x = 0$, then due to the factor $|x'| = 0$, the integral (3.28) also vanishes. Therefore, we only need to consider the case $x \neq 0$.

Assume $|x| \sim 2^{-l}$. We discuss two cases.

Case 1: $2^k \gg 2^{l+n_0}$. In this case $|x| \gg |y|$ (recall $|y| \lesssim 2^{-k+n_0}$) and $|x-y| \sim |x|$. Therefore,

$$\begin{aligned}
(3.28) &\lesssim \|K_1(y)\|_{L^\infty(|y|\sim 2^{-l})} \cdot 2^{-l} \cdot \frac{\text{Leb}\{y \in \mathbb{R}^5 : \ 2^{-k-n_0} \lesssim |y| \lesssim 2^{-k+n_0}\}}{2^{-k}} \\
&\lesssim 2^{5l} \cdot 2^{-l} \cdot \frac{(2^{-k+n_0})^5}{2^{-k}} \\
&\lesssim 2^{-4k+4l+5n_0}.
\end{aligned}$$

Case 2: $2^k \lesssim 2^{l+n_0}$. In this case we note that the kernel K_1 in (3.28) corresponds to a Riesz-type operator on $\mathbb{R}^5$. By using the interpolation inequality

$$\|\mathcal{R}_{ij} f\|_{L^\infty(\mathbb{R}^5)} \lesssim \|f\|_{L^5(\mathbb{R}^5)}^{\frac{1}{2}} \cdot \|\nabla f\|_{L^\infty(\mathbb{R}^5)}^{\frac{1}{2}},$$

we can bound (3.28) as

$$\begin{aligned}
(3.28) &\lesssim 2^{-l} \cdot \left| \int_{\mathbb{R}^5} K_1(x-y) \cdot \frac{\eta_k(\phi(|y'|, y_5))}{\phi^r(|y'|, y_5)} \, \mathrm{d}y \right| \\
&\lesssim 2^{-l} \cdot \left\| \frac{1}{2^{-k}} \right\|_{L^5(\,y\in\mathbb{R}^5:\, 2^{-k-n_0} \lesssim |y| \lesssim 2^{-k+n_0})}^{\frac{1}{2}} \\
&\quad \cdot \left(\left\| \partial_r \left(\frac{\eta_k(\phi(r,z))}{\phi^r(r,z)} \right) \right\|_\infty + \left\| \partial_z \left(\frac{\eta_k(\phi(r,z))}{\phi^r(r,z)} \right) \right\|_\infty \right)^{\frac{1}{2}} \\
&\lesssim 2^{-l} \cdot 2^{\frac{k}{2}} \cdot (2^{-k+n_0})^{\frac{1}{2}} \cdot (2^{n_0} \cdot 2^{2k})^{\frac{1}{2}} \\
&\lesssim 2^{n_0-l+k}.
\end{aligned}$$

Collecting the estimates, we have

$$\begin{aligned}
\sum_{A \le k \le 2A} (3.28) &\lesssim \sum_{2^k \gg 2^{l+n_0}} 2^{-4k+4l+5n_0} + \sum_{2^k \lesssim 2^{l+n_0}} 2^{n_0-l+k} \\
&\lesssim 2^{2n_0}.
\end{aligned}$$

Hence, the estimate (3.26) follows.

By (3.22) and parity of ϕ and η_k, we have

$$
\begin{aligned}
-\left(\partial_r\left(\Delta-\frac{1}{r^2}\right)^{-1}\partial_z\omega\right)(0,0) &= C\int_{\mathbb{R}^5}\frac{y_5}{|y|^5}\cdot\frac{1}{|y'|}\omega(y)\,\mathrm{d}y\\
&= C\int_{\mathbb{R}^5}\frac{y_5}{|y|^5}\cdot\frac{\omega_0(\phi(|y'|,y_5))}{\phi^r(|y'|,y_5)}\,\mathrm{d}y\\
&= C\frac{\sqrt{\log A}}{A}\cdot\sum_{A\le k\le 2A}\int_{\mathbb{R}^5}\frac{y_5}{|y|^5}\cdot\frac{\eta_k(\phi(|y'|,y_5))}{\phi^r(|y'|,y_5)}\,\mathrm{d}y \qquad (3.30)\\
&= 2C\frac{\sqrt{\log A}}{A}\cdot\sum_{A\le k\le 2A}\int_{y\in\mathbb{R}^5:\,y_5>0}\frac{y_5}{|y|^5}\cdot\frac{\eta_k(\phi(|y'|,y_5))}{\phi^r(|y'|,y_5)}\,\mathrm{d}y\\
&\gtrsim \frac{\sqrt{\log A}}{A}\sum_{A\le k\le 2A}\frac{2^{-k-n_0}}{(2^{-k+n_0})^5}\cdot\frac{1}{2^{-k}}\cdot\int\eta_k(\phi(r,z))r^3\,\mathrm{d}r\,\mathrm{d}z.
\end{aligned}
$$

Since the map ϕ preserves the measure $r\,\mathrm{d}r\,\mathrm{d}z$, the inverse map $\tilde{\phi}$ also preserves the same measure. By the change of variable $(r,z)\to\tilde{\phi}(r,z)$, we have

$$
\begin{aligned}
\int\eta_k(\phi(r,z))r^3\,\mathrm{d}r\,\mathrm{d}z &= \int\eta_k(r,z)\tilde{\phi}^r(r,z)^2 r\,\mathrm{d}r\,\mathrm{d}z\\
&\gtrsim (2^{-k-n_0})^2\|\eta_k\|_{L^1(\mathbb{R}^3)}\\
&\gtrsim 2^{-5k-2n_0}.
\end{aligned}
$$

Plugging this estimate into (3.30), we obtain

$$
\begin{aligned}
-\left(\partial_r\left(\Delta-\frac{1}{r^2}\right)^{-1}\partial_z\omega\right)(0,0) &\gtrsim \frac{\sqrt{\log A}}{A}\sum_{A\le k\le 2A}2^{-8n_0}\\
&\gtrsim \sqrt{\log A}\cdot 2^{-8n_0}.
\end{aligned}
$$

Hence, (3.27) is proved. ■

We now prove the existence of large deformation in the 3D axisymmetric case. To fix the notation, consider the 3D axisymmetric Euler equation without swirl in simplified form

$$\begin{cases} \partial_t \left(\frac{\omega^\theta}{r}\right) + (u\cdot\nabla)\left(\frac{\omega^\theta}{r}\right) = 0, & (r,z) \in (0,\infty)\times\mathbb{R},\ 0 < t \le 1, \\ \omega^\theta\Big|_{t=0} = g_A, \end{cases}$$

where $u = u^r e_r + u^z e_z$ and g_A is defined in (3.11). Note that here ω^θ is a scalar valued function that is related to u by the relation $\mathrm{curl}(u) = \omega^\theta e_\theta$.

Let $\phi = \phi(t,r,z)$ be the forward characteristic line as defined in (3.2) and $\tilde\phi = \tilde\phi(t,r,z)$ be the inverse map. Then

Proposition 3.9. For all A sufficiently large, we have

$$\max_{0\le t\le \frac{1}{\log\log A}} (\|D\phi(t)\|_\infty + \|D\tilde\phi(t)\|_\infty) \ge \log\log A.$$

Proof of Proposition 3.9 We argue by contradiction. Assume

$$\max_{0\le t\le \frac{1}{B}} \left(\|D\phi(t)\|_\infty + \|D\tilde\phi(t)\|_\infty\right) \le B, \tag{3.31}$$

where for simplicity of the notation we denote $B = \log\log A$.

Denote $D(t) = D(t,r,z) = (D\phi)(t,r,z)$. By (3.2), we have

$$\begin{aligned} \partial_t D(t) &= \begin{pmatrix} \partial_r u^r & \partial_z u^r \\ \partial_r u^z & \partial_z u^z \end{pmatrix} D(t) \\ &= \begin{pmatrix} \partial_r u^r & 0 \\ 0 & \partial_z u^z \end{pmatrix} D(t) + P(t)D(t), \end{aligned} \tag{3.32}$$

where we denote

$$P(t) = P(t,r,z) = \begin{pmatrix} 0 & (\partial_z u^r)(t,\phi(t,r,z)) \\ (\partial_r u^z)(t,\phi(t,r,z)) & 0 \end{pmatrix}.$$

Now since

$$\frac{\omega^\theta(t,\phi(t,r,z))}{\phi^r(t,r,z)} = \frac{\omega^\theta_0(t,r,z)}{r},$$

we have

$$\omega^\theta(t,r,z) = \frac{\omega_0^\theta(\tilde{\phi}(t,r,z))}{\tilde{\phi}^r(t,r,z)} r.$$

By (3.3), we have

$$\begin{aligned} r &= \phi^r(t, \tilde{\phi}^r(t,r,z), \tilde{\phi}^z(t,r,z)) - \phi^r(t, 0, \tilde{\phi}^z(t,r,z)) \\ &\le \|\partial_r \phi^r(t,\cdot)\|_\infty \tilde{\phi}^r(t,r,z) \\ &\le B\tilde{\phi}^r(t,r,z). \end{aligned}$$

Therefore,

$$\max_{0\le t\le \frac{1}{B}} \|\omega(t,\cdot)\|_\infty \le \frac{\sqrt{\log A}}{A} B. \tag{3.33}$$

Since

$$\omega^\theta = \partial_r u^z - \partial_z u^r, \tag{3.34}$$

$$\frac{1}{r}\partial_r(ru^r) + \partial_z u^z = 0, \tag{3.35}$$

it is not difficult to check that

$$u^r = -\Big(\Delta - \frac{1}{r^2}\Big)^{-1} \partial_z \omega^\theta, \tag{3.36}$$

and

$$\partial_z u^r = -\partial_{zz}\Big(\Delta - \frac{1}{r^2}\Big)^{-1} \omega^\theta.$$

By (3.26) and (3.31), we get

$$\|(\partial_z u^r)(t)\|_\infty \lesssim B^2 \frac{\sqrt{\log A}}{A}.$$

By (3.34) and (3.33), we also get

$$\begin{aligned}\|\partial_r u^z(t)\|_\infty &\lesssim \|\omega^\theta(t)\|_\infty + \|(\partial_z u^r)(t)\|_\infty \\ &\lesssim B^2 \frac{\sqrt{\log A}}{A}.\end{aligned}$$

Hence,

$$\|P(t)\|_\infty \lesssim B^2 \frac{\sqrt{\log A}}{A}.$$

Denote

$$\lambda(t,r,z) = -(\partial_r u^r)(t,\phi(t,r,z)).$$

By (3.36) and (3.27), we have

$$-\lambda(t,0,0) \gtrsim \sqrt{\log A}\cdot B^{-8}, \quad \forall\, 0 \le t \le \frac{1}{B}. \tag{3.37}$$

By (3.35), we can write

$$\begin{aligned}(\partial_z u^z)(t,\phi(t,r,z)) &= -(\partial_r u^r)(t,\phi) - \frac{1}{\phi^r} u^r(t,\phi) \\ &= \lambda(t,r,z) - \frac{1}{\phi^r(t,r,z)} u^r(t,\phi(t,r,z)).\end{aligned}$$

Using the above computation and integrating (3.32) in time, we get

$$D(t) = \begin{pmatrix} e^{-\int_0^t \lambda} & 0 \\ 0 & e^{\int_0^t \lambda - \int_0^t \frac{1}{\phi^r} u^r(s,\phi)\,\mathrm{d}s} \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-\int_\tau^t \lambda} & 0 \\ 0 & e^{\int_\tau^t \lambda - \int_\tau^t \frac{1}{\phi^r} u^r(s,\phi)\,\mathrm{d}s} \end{pmatrix} P(\tau) D(\tau)\,\mathrm{d}\tau.$$

By (3.5), we have

$$\max_{0\le t\le \frac{1}{B}} e^{\left|\int_0^t \frac{u^r(s,\phi)}{\phi^r}\right|} \lesssim B.$$

Therefore, we get

$$\frac{1}{B} e^{|\int_0^t \lambda|} \lesssim B + \max_{0\le\tau\le t}\left(e^{2|\int_0^\tau \lambda|}\right)\cdot B^{10}\cdot \frac{\sqrt{\log A}}{A}$$

or

$$e^{|\int_0^t \lambda|} \lesssim B^2 + \max_{0\le\tau\le t}\left(e^{2|\int_0^\tau \lambda|}\right)\cdot B^{11}\cdot \frac{\sqrt{\log A}}{A}.$$

Since $B^{12} \ll A/\sqrt{\log A}$, a standard continuity argument then gives

$$e^{|\int_0^t \lambda(s,r,z)\,\mathrm{d}s|} \lesssim B^2, \quad \forall\, 0 \le t \le \frac{1}{B}, \, r \ge 0, \, z \in \mathbb{R}.$$

But this obviously contradicts (3.37). ■

Lemma 3.10 (Vanishing near $r = 0$). Let $U = U^r e_r + U^z e_z$ be a (possibly time-dependent) given smooth axisymmetric without swirl velocity field such that $\nabla \cdot U = 0$ and

$$\max_{0\le t\le 1} (\|D^2 U(t)\|_4 + \|DU(t)\|_\infty + \|U(t)\|_\infty) \le C_1 < \infty. \tag{3.38}$$

Suppose ω is a smooth solution to the *linear* system

$$\begin{cases} \partial_t\left(\frac{\omega}{r}\right) + (U\cdot\nabla)\left(\frac{\omega}{r}\right) = 0, \quad x = (x_1, x_2, z),\, r = \sqrt{x_1^2 + x_2^2}, \\ \omega\Big|_{t=0} = \omega_0. \end{cases} \tag{3.39}$$

Here the initial data $\omega_0 \in C_c^\infty(\mathbb{R}^3)$ and have the form

$$\omega_0 = \omega_0^\theta e_\theta,$$

where $\omega_0^\theta = \omega_0^\theta(r,z)$ is scalar valued and

$$e_\theta = \frac{1}{r}\big(-x_2,\, x_1,\, 0\big).$$

Assume that ω_0 vanishes near $r = 0$, that is, there exists a constant $r_0 > 0$ such that

$$\mathrm{supp}(\omega_0(r,z)) \subset \left\{(r,z) : \, r_0 \le r \le \frac{1}{r_0}\right\}.$$

Then there exists a constant $R_0 = R_0(r_0, C_1) > 0$ such that

$$\mathrm{supp}(\omega(t,r,z)) \subset \left\{(r,z) : \, R_0 \le r \le \frac{1}{R_0}\right\}, \quad \forall\, 0 \le t \le 1. \tag{3.40}$$

Furthermore we have the estimate

$$\max_{0\le t\le 1}\|\omega(t)\|_{H^2}\le C_2, \tag{3.41}$$

where the constant C_2 only depends on $(\|\omega_0\|_{H^2}, r_0, C_1)$.

Proof of Lemma 3.10 The property (3.40) follows easily from finite-speed propagation of the transport equation. For example by (3.9) (with u^r replaced by U^r) and (3.38), we have for some $C_3 = C_3(C_1) > 0$

$$\frac{1}{C_3}\le \phi^r(t,r,z)/r \le C_3, \quad \forall\, 0\le t\le 1.$$

This shows that the boundary of the support is supported away from $r = 0$. Clearly (3.40) follows.

Denote $g = \frac{\omega}{r}$ and $g_0 = \frac{\omega_0}{r}$. Since ω is supported away from $r = 0$, obviously we have

$$\|g_0\|_{H^2}\lesssim_{r_0}\|\omega_0\|_{H^2}.$$

Since

$$\partial_t g + (U\cdot\nabla)g = 0,$$

a simple H^2 energy estimate then gives

$$\begin{aligned}\partial_t\big(\|g\|_{H^2}^2\big) &\lesssim \|D^2U\|_4\|\nabla g\|_4\|g\|_{H^2} + \|DU\|_\infty\|g\|_{H^2}^2\\ &\lesssim_{C_1}\|g\|_{H^2}^2.\end{aligned}$$

Therefore,

$$\max_{0\le t\le 1}\|g(t)\|_{H^2}\lesssim_{C_1,r_0}\|\omega_0\|_{H^2}.$$

Since $\omega = rg$ and ω is supported on $r\sim_{r_0,C_1} 1$, obviously (3.41) follows. ■

Proposition 3.11. Suppose ω is a smooth solution to the axisymmetric (without swirl) Euler equation in the form

$$\begin{cases} \partial_t \left(\frac{\omega}{r}\right) + (u\cdot\nabla)\left(\frac{\omega}{r}\right) = 0, \quad 0 < t \le 1,\, x = (x_1, x_2, z),\, r = \sqrt{x_1^2 + x_2^2}, \\ u = -\Delta^{-1}\nabla \times \omega, \\ \omega\Big|_{t=0} = \omega_0 \end{cases} \tag{3.42}$$

and it satisfies the following conditions:

- $\omega_0 \in C_c^\infty(\mathbb{R}^3)$ and has the form

$$\omega_0 = \omega_0^\theta e_\theta, \tag{3.43}$$

  where $\omega_0^\theta = \omega_0^\theta(r,z)$ is scalar valued and ω_0 vanishes near $r = 0$, that is, there exists a constant $r_0 > 0$ such that

$$\operatorname{supp}(\omega_0(r,z)) \subset \left\{(r,z) : r > r_0\right\}.$$

- Let $\phi = (\phi^r, \phi^z)$ be the characteristic lines defined in (3.2) and $\tilde{\phi}$ be the inverse. For some $0 < t_0 \le 1$, $\tilde{r}_* \ge 0$, $\tilde{z}_* \in \mathbb{R}$, we have

$$\left\|(D\tilde{\phi})(t_0, \tilde{r}_*, \tilde{z}_*)\right\|_\infty = \sup_{r\ge 0, z\in\mathbb{R}} \left\|(D\tilde{\phi})(t_0, r, z)\right\|_\infty =: M \gg 1. \tag{3.44}$$

  Here in (3.44), $\|\cdot\|_\infty$ denotes the matrix max norm defined by $\|A\|_\infty = \max(|a_{ij}|)$ $(A = (a_{ij}))$. The notation $M \gg 1$ means that M is sufficiently large than an absolute constant.

Then we can find a smooth solution also solving the axisymmetric (without swirl) Euler equation

$$\begin{cases} \partial_t \left(\frac{\tilde{\omega}}{r}\right) + (\tilde{u}\cdot\nabla)\left(\frac{\tilde{\omega}}{r}\right) = 0, \quad 0 < t \le 1, \\ \tilde{u} = -\Delta^{-1}\nabla \times \tilde{\omega}, \\ \tilde{\omega}\Big|_{t=0} = \tilde{\omega}_0 \end{cases}$$

such that the following hold:

1. $\tilde{\omega}_0 \in C_c^\infty(\mathbb{R}^3)$ and has the form

$$\tilde{\omega}_0 = \tilde{\omega}_0^\theta e_\theta,$$

where $\tilde{\omega}_0^\theta = \tilde{\omega}_0^\theta(r,z)$. The function $\tilde{\omega}_0$ also vanishes near $r = 0$, that is, there exists a constant $\tilde{r}_0 > 0$ such that

$$\text{supp}(\tilde{\omega}_0(r,z)) \subset \big\{(r,z) : r > \tilde{r}_0\big\}.$$

Furthermore,

$$\begin{aligned}
\left\|\frac{\tilde{\omega}_0}{r}\right\|_{L^1(\mathbb{R}^3)} &\le 2\left\|\frac{\omega_0}{r}\right\|_{L^1(\mathbb{R}^3)}, \\
\left\|\frac{\tilde{\omega}_0}{r}\right\|_{L^\infty(\mathbb{R}^3)} &\le 2\left\|\frac{\omega_0}{r}\right\|_{L^\infty(\mathbb{R}^3)}, \\
\left\|\frac{\tilde{\omega}_0}{r}\right\|_{L^{3,1}(\mathbb{R}^3)} &\le 2\left\|\frac{\omega_0}{r}\right\|_{L^{3,1}(\mathbb{R}^3)}.
\end{aligned} \tag{3.45}$$

2. $\tilde{\omega}_0$ is a small perturbation of ω_0:

$$\begin{aligned}
\|\tilde{\omega}_0\|_{L^1(\mathbb{R}^3)} &\le 2\|\omega_0\|_{L^1(\mathbb{R}^3)}, \\
\|\tilde{\omega}_0\|_{L^\infty(\mathbb{R}^3)} &\le 2\|\omega_0\|_{L^\infty(\mathbb{R}^3)}, \\
\|\tilde{\omega}_0\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} &\le \|\omega_0\|_{\dot{H}^{\frac{3}{2}}} + \frac{\tilde{C}}{M^{\frac{1}{6}}}.
\end{aligned} \tag{3.46}$$

Here $\tilde{C} > 0$ is an absolute constant.

3. For the same t_0 as in (3.44), we have

$$\|\tilde{\omega}(t_0)\|_{\dot{H}^{\frac{3}{2}}} > M^{\frac{1}{6}}. \tag{3.47}$$

Proof of Proposition 3.11 We begin with a general derivation. Let W be a smooth solution to the system

$$\begin{cases}
\partial_t\left(\frac{W}{r}\right) + (U\cdot\nabla)\left(\frac{W}{r}\right) = 0, \\
U = -\Delta^{-1}\nabla\times W, \\
W\Big|_{t=0} = W_0 = fe_\theta.
\end{cases} \tag{3.48}$$

Here $f = f(r,z)$ is scalar valued. Define the corresponding forward characteristic lines $\Phi = (\Phi^r, \Phi^z)$ in the same way as (3.2) and let $\tilde{\Phi}$ be the inverse map. Then we have

$$W(t,x) = W^\theta(t,r,z)e_\theta,$$

where W^θ is scalar valued and

$$\frac{W^\theta(t,\Phi(t,r,z))}{\Phi^r(t,r,z)} = \frac{f(r,z)}{r}.$$

Therefore,

$$W^\theta(t,r,z) = \frac{f(\tilde\Phi(t,r,z))}{\tilde\Phi^r(t,r,z)} r \tag{3.49}$$

and

$$W(t,x) = \frac{f(\tilde\Phi(t,r,z))}{\tilde\Phi^r(t,r,z)} r e_\theta, \quad x=(x_1,x_2,z),\ r=\sqrt{x_1^2+x_2^2}. \tag{3.50}$$

Now we discuss two cases:

`Case 1`: $\|\omega(t_0,\cdot)\|_{\dot H^{\frac32}} > M^{\frac16}$. In this case we just set $\tilde\omega = \omega$ and no work is needed.

`Case 2`:

$$\|\omega(t_0,\cdot)\|_{\dot H^{\frac32}} \le M^{\frac16}. \tag{3.51}$$

In this case in order not to confuse with some notations later on we shall denote $\tilde\omega_0$ as W_0 and $\tilde\omega$ as W. We take the initial data W_0 in (3.48) as

$$W_0 = \omega_0 + k^{-\frac32} G_0, \tag{3.52}$$

where ω_0 is the same as in (3.42). The function G_0 has the form

$$G_0(x) = g_0(r,z) e_\theta \tag{3.53}$$

where g_0 is scalar valued. The detailed form of g_0 will be specified later in the course of the proof.

We shall take the parameter k sufficiently large. In the rest of this proof, to simplify the presentation, we shall use the notation $X = O(k^\alpha)$ (α is a real number) if the quantity X obeys the bound $X \le C_1 k^\alpha$ and the positive constant C_1 can depend on all other parameters except k.

Now we assume G_0 in (3.53) is a smooth compactly supported function that obeys the following bounds:

$$\|G_0\|_{L^1(\mathbb{R}^3)} + \|G_0\|_{L^\infty(\mathbb{R}^3)} + \left\|\frac{G_0}{r}\right\|_{L^1(\mathbb{R}^3)} + \left\|\frac{G_0}{r}\right\|_{L^\infty(\mathbb{R}^3)} = O(1),$$
$$\|DG_0\|_{L^1(\mathbb{R}^3)} + \|DG_0\|_{L^\infty(\mathbb{R}^3)} = O(k),$$
$$\|D^2G_0\|_{L^1(\mathbb{R}^3)} + \|D^2G_0\|_{L^\infty(\mathbb{R}^3)} = O(k^2). \tag{3.54}$$

By (3.50), (3.52), (3.53), and (3.43), we have

$$\begin{aligned} W(t,x) &= \frac{\omega_0^\theta(\tilde{\Phi}(t,r,z))}{\tilde{\Phi}^r(t,r,z)} re_\theta + k^{-\frac{3}{2}} \frac{g_0(\tilde{\Phi}(t,r,z))}{\tilde{\Phi}^r(t,r,z)} re_\theta \\ &= \frac{\omega_0^\theta(\tilde{\phi}(t,r,z))}{\tilde{\phi}^r(t,r,z)} re_\theta + k^{-\frac{3}{2}} \frac{g_0(\tilde{\phi}(t,r,z))}{\tilde{\phi}^r(t,r,z)} re_\theta + E_1 + E_2, \end{aligned} \tag{3.55}$$

where $\tilde{\phi}$ is the same as in (3.44) and

$$E_1 = \frac{\omega_0^\theta(\tilde{\Phi}(t,r,z))}{\tilde{\Phi}^r(t,r,z)} re_\theta - \frac{\omega_0^\theta(\tilde{\phi}(t,r,z))}{\tilde{\phi}^r(t,r,z)} re_\theta,$$
$$E_2 = k^{-\frac{3}{2}} \left(\frac{g_0(\tilde{\Phi}(t,r,z))}{\tilde{\Phi}^r(t,r,z)} re_\theta - \frac{g_0(\tilde{\phi}(t,r,z))}{\tilde{\phi}^r(t,r,z)} re_\theta \right).$$

We now show that the terms E_1 and E_2 in (3.55) are negligible in the computation of $H^{\frac{3}{2}}$-norm of W. More precisely, we shall show for some $\alpha > 0$,

$$\max_{0\le t\le 1} \|E_1(t)\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} + \max_{0\le t\le 1} \|E_2(t)\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} = O(k^{-\alpha}). \tag{3.56}$$

To show (3.56), let us introduce ω_2, W_1, and W_2 that solve the following *linear* systems:

$$\begin{cases} \partial_t\left(\frac{\omega_2}{r}\right) + (U\cdot\nabla)\left(\frac{\omega_2}{r}\right) = 0, \\ \omega_2\Big|_{t=0} = \omega_0; \end{cases} \tag{3.57}$$

$$\begin{cases} \partial_t\left(\frac{W_1}{r}\right) + (u\cdot\nabla)\left(\frac{W_1}{r}\right) = 0, \\ W_1\Big|_{t=0} = k^{-\frac{3}{2}} G_0; \end{cases} \tag{3.58}$$

$$\begin{cases} \partial_t\left(\frac{W_2}{r}\right) + (U\cdot\nabla)\left(\frac{W_2}{r}\right) = 0, \\ W_2\Big|_{t=0} = k^{-\frac{3}{2}} G_0. \end{cases} \tag{3.59}$$

Here the drift terms u and U and the data ω_0 and G_0 are the same as in the nonlinear systems (3.42) and (3.48).

It is not difficult to check that

$$W = \omega + W_1 + E_1 + E_2,$$

$$E_1 = \omega_2 - \omega,$$

$$E_2 = W_2 - W_1.$$

Therefore, we only need to run perturbation arguments between the nonlinear systems (3.42) and (3.48) and the linear systems (3.57)–(3.59).

We first control the drift difference $u - U$.

By (3.52) and (3.54), we have $\|W_0\|_{W^{1,p}} = O(1)$ for any $1 \le p \le \infty$.

Thanks to the axisymmetry without swirl, we may write the system (3.48) as either

$$\partial_t W + (U\cdot\nabla)W = (W\cdot\nabla)U \tag{3.60}$$

or

$$\partial_t W + (U\cdot\nabla)W = \frac{U^r}{r} W. \tag{3.61}$$

Take any $3 < p < \infty$. A standard energy estimate on (3.60) in $W^{1,p}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|W(t)\|_{W^{1,p}}^p\right) \lesssim \left(\|Du(t)\|_\infty + \|W(t)\|_\infty\right)\|W(t)\|_{W^{1,p}}^p. \tag{3.62}$$

Note that by (3.54) and Lemma 3.2,

$$\max_{0\le t\le 1}\left\|\frac{U^r(t)}{r}\right\|_\infty \lesssim \left\|\frac{W_0}{r}\right\|_{L^{3,1}} = O(1).$$

Therefore by (3.61), we have

$$\max_{0\le t\le 1}\left(\|W(t)\|_2 + \|W(t)\|_\infty\right) = O(1).$$

By the usual log-interpolation inequality, we have

$$\|DU(t)\|_\infty \lesssim \|W(t)\|_2 + \log\big(10 + \|W(t)\|_{W^{1,p}}^p\big)\|W(t)\|_\infty$$
$$\lesssim O(1)\cdot \log\big(10+\|W(t)\|_{W^{1,p}}^p\big).$$

Plugging the last estimate into (3.62), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|W(t)\|_{W^{1,p}}^p\big) \lesssim O(1)\cdot\log\big(10+\|W(t)\|_{W^{1,p}}^p\big)\|W(t)\|_{W^{1,p}}^p.$$

Integrating in time then gives

$$\max_{0\le t\le 1}\|W(t)\|_{W^{1,p}} = O(1), \quad \forall\, 3<p<\infty. \tag{3.63}$$

By Sobolev embedding, we get

$$\max_{0\le t\le 1}\big(\|D^2U(t)\|_p + \|DU(t)\|_\infty\big) = O(1), \quad \forall\, 3<p<\infty. \tag{3.64}$$

Similarly using (3.54) we can derive

$$\max_{0\le t\le 1}\|W(t)\|_{H^2} = O(k^{\frac12}). \tag{3.65}$$

Note that the system (3.42) is independent of the parameter k; therefore, we have

$$\max_{0\le t\le 1}\|u(t)\|_{W^{20,p}} = O(1), \quad \forall\, 2\le p<\infty. \tag{3.66}$$

Now to control the difference, we recall

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u,\\ \partial_t W + (U\cdot\nabla)W = (W\cdot\nabla)U,\\ (W-\omega)\Big|_{t=0} = k^{-\frac32}G_0. \end{cases}$$

Obviously

$$\partial_t(W-\omega) + (U\cdot\nabla)(W-\omega) + \Big((U-u)\cdot\nabla\Big)\omega = (W\cdot\nabla)(U-u) + \Big((W-\omega)\cdot\nabla\Big)u.$$

By (3.63), (3.64), (3.66), and Sobolev embedding, we then obtain

$$\partial_t\Big(\|W-\omega\|_2^2\Big) \lesssim \|U-u\|_6\cdot\|W-\omega\|_2\cdot\|\nabla\omega\|_3+\|W\|_\infty\cdot\|D(U-u)\|_2 \cdot\|W-\omega\|_2+\|Du\|_\infty\cdot\|W-\omega\|_2^2 \lesssim O(1)\cdot\|W-\omega\|_2^2.$$

Therefore,

$$\max_{0\le t\le 1}\|W(t)-\omega(t)\|_2 = O(k^{-\frac{3}{2}}).$$

In a similar way, we can derive

$$\max_{0\le t\le 1}\|W(t)-\omega(t)\|_p = O(k^{-\frac{3}{2}}), \quad \forall\, 1<p<\infty,$$

$$\max_{0\le t\le 1}\Big(\|U(t)-u(t)\|_p+\|\nabla(U(t)-u(t))\|_p\Big) = O(k^{-\frac{3}{2}}), \quad \forall\, 2\le p<\infty. \tag{3.67}$$

We are now ready to control $E_1=\omega_2-\omega$. By (3.57), (3.64), and Lemma 3.10, we have

$$\max_{0\le t\le 1}\|\omega_2(t)\|_{H^2} = O(1).$$

By (3.66), we get

$$\max_{0\le t\le 1}\|\omega_2(t)-\omega(t)\|_{H^2} = O(1).$$

On the other hand, using (3.67), it is not difficult to check that

$$\max_{0\le t\le 1}\|\omega_2(t)-\omega(t)\|_2 = O(k^{-\frac{3}{2}}).$$

Interpolating the above two bounds then gives

$$\max_{0\le t\le 1}\|\omega_2(t)-\omega(t)\|_{H^{\frac{3}{2}}} = O(k^{-\frac{3}{8}}).$$

Therefore, E_1 is OK for us.

To control E_2, we note that by (3.58)–(3.59), we have

$$\partial_t W_1 + (u\cdot\nabla)W_1 = (W_1\cdot\nabla)u, \tag{3.68}$$

$$\partial_t W_2 + (U\cdot\nabla)W_2 = (W_2\cdot\nabla)U, \tag{3.69}$$

$$\begin{aligned}&\partial_t(W_1-W_2)+((u-U)\cdot\nabla)W_1+(U\cdot\nabla)(W_1-W_2)\\ &= ((W_1-W_2)\cdot\nabla)u+(W_2\cdot\nabla)(u-U).\end{aligned} \tag{3.70}$$

For (3.68), a simple energy estimate using (3.54) and (3.66) gives

$$\begin{aligned}&\max_{0\le t\le 1}\|W_1\|_2 = O(k^{-\frac{3}{2}}),\\ &\max_{0\le t\le 1}\|\nabla W_1\|_4 = O(k^{-\frac{1}{2}}),\\ &\max_{0\le t\le 1}\|W_1\|_{H^2} = O(k^{\frac{1}{2}}).\end{aligned} \tag{3.71}$$

Similarly for (3.69), we use (3.54), (3.64), and (3.65) to get

$$\begin{aligned}&\max_{0\le t\le 1}\|W_2\|_4 = O(k^{-\frac{3}{2}}),\\ &\max_{0\le t\le 1}\|W_2\|_{H^2} = O(k^{\frac{1}{2}}).\end{aligned} \tag{3.72}$$

For (3.70), a simple L^2 estimate using (3.67), (3.71), and (3.72) gives

$$\begin{aligned}\partial_t(\|W_1-W_2\|_2^2) &\lesssim \|W_1-W_2\|_2\cdot\|u-U\|_4\cdot\|\nabla W_1\|_4+\|W_1-W_2\|_2^2\cdot\|\nabla u\|_\infty\\ &\quad+\|W_2\|_4\cdot\|\nabla(u-U)\|_4\cdot\|W_1-W_2\|_2\\ &\lesssim O(k^{-2})\cdot\|W_1-W_2\|_2+O(1)\cdot\|W_1-W_2\|_2^2+O(k^{-3})\cdot\|W_1-W_2\|_2.\end{aligned}$$

Gronwall in time then gives

$$\max_{0\le t\le 1}\|W_1(t)-W_2(t)\|_2 = O(k^{-2}).$$

Interpolating this with the trivial estimate

$$\max_{0\le t\le 1}\|W_1(t)-W_2(t)\|_{H^2} = O(k^{\frac{1}{2}})$$

then yields

$$\max_{0\le t\le 1}\|W_1(t)-W_2(t)\|_{H^{\frac{3}{2}}} = O(k^{-\frac{1}{8}}).$$

This shows that $\|E_2\|_{H^{\frac{3}{2}}} = O(k^{-\frac{1}{8}})$ and we have finished the proof of (3.56).

We now specify the choice of g_0 in (3.53).

By (3.44), we have

$$\max\left\{|(\partial_r\tilde{\phi}^r)(t_0,\tilde{r}_*,\tilde{z}_*)|,\ |(\partial_z\tilde{\phi}^r)(t_0,\tilde{r}_*,\tilde{z}_*)|,\ |(\partial_r\tilde{\phi}^z)(t_0,\tilde{r}_*,\tilde{z}_*)|,\ |(\partial_z\tilde{\phi}^z)(t_0,\tilde{r}_*,\tilde{z}_*)|\right\} = M.$$

WLOG we assume

$$\left|(\partial_r\tilde{\phi}^r)(t_0,\tilde{r}_*,\tilde{z}_*)\right| = M. \tag{3.73}$$

The other cases are similarly treated.

Let (r_*,z_*) be the pre-image of $(\tilde{r}_*,\tilde{z}_*)$, that is, $\tilde{r}_* = \phi^r(t_0,r_*,z_*)$, $\tilde{z}_* = \phi^z(t_0,r_*,z_*)$. By (3.7), we have

$$\left|\det\Big((D\tilde{\phi})(t_0,\phi(t_0,r_*,z_*))\Big)\right| = \frac{\phi^r(t_0,r_*,z_*)}{r_*} =: N_* > 0. \tag{3.74}$$

By the fundamental theorem of calculus and (3.3), we have

$$\begin{aligned} r_* &= \tilde{\phi}^r(t_0,\phi^r(t_0,r_*,z_*),\phi^z(t_0,r_*,z_*)) - \tilde{\phi}^r(t_0,0,\phi^z(t_0,r_*,z_*)) \\ &\le \|\partial_r\tilde{\phi}^r\|_\infty \cdot \phi^r(t_0,r_*,z_*) \\ &\le M\cdot\phi^r(t_0,r_*,z_*). \end{aligned}$$

Therefore,

$$N_*M \ge 1. \tag{3.75}$$

This relation will be used later.

By (3.73), (3.74), and continuity, we can find a nonempty open set Ω_0 around the point (r_*,z_*) such that

$$\begin{aligned} &\frac{M}{2} < |(\partial_r\tilde{\phi}^r)(t_0,\phi(t_0,r,z))| < 2M, \\ &\frac{N_*}{2} < \frac{\phi^r(t_0,r,z)}{r} = \left|\det((D\tilde{\phi})(t_0,\phi(t_0,r,z)))\right| < 2N_*, \quad \forall\,(r,z)\in\Omega_0. \end{aligned} \tag{3.76}$$

Furthermore, we may shrink Ω_0 slightly if necessary such that for some $\delta_1 > 0$,

$$\Omega_0 \cap \left\{(r,z):\ 0\le r\le\delta_1\right\} = \emptyset.$$

In yet other words, if $(r,z)\in\Omega_0$, then we must have $r > \delta_1$.

Now choose $b \in C_c^\infty(\Omega_0)$ such that

$$\int |b(r,z)|^2 r \,\mathrm{d}r \,\mathrm{d}z = 1. \tag{3.77}$$

Since by our choice Ω_0 stays away from the axis $r = 0$, the function b can be naturally regarded as a smooth function on $\mathbb{R}^3$.

We now let

$$g_0(r,z) = \frac{1}{M^{\frac{1}{6}}} \cos(kr) b(r,z) \tag{3.78}$$

and recall from (3.53)

$$\begin{aligned} G_0(x) &= g_0(r,z) e_\theta \\ &= \frac{1}{M^{\frac{1}{6}}} \cos(kr) b(r,z) e_\theta. \end{aligned}$$

By (3.77), it is not difficult to check that (3.54) is satisfied.

Since

$$W_0 = \omega_0 + k^{-\frac{3}{2}} G_0,$$

by taking k sufficiently large, obviously we can have

$$\begin{aligned} \|W_0\|_{L^1(\mathbb{R}^3)} &\le 2\|\omega_0\|_{L^1(\mathbb{R}^3)}, \\ \|W_0\|_{L^\infty(\mathbb{R}^3)} &\le 2\|\omega_0\|_{L^\infty(\mathbb{R}^3)}, \\ \left\|\frac{W_0}{r}\right\|_{L^1(\mathbb{R}^3)} &\le 2\left\|\frac{\omega_0}{r}\right\|_{L^1(\mathbb{R}^3)}, \\ \left\|\frac{W_0}{r}\right\|_{L^\infty(\mathbb{R}^3)} &\le 2\left\|\frac{\omega_0}{r}\right\|_{L^\infty(\mathbb{R}^3)}, \\ \left\|\frac{W_0}{r}\right\|_{L^{3,1}(\mathbb{R}^3)} &\le 2\left\|\frac{\omega_0}{r}\right\|_{L^{3,1}(\mathbb{R}^3)}. \end{aligned}$$

Therefore (3.45) and the 1st two conditions in (3.46) are easily satisfied. To check the 3rd condition therein, we note that by (3.77) and for k sufficiently large,

$$\begin{aligned} \|G_0\|_{L^2(\mathbb{R}^3)} &\lesssim \frac{1}{M^{\frac{1}{6}}}, \\ \|G_0\|_{H^2(\mathbb{R}^3)} &\lesssim \frac{1}{M^{\frac{1}{6}}} \cdot k^2. \end{aligned}$$

Here, the implied constants are absolute constants. Interpolation then gives

$$\|G_0\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \lesssim \frac{1}{M^{\frac{1}{6}}} k^{\frac{3}{2}}.$$

Thus all conditions in (3.45) and (3.46) are satisfied.

It remains to show (3.47).

By (3.51), (3.55), and (3.56), we have

$$\begin{aligned}
\|W(t_0,\cdot)\|_{\dot{H}^{\frac{3}{2}}} &\geq \left\| k^{-\frac{3}{2}} \cdot \frac{g_0(\tilde{\phi}(t_0))}{\tilde{\phi}^r(t_0)} r e_\theta \right\|_{\dot{H}^{\frac{3}{2}}} - \left\| \frac{\omega_0^\theta(\tilde{\phi}(t_0))}{\tilde{\phi}^r(t_0)} r e_\theta \right\|_{\dot{H}^{\frac{3}{2}}} - \|E_1\|_{\dot{H}^{\frac{3}{2}}} - \|E_2\|_{\dot{H}^{\frac{3}{2}}} \\
&\geq \left\| k^{-\frac{3}{2}} \cdot \frac{g_0(\tilde{\phi}(t_0))}{\tilde{\phi}^r(t_0)} r e_\theta \right\|_{\dot{H}^{\frac{3}{2}}} - M^{\frac{1}{6}} - O(k^{-\alpha}).
\end{aligned}$$

Therefore, (3.47) will be established once we prove the stronger estimate

$$\left\| k^{-\frac{3}{2}} \cdot \frac{g_0(\tilde{\phi}(t_0))}{\tilde{\phi}^r(t_0)} r e_\theta \right\|_{\dot{H}^{\frac{3}{2}}} \gtrsim M^{\frac{1}{3}}. \tag{3.79}$$

We shall prove this via interpolation and inflation of H^1 norm.

By (3.78), we have

$$\begin{aligned}
\left\| k^{-\frac{3}{2}} \cdot \frac{g_0(\tilde{\phi}(t_0))}{\tilde{\phi}^r(t_0)} r e_\theta \right\|_{L^2(\mathbb{R}^3)} &\lesssim k^{-\frac{3}{2}} \left(\int \left| \frac{g_0(\tilde{\phi}(t_0))}{\tilde{\phi}^r(t_0)} r \right|^2 r\,\mathrm{d}r\,\mathrm{d}z \right)^{\frac{1}{2}} \\
&\lesssim k^{-\frac{3}{2}} \left(\int \left| \frac{g_0(r,z)\phi^r(t_0,r,z)}{r} \right|^2 r\,\mathrm{d}r\,\mathrm{d}z \right)^{\frac{1}{2}} \\
&\lesssim \frac{k^{-\frac{3}{2}}}{M^{\frac{1}{6}}} \left(\int \frac{\cos^2(kr) b^2(r,z) (\phi^r(t_0,r,z))^2}{r^2} r\,\mathrm{d}r\,\mathrm{d}z \right)^{\frac{1}{2}} \\
&\lesssim \frac{k^{-\frac{3}{2}}}{M^{\frac{1}{6}}} \left\| \frac{b\phi^r(t_0)}{r} \right\|_{L^2(\mathbb{R}^3)} \lesssim \frac{k^{-\frac{3}{2}}}{M^{\frac{1}{6}}} N_*,
\end{aligned} \tag{3.80}$$

where in the last inequality we have used (3.76) and (3.77).

Now introduce

$$g_1(r,z) = \sin(k\tilde{\phi}^r(t_0,r,z)) \frac{b(\tilde{\phi}(t_0,r,z))}{\tilde{\phi}^r(t_0,r,z)} (\partial_r \tilde{\phi}^r)(t_0,r,z) r e_\theta.$$

By (3.76) and a similar calculation as in (3.80), we have for k sufficiently large,

$$\begin{aligned}\|g_1\|_{L^2(r\,\mathrm{d}r\,\mathrm{d}z)} &\geq \left(\int \frac{\sin^2(kr)b^2(r,z)((\partial_r\tilde{\phi}^r)(t_0,\phi(t_0,r,z)))^2}{r^2}(\phi^r(t_0,r,z))^2 r\,\mathrm{d}r\,\mathrm{d}z\right)^{\frac{1}{2}}\\ &\geq M\left\|\frac{b\phi^r(t_0)}{r}\right\|_{L^2(\mathbb{R}^3)} - O(k^{-\alpha})\\ &\geq \frac{2}{3}M\left\|\frac{b\phi^r(t_0)}{r}\right\|_{L^2(\mathbb{R}^3)} \gtrsim M\cdot N_*.\end{aligned} \tag{3.81}$$

Now for the $\dot{H}^1$-norm, by using (3.81), we have

$$\begin{aligned}\left\|k^{-\frac{3}{2}}\cdot\frac{g_0(\tilde{\phi}(t_0))}{\tilde{\phi}^r(t_0)}re_\theta\right\|_{\dot{H}^1(\mathbb{R}^3)} &\geq k^{-\frac{3}{2}}\left\|\partial_r\Big(\frac{g_0(\tilde{\phi}(t_0))}{\tilde{\phi}^r(t_0)}re_\theta\Big)\right\|_{L^2(r\,\mathrm{d}r\,\mathrm{d}z)}\\ &\geq \frac{k^{-\frac{3}{2}}}{M^{\frac{1}{6}}}\cdot(k\|g_1\|_{L^2(r\,\mathrm{d}r\,\mathrm{d}z)}+O(1))\\ &\geq \frac{1}{2}k^{-\frac{1}{2}}M^{\frac{5}{6}}N_*,\end{aligned} \tag{3.82}$$

where again we need to take k sufficiently large.

We are now ready to prove (3.79).

By the usual interpolation inequality

$$\|f\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|f\|^{\frac{1}{3}}_{L^2(\mathbb{R}^3)}\cdot\|f\|^{\frac{2}{3}}_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)}$$

and (3.80) and (3.82), we have

$$k^{-\frac{1}{2}}M^{\frac{5}{6}}N_* \lesssim \left(\frac{k^{-\frac{3}{2}}N_*}{M^{\frac{1}{6}}}\right)^{\frac{1}{3}}\cdot\left\|k^{-\frac{3}{2}}\cdot\frac{g_0(\tilde{\phi}(t_0))}{\tilde{\phi}^r(t_0)}re_\theta\right\|^{\frac{2}{3}}_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)}.$$

By (3.75), we then have

$$\begin{aligned}\left\|k^{-\frac{3}{2}}\cdot\frac{g_0(\tilde{\phi}(t_0))}{\tilde{\phi}^r(t_0)}re_\theta\right\|^{\frac{2}{3}}_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} &\gtrsim M^{\frac{8}{9}}N_*^{\frac{2}{3}}\\ &\gtrsim M^{\frac{2}{9}}.\end{aligned}$$

Hence,

$$\left\| k^{-\frac{3}{2}} \cdot \frac{g_0(\tilde{\phi}(t_0))}{\tilde{\phi}^r(t_0)} r e_\theta \right\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} \gtrsim M^{\frac{1}{3}} \gg M^{\frac{1}{6}}.$$

This ends the estimate of (3.79). ■

Proposition 3.12. For any $A \gg 1$, there exist $\delta_0 = \delta_0(A) \to 0$, $t_0 = t_0(A) \to 0$, $M_0 = M_0(A) \to \infty$ (as $A \to \infty$), and a smooth solution ω to the axisymmetric (without swirl) Euler equation

$$\begin{cases} \partial_t \left(\frac{\omega}{r}\right) + (u \cdot \nabla)\left(\frac{\omega}{r}\right) = 0, \quad 0 < t \le 1,\ x = (x_1, x_2, z),\ r = \sqrt{x_1^2 + x_2^2}, \\ u = -\Delta^{-1}\nabla \times \omega, \\ \omega\Big|_{t=0} = \omega_0 \end{cases}$$

such that the following conditions are satisfied:

1. $\omega_0 \in C_c^\infty(\mathbb{R}^3)$, $\omega_0 = \omega_0^\theta(r,z) e_\theta$, and for some $r_0 > 0$

$$\operatorname{supp}(\omega_0^\theta(r,z)) \subset \left\{(r,z) : r > r_0\right\}. \tag{3.83}$$

2. The L^∞ norm of ω is uniformly small on the interval $[0, 1]$:

$$\max_{0 \le t \le 1} \|\omega(t)\|_{L^\infty} \le \delta_0(A). \tag{3.84}$$

3. The support of $\omega(t)$ remains close to the origin:

$$\operatorname{supp}(\omega(t,\cdot)) \subset \left\{x : \quad |x| < \delta_0(A)\right\}, \quad \forall\, 0 \le t \le 1. \tag{3.85}$$

4. The $\dot{H}^{\frac{3}{2}}$-norm of ω is inflated rapidly from $t = 0$ to $t = t_0$:

$$\|\omega_0\|_{\dot{H}^{\frac{3}{2}}} < \delta_0(A),$$

$$\|\omega(t_0)\|_{\dot{H}^{\frac{3}{2}}} > M_0(A). \tag{3.86}$$

Proof of Proposition 3.12 We first note that it suffices to construct the solution ω satisfying all other conditions except (3.85). Indeed if ω is such a solution, then for any $\lambda > 0$,

$$\omega_\lambda(t,x) := \omega(t, \lambda x)$$

is also a solution to the Euler equation. By finite speed propagation, we have

$$\mathrm{supp}(\omega(t)) \subset K, \quad \forall\, 0 \le t \le 1,$$

where K is a fixed compact set. On the other hand

$$\mathrm{supp}(\omega_\lambda(t)) \subset \frac{1}{\lambda}K = \left\{\frac{1}{\lambda}x :\ x \in K\right\}, \quad \forall\, 0 \le t \le 1.$$

Obviously by taking λ sufficiently large we can satisfy (3.85). Note that (3.84) and (3.86) are invariant under the scaling transformation $x \to \lambda x$. Therefore, in the rest of this proof we shall ignore (3.85).

For $A \gg 1$, we choose g_A as in (3.11) and denote by W the corresponding smooth solution to the Euler equation:

$$\begin{cases} \partial_t\left(\frac{W}{r}\right) + (U\cdot\nabla)\left(\frac{W}{r}\right) = 0, \quad -2 \le t \le 2, \\ \nabla \cdot U = 0, \\ W\Big|_{t=0} = W_0 = g_A e_\theta. \end{cases}$$

By (3.16) we have (recall $U = U^r e_r + U^z e_z$)

$$\begin{aligned} \left\|\frac{U^r(t)}{r}\right\|_\infty &\le C\left\|\frac{W(t)}{r}\right\|_{L^{3,1}} \\ &\le C\sqrt{\log A}, \quad \forall\, t \in \mathbb{R}, \end{aligned}$$

where $C > 0$ is an absolute constant and we have used the $L^{3,1}$-preservation of W/r:

$$\left\|\frac{W(t)}{r}\right\|_{L^{3,1}} = \left\|\frac{W_0}{r}\right\|_{L^{3,1}}, \quad \forall\, t \in \mathbb{R}.$$

Since

$$\partial_t W + (U\cdot\nabla)W = \frac{U^r}{r}W,$$

we get

$$\begin{aligned} \max_{-2\le t\le 2} \|W(t)\|_\infty &\le \|W_0\|_\infty e^{\max_{-2\le t\le 2}\|\frac{U^r}{r}\|_\infty} \\ &\le \frac{\sqrt{\log A}}{A} e^{C\sqrt{\log A}} < A^{-\frac{1}{2}}, \end{aligned} \tag{3.87}$$

for A sufficiently large.

By definition of g_A, the condition (3.83) is trivially satisfied. It remains to check (3.86). By (3.17), we have

$$\|W_0\|_{\dot{H}^{\frac{3}{2}}} \lesssim \frac{\sqrt{\log A}}{\sqrt{A}}.$$

Let $\Phi = (\Phi^r, \Phi^z)$ be the forward characteristic lines as in (3.2) and let $\tilde{\Phi}$ be the inverse. By Proposition 3.9, we have for some $0 < t_1 \le \frac{1}{\log\log A}$,

$$\|D\Phi(t_1)\|_\infty + \|D\tilde{\Phi}(t_1)\|_\infty \ge \log\log A. \tag{3.88}$$

By differentiating the identity $\Phi \circ \tilde{\Phi} = id$ and using (3.7), we have

$$\begin{aligned}(D\Phi)(\tilde{\Phi}(r,z)) &= \Big((D\tilde{\Phi})(r,z)\Big)^{-1} \\ &= \frac{1}{\det(D\tilde{\Phi}(r,z))}\operatorname{adj}((D\tilde{\Phi})(r,z)) \\ &= \frac{\tilde{\Phi}^r(r,z)}{r}\operatorname{adj}((D\tilde{\Phi})(r,z)),\end{aligned}$$

where $\operatorname{adj}((D\tilde{\Phi})(r,z))$ is the adjugate matrix of $D\tilde{\phi}(r,z)$. Recall that for any 2×2 matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we have

$$\operatorname{adj}(B) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and obviously

$$\begin{aligned}\|B\|_\infty &= \max\left\{|a|, |b|, |c|, |d|\right\} \\ &= \|\operatorname{adj}(B)\|_\infty.\end{aligned}$$

Therefore,

$$\|D\Phi(t_1)\|_\infty \leq \left\| \frac{\tilde{\Phi}^r}{r} \right\|_\infty \|\mathrm{adj}(D\tilde{\Phi})\|_\infty$$
$$\leq \|D\tilde{\Phi}(t_1)\|_\infty^2$$

Consequently, we have

$$\|D\tilde{\Phi}(t_1)\|_\infty \gtrsim \sqrt{\log\log A}. \tag{3.89}$$

We can then apply Proposition 3.11 (with W as the input solution ω) and obtain $\tilde{\omega}$ as the desired solution (note that $\|\frac{\tilde{\omega}_0}{r}\|_{L^{3,1}} \lesssim \sqrt{\log A}$, $\|\tilde{\omega}_0\|_\infty \lesssim \frac{\sqrt{\log A}}{A}$ so that we can repeat the computation of (3.87) and still have $\max_{0\leq t\leq 1} \|\tilde{\omega}(t)\|_\infty \lesssim A^{-\frac{1}{2}}$.) ■

Lemma 3.13. Suppose ω^1 and ω^2 are given smooth solutions to the 3D Euler equations (in vorticity form):

$$\begin{cases} \partial_t \omega^j + (u^j \cdot \nabla)\omega^j = (\omega^j \cdot \nabla) u^j, \qquad 0 < t \leq 1, \\ u^j = -\Delta^{-1}\nabla \times \omega^j, \\ \omega^j\Big|_{t=0} = \omega_0^j \in C_c^\infty(\mathbb{R}^3), \quad j = 1, 2. \end{cases}$$

Here we assume the lifespan of each ω^j is at least $[0, 1]$.

Define

$$r_0 = \max_{j=1,2} \max_{0\leq t\leq 1} \|u^j(t)\|_\infty. \tag{3.90}$$

Consider the problem

$$\begin{cases} \partial_t W + (U \cdot \nabla) W = (W \cdot \nabla) U, \\ U = -\Delta^{-1}\nabla \times W, \\ W\Big|_{t=0} = W_0, \end{cases} \tag{3.91}$$

where

$$W_0(x) = \omega_0^1(x) + \omega_0^2(x - x_W),$$

and $x_W \in \mathbb{R}^3$ is a vector that controls the mutual distance between ω_0^1 and ω_0^2.

For any $\epsilon > 0$, there exists $R_\epsilon = R_\epsilon(\epsilon, \max_{j=1,2}\max_{0\le t\le 1}\|u^j(t)\|_{H^4}) > 100r_0$ sufficiently large, such that if $|x_W| \ge R_\epsilon$, then the following hold:

(1) There exists a unique smooth solution W to (3.91) on the time interval $[0,1]$. Furthermore for any $0 \le t \le 1$ it has the decomposition

$$W(t) = W^1(t) + W^2(t), \tag{3.92}$$

where

$$\begin{aligned}
&\mathrm{supp}(W^1(t)) \subset \Omega_1^\epsilon,\\
&\quad \Omega_1^\epsilon := \Big\{x \in \mathbb{R}^3 : \, d\big(x, \mathrm{supp}(\omega_1^0)\big) < r_0 + \epsilon\Big\},\\
&\mathrm{supp}(W^2(t)) \subset \Omega_W^\epsilon,\\
&\quad \Omega_W^\epsilon := \Big\{y = x + x_W : \, d\big(x, \mathrm{supp}(\omega_2^0)\big) < r_0 + \epsilon\Big\}.
\end{aligned}$$

(2) The flow W is uniformly close to $\omega^1(\cdot) + \omega^2(\cdot - x_W)$:

$$\begin{aligned}
\max_{0\le t\le 1}\big\|W^1(t,\cdot) - \omega^1(t,\cdot)\big\|_{H^2} &< \epsilon,\\
\max_{0\le t\le 1}\big\|W^2(t,\cdot) - \omega^2(t,\cdot - x_W)\big\|_{H^2} &< \epsilon
\end{aligned} \tag{3.93}$$

(3) All higher Sobolev norms of W^1 and W^2 can be controlled in terms of ω_0^1 and ω_0^2, respectively: let

$$N = \max_{0\le t\le 1}\big(\|\omega^1(t,\cdot)\|_\infty + \|\omega^2(t,\cdot)\|_\infty\big) + \|u_0^1\|_2 + \|u_0^2\|_2.$$

Here u_0^1 and u_0^2 are the velocity fields corresponding to the vorticity ω_0^1 and ω_0^2, respectively. Then for any $k \ge 3$,

$$\begin{aligned}
\max_{0\le t\le 1}\|W^1(t,\cdot)\|_{H^k} &\le C\big(k, \|\omega_0^1\|_{H^k}, N\big) < \infty,\\
\max_{0\le t\le 1}\|W^2(t,\cdot)\|_{H^k} &\le C\big(k, \|\omega_0^2\|_{H^k}, N\big) < \infty.
\end{aligned} \tag{3.94}$$

Proof of Lemma 3.13 Let $R = |x_W|$ and denote

$$M_0 = 100\big(\|u_0^1\|_{H^4} + \|u_0^2\|_{H^4}\big),$$
$$M_1 = \max_{0\le t\le 1} \big(\|\omega^1(t)\|_\infty + \|\omega^2(t)\|_\infty + 1\big). \tag{3.95}$$

Consider the 3D Euler equation (in velocity formulation)

$$\begin{cases} \partial_t u + (u\cdot\nabla)u = -\nabla p, \\ \nabla\cdot u = 0, \\ u\Big|_{t=0} = u_0. \end{cases}$$

Suppose

$$\|u_0\|_{H^4} \le M_0,$$

and on some time interval $[0,\tau]$, $\tau \le 1$,

$$\max_{0\le t\le\tau} \|\omega(t)\|_\infty \le M_1,$$

where $\omega = \mathrm{curl}(u)$. Then by a simple energy estimate, we have

$$\max_{0\le t\le\tau} \|u(t)\|_{H^4} \le M_2, \tag{3.96}$$

where $M_2 = M_2(M_0, M_1) > 0$ can be taken as a constant that is uniform for all $\tau \le 1$. We shall need this constant below. Also by standard local well-posedness theory, if for some t_0 we have $\|u(t_0)\|_{H^4} \le M_2$, then there exists $\tau_0 = \tau_0(M_2) > 0$ such that the corresponding local solution has lifespan at least $[t_0, t_0+\tau_0]$ and

$$\max_{t_0\le t\le t_0+\tau_0} \|u(t)\|_{H^4} \le 2M_2.$$

This fact will also be used below.

Now let $0 = t_0 < t_1 < \cdots < t_{L-1} < t_L = 1$ be a partition of the time interval $[0,1]$ such that

$$\max_{0\le i\le L-1} (t_{i+1} - t_i) < \tau_0.$$

We now inductively check the following

Claim: For each $i = 0, 1, \cdots, L$, there exists $R_i > 0$ sufficiently large such that if $R > R_i$, then the following hold:

(1) $W(t)$ has the decomposition (3.92) for all $0 \le t \le t_i$.

(2)

$$\max_{0 \le t \le t_i} \|W^1(t,\cdot) - \omega^1(t,\cdot)\|_2 < R^{-\frac{1}{4}}, \tag{3.97}$$

$$\max_{0 \le t \le t_i} \|W^2(t,\cdot) - \omega^2(t,\cdot - x_W)\|_2 < R^{-\frac{1}{4}}. \tag{3.98}$$

(3)

$$\max_{0 \le t \le t_i} \|U(t)\|_{H^4} \le M_2. \tag{3.99}$$

Indeed the claim holds trivially for $i = 0$. Now assume the claim holds for all $i \le l-1$ $(l \ge 1)$, and we need to prove the claim for $i = l$. Since $\|U(t_{l-1})\|_{H^4} \le M_2$, by our choice of τ_0, $U(t)$ can be extended to $[t_{l-1}, t_l]$, and

$$\begin{aligned} \max_{0 \le t \le t_l} \|U(t)\|_{H^4} &\le \max\left\{ \max_{0 \le t \le t_{l-1}} \|U(t)\|_{H^4}, \max_{t_{l-1} \le t \le t_l} \|U(t)\|_{H^4} \right\} \\ &\le 2M_2. \end{aligned} \tag{3.100}$$

By the inductive assumption, we have

$$W(t_{l-1}) = W^1(t_{l-1}) + W^2(t_{l-1}),$$

and for R sufficiently large,

$$\mathrm{dist}\Big(\mathrm{supp}(W^1(t_{l-1})), \mathrm{supp}(W^2(t_{l-1}))\Big) > \frac{2}{3}R.$$

By (3.100) and finite speed propagation, we then have for R sufficiently large,

$$\mathrm{dist}\Big(\mathrm{supp}(W^1(t)), \mathrm{supp}(W^2(t))\Big) > \frac{1}{3}R, \qquad \forall\, 0 \le t \le t_l. \tag{3.101}$$

Denote by U^1 and U^2 the velocity fields corresponding to the vorticity W^1 and W^2, respectively. By (3.101) and an argument similar to the 2D case in [2], it is not difficult to check that

$$\max_{0\le t\le t_l}\max_{|\alpha|\le 3}\|D^\alpha U^2(t,\cdot)\|_{L^\infty(x\in \mathrm{supp}(W^1(t)))}\le R^{-\frac{1}{3}}, \tag{3.102}$$

where again we need to take R sufficiently large (to kill some prefactors).

Now observe that

$$\begin{cases}\partial_t W^1+(U^1\cdot\nabla)W^1=(W^1\cdot\nabla)U^1-(U^2\cdot\nabla)W^1+(W^1\cdot\nabla)U^2,\\ \partial_t\omega^1+(u^1\cdot\nabla)\omega^1=(\omega^1\cdot\nabla)u^1,\\ W^1\Big|_{t=0}=\omega^1\Big|_{t=0}=\omega^1_0.\end{cases}$$

Set $\eta=W^1-\omega^1$, $v=U^1-u^1$. Then clearly

$$\partial_t\eta+(v\cdot\nabla)W^1+(u^1\cdot\nabla)\eta=(\eta\cdot\nabla)U^1+(\omega^1\cdot\nabla)v-(U^2\cdot\nabla)W^1+(W^1\cdot\nabla)U^2.$$

A simple L^2 estimate using (3.100) and (3.102) then gives for $0<t\le t_l$:

$$\begin{aligned}\partial_t(\|\eta\|_2^2)&\lesssim\|\eta\|_2\cdot\|v\|_6\cdot\|\nabla W^1\|_3+\|\eta\|_2^2\cdot\|\nabla U^1\|_\infty+\|\nabla v\|_2\cdot\|\eta\|_2\cdot\|\omega^1\|_\infty\\ &\quad+R^{-\frac{1}{3}}\|\nabla W^1\|_2\cdot\|\eta\|_2+R^{-\frac{1}{3}}\|W^1\|_2\cdot\|\eta\|_2\\ &\lesssim_{M_2,\omega^1}\ \|\eta\|_2^2+R^{-\frac{1}{3}}\|\eta\|_2.\end{aligned}$$

Integrating in time up to t_l and taking R sufficiently large then gives

$$\max_{0\le t\le t_l}\|\eta(t)\|_2<R^{-\frac{1}{4}}.$$

This settles (3.97) for $i=l$. The inequality (3.98) is proved similarly. Interpolating (3.97), (3.98) with (3.100) then easily yields that (see (3.95))

$$\max_{0\le t\le t_l}\|W(t)\|_\infty\le M_1.$$

Therefore, by (3.96), we can upgrade the rough estimate (3.100) to (3.99) for $i=l$. By (3.97)–(3.98), interpolation, and taking R sufficiently large, we can easily have (see (3.90))

$$\max_{0\le t\le t_l}\|U(t,\cdot)\|_\infty\le r_0+\epsilon/2.$$

Hence, the decomposition (3.92) follows. We have completely proved the claim.

By using the claim and a simple interpolation argument, it is not difficult to check that (3.93) holds. Finally (3.94) follows from a simple energy estimate using the disjointness of the support of W^1 and W^2 and an estimate similar to (3.102). The lemma is proved. ■

Proposition 3.14. Assume $\{\omega^j\}_{j=1}^{\infty}$ is a sequence of smooth functions each of which solves the 3D incompressible Euler equation (in vorticity form)

$$\begin{cases} \partial_t \omega^j + (u^j \cdot \nabla)\omega^j = (\omega^j \cdot \nabla) u^j, \quad 0 < t \le 1, \\ u^j = -\Delta^{-1} \nabla \times \omega^j, \\ \omega^j \Big|_{t=0} = \omega_0^j \in C_c^{\infty}(\mathbb{R}^3), \end{cases}$$

and satisfies the following condition:

For each $j \ge 1$, $\operatorname{supp}(\omega^j(t)) \subset B(0, 2^{-10j})$ for any $0 \le t \le 1$ and

$$\|u_0^j\|_{H^{\frac{5}{2}}} + \max_{0 \le t \le 1} (\|\omega^j(t)\|_\infty + \|u^j(t)\|_\infty) \le 2^{-10j}. \tag{3.103}$$

Here u_0^j is the velocity corresponding to the vorticity ω_0^j.

Then there exist centers $x_j \in \mathbb{R}^3$ whose mutual distance are sufficiently large (i.e., $|x_j - x_k| \gg 1$ if $j \ne k$) such that the following hold:

(1) Take the initial data (vorticity)

$$W_0(x) = \sum_{j=1}^{\infty} \omega_0^j (x - x_j),$$

then $W_0 \in L^1 \cap L^\infty \cap H^{\frac{3}{2}} \cap C^\infty$. The corresponding initial velocity $U_0 \in H^{\frac{5}{2}} \cap C^\infty$. Furthermore, for any $j \ne k$,

$$B(x_j, 100) \cap B(x_k, 100) = \varnothing.$$

(2) With W_0 as initial data, there exists a unique smooth solution W to the Euler equation (in vorticity form)

$$\begin{cases} \partial_t W + (U \cdot \nabla) W = (W \cdot \nabla) U, \\ U = -\Delta^{-1} \nabla \times W, \\ W \Big|_{t=0} = W_0, \end{cases}$$

on the time interval $[0, 1]$ satisfying $W \in L_t^\infty L_x^1 \cap L_t^\infty L_x^\infty \cap C^\infty$, $U \in C^\infty \cap L_t^\infty L_x^2$. Moreover for any $0 \le t \le 1$,

$$\mathrm{supp}(W(t,\cdot)) \subset \bigcup_{j=1}^{\infty} B(x_j, 1). \tag{3.104}$$

(3) For any $\epsilon > 0$, there exists an integer J_ϵ sufficiently large such that if $j \ge J_\epsilon$, then

$$\max_{0\le t\le 1} \|W(t,\cdot) - \omega^j(t, \cdot - x_j)\|_{H^2(B(x_j,1))} < \epsilon. \tag{3.105}$$

Proof of Proposition 3.14. Define $x_1 = 0$. By recursively applying Lemma 3.13, we can choose centers x_j whose mutual distance is sufficiently large such that for each $l \ge 2$, we can find a unique smooth solution W^l solving the system

$$\begin{cases} \partial_t W^l + (U^l \cdot \nabla) W^l = (W^l \cdot \nabla) U^l, \quad 0 < t \le 1, \\ U^l = -\Delta^{-1} \nabla \times W^l, \\ W^l \Big|_{t=0} = W_0^l, \end{cases}$$

where

$$W_0^l = \sum_{j=1}^{l} \omega_0^j (x - x_j).$$

Furthermore, W^l satisfies

- $\mathrm{supp}(W^l(t)) \subset \bigcup_{j=1}^{l} B(x_j, \frac{1}{2})$, for all $0 \le t \le 1$.
- $\max_{0\le t\le 1} \|W^l(t,\cdot) - \omega^j(t, \cdot - x_j)\|_{H^2(B(x_j,1))} < 2^{-j}$, for any $1 \le j \le l$.
- $\max_{0\le t\le 1} \|W^{l+1}(t,\cdot) - W^l(t,\cdot)\|_{H^2(\bigcup_{j=1}^{l} B(x_j,1))} < 2^{-l}$.
- $\max_{0\le t\le 1} \|W^{l+1}(t,\cdot) - W^l(t,\cdot)\|_{L^2} < 2^{-l}$.
- $\max_{0\le t\le 1} \|W^l(t,\cdot)\|_{H^k(B(x_j,1))} \le C_k = C_k(k, \|\omega_0^j\|_{H^k}) < \infty$, for any $1 \le j \le l$.

Note that in the last inequality above we have no dependence on other constants, thanks to the strong assumption (3.103).

Now define

$$W(t,x) = \begin{cases} \lim_{l\to\infty} W^l(t,x), & \text{if } x \in \bigcup_{j=1}^{\infty} B(x_j, 1), \\ 0, & \text{otherwise}. \end{cases}$$

Fix any $j_0 \geq 1$. By using the properties of W^l listed above, we have

$$\max_{0\le t\le 1} \left\| W^{l+1}(t,\cdot) - W^l(t,\cdot) \right\|_{H^2(B(x_{j_0},1))} \le 2^{-l}, \quad \text{if } l \ge j_0+1.$$

Also for any $k \geq 3$,

$$\max_{0\le t\le 1} \| W^l(t,\cdot)\|_{H^k(B(x_{j_0},1))} \le C_k, \quad \forall\, l \ge j_0+1.$$

Therefore, $(W^l(t))$ is Cauchy in $H^k(B(x_{j_0},1))$ for any $k \geq 2$. Hence, W^l converges uniformly to $W \in C^\infty((B(x_{j_0},1))$. Since j_0 is arbitrary, we obtain $W \in C^\infty(\mathbb{R}^3)$. Similarly fix any $j_0 \geq 1$. By Sobolev embedding, we have

$$\begin{aligned} \max_{0\le t\le 1} \left\| W^l(t,\cdot) - \omega^{j_0}(t,\cdot) \right\|_{L^\infty(B(x_{j_0},1))} &\lesssim \max_{0\le t\le 1} \left\| W^l(t,\cdot) - \omega^{j_0}(t,\cdot) \right\|_{H^2(B(x_{j_0},1))} \\ &\lesssim 2^{-j_0}, \quad \forall\, l \ge j_0+1. \end{aligned}$$

By (3.103) and sending $l \to \infty$, we obtain $\max_{0\le t\le 1} \|W(t,\cdot)\|_{L^\infty} \lesssim 1$. Similarly it is also easy to check that $W \in L_t^\infty L_x^1$. Since W^l is Cauchy in L^2, by Sobolev embedding, we have U^l is Cauchy in L^6 and converges to the limit U. It is not difficult to check that U is smooth and W is the desired solution. The estimate (3.105) follows obviously from the property of W^l and passing l to the limit. The proposition is proved. ■

We are now ready to complete the

Proof of Theorem 1.2 It suffices for us to prove the case $\omega_0^{(g)} \equiv 0$. The case for nonzero $\omega_0^{(g)}$ is a simple modification of the proof below.

For each $j \geq 1$, by using Proposition 3.12, we can find a smooth solution ω^j solving the system

$$\begin{cases} \partial_t \omega^j + (u^j\cdot\nabla)\omega^j = (\omega^j\cdot\nabla)u^j, \quad 0 < t \le 1, \\ u^j = -\Delta^{-1}\nabla\times\omega^j, \\ \omega^j\Big|_{t=0} = \omega_0^j, \end{cases}$$

such that the following hold:

- $\operatorname{supp}(\omega^j(t,\cdot)) \subset \{x,\ |x| < 2^{-100j}\}$, for any $0 \le t \le 1$.
- $\max_{0\le t\le 1}(\|\omega^j(t)\|_{L^\infty} + \|u^j(t)\|_{L^\infty}) \le 2^{-100j}$.

- Let u_0^j be the velocity corresponding to the vorticity ω_0^j, then

$$\|u_0^j\|_{H^{\frac{5}{2}}} < 2^{-100j}.$$

- For some $0 < t_j^0 < \frac{1}{j}$, we have

$$\|\omega^j(t_j^0, \cdot)\|_{\dot{H}^{\frac{3}{2}}} > 2^j.$$

By continuity and the last inequality above, we can find $0 < t_j^1 < t_j^2 < \frac{1}{j}$ such that

$$\|\omega^j(t, \cdot)\|_{\dot{H}^{\frac{3}{2}}} > 2^j, \quad \forall\, t_j^1 \le t \le t_j^2. \tag{3.106}$$

By Proposition 3.14, we can then find centers x_j and build a smooth solution W having initial data

$$W(0, x) = \sum_{j=1}^{\infty} \omega_0^j(x - x_j).$$

The regularity properties of W are simple consequences of Proposition 3.14. By (3.104), we can write

$$W(t, x) = \sum_{j=1}^{\infty} W^j(t, x),$$

where $W^j \in C_c^\infty(B(x_j, 1))$.

Now we make the following:

Claim: there exists an integer $J_1 > 0$ and constants $C_1 > 0$, $C_2 > 0$ such that the following hold: for any $0 \le \tau_0 \le 1$, if $\|W(\tau_0, \cdot)\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} < \infty$, then

$$\|W(\tau_0, \cdot)\|_{\dot{H}^{\frac{3}{2}}} \ge C_1 \|\omega^j(\tau_0, \cdot)\|_{\dot{H}^{\frac{3}{2}}} - C_2, \quad \forall j \ge J_1. \tag{3.107}$$

Here the constant $C_1 > 0$ is actually an absolute constant. The constant C_2 depends on $\max_{0 \le t \le 1} \|W(t, \cdot)\|_2$.

To prove the claim, fix a smooth cut-off function $\phi \in C_c^\infty(\mathbb{R}^3)$ such that $\phi(x) = 1$ for $|x| \le 1$ and $\phi(x) = 0$ for $|x| \ge 2$. Since $|x_j - x_k| \gg 1$ for $j \ne k$, by (3.104), we have for any $j \ge 1$, we have

$$W^j(\tau_0, x) = W(\tau_0, x)\phi(x - x_j) = W(\tau_0, x)\phi_j(x), \quad \text{here } \phi_j(x) := \phi(x - x_j).$$

Fourier transform and the triangle inequality then give

$$\begin{aligned} |\xi|^{\frac{3}{2}} |\widehat{W^j}(\tau_0, \xi)| &\lesssim |\xi|^{\frac{3}{2}} \int_{\mathbb{R}^3} |\hat{W}(\tau_0, \xi - \eta)||\hat{\phi}_j(\eta)| \,\mathrm{d}\eta \\ &\lesssim \int_{\mathbb{R}^3} |\xi - \eta|^{\frac{3}{2}} |\hat{W}(\tau_0, \xi - \eta)||\hat{\phi}_j(\eta)| \,\mathrm{d}\eta + \int_{\mathbb{R}^3} |\hat{W}(\tau_0, \xi - \eta)||\eta|^{\frac{3}{2}} |\hat{\phi}_j(\eta)| \,\mathrm{d}\eta. \end{aligned}$$

Young's inequality then gives for any $j \ge 1$,

$$\|W^j(\tau_0, \cdot)\|_{\dot{H}^{\frac{3}{2}}} \lesssim \|W(\tau_0, \cdot)\|_{\dot{H}^{\frac{3}{2}}} + \|W(\tau_0, \cdot)\|_{L^2}.$$

It is easy to check that the implied constants in the above inequalities are only absolute constants (they depend only on the cut-off function ϕ). By (3.105) and choosing $\epsilon = 1$, we get for any $j \ge J_1$,

$$\|\omega^j(\tau_0, \cdot)\|_{\dot{H}^{\frac{3}{2}}} \le \tilde{C}_1 \|W(\tau_0, \cdot)\|_{\dot{H}^{\frac{3}{2}}} + \tilde{C}_2,$$

where $\tilde{C}_1 > 0$ is an absolute constant and $\tilde{C}_2$ depends only on $\max_{0 \le t \le 1} \|W(t, \cdot)\|_{L^2}$. The claim is proved.

With (3.107) in hand, we now argue by contradiction to finish the proof of the theorem. Assume for some $t_0 < 1$, we have

$$L_0 := \text{ess-sup}_{0 \le t \le t_0} \|W(t, \cdot)\|_{\dot{H}^{\frac{3}{2}}} < \infty. \tag{3.108}$$

By (3.106), we choose $j \gg 1$ sufficiently large such that

$$C_1 2^j - C_2 > 2L_0,$$
$$t_j^2 < t_0.$$

By (3.107), for any $t_j^1 \le t \le t_j^2$, we must have

$$2L_0 \le \|W(t, \cdot)\|_{\dot{H}^{\frac{3}{2}}} < \infty, \quad \text{or} \quad \|W(t, \cdot)\|_{\dot{H}^{\frac{3}{2}}} = +\infty.$$

This obviously contradicts (3.108). The theorem is proved. ■

4 3D Compactly supported case

Lemma 4.1. Let $f \in C_c^\infty(B(0,100))$ and $g \in C_c^\infty(B(0,100))$ be axisymmetric functions on $\mathbb{R}^3$ having the form:

$$f(x) = f^\theta(r,z)e_\theta, \quad g(x) = g^\theta(r,z)e_\theta, \quad x = (x_1, x_2, z),\ r = \sqrt{x_1^2 + x_2^2},$$

where f^θ and g^θ are scalar valued and vanish near $r = 0$, that is, for some $r_0 > 0$,

$$\operatorname{supp}(f^\theta) \subset \{(r,z) : r > r_0\},$$
$$\operatorname{supp}(g^\theta) \subset \{(r,z) : r > r_0\}.$$

Let ω^a and ω be smooth solutions to the following axisymmetric (without swirl) Euler equations:

$$\begin{cases} \partial_t\left(\frac{\omega^a}{r}\right) + (u^a \cdot \nabla)\left(\frac{\omega^a}{r}\right) = 0, \\ u^a = -\Delta^{-1}\nabla \times \omega^a, \\ \omega^a\Big|_{t=0} = f. \end{cases} \tag{4.1}$$

$$\begin{cases} \partial_t\left(\frac{\omega}{r}\right) + (u \cdot \nabla)\left(\frac{\omega}{r}\right) = 0, \\ u = -\Delta^{-1}\nabla \times \omega, \\ \omega\Big|_{t=0} = f + g. \end{cases} \tag{4.2}$$

For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon, f) > 0$ sufficiently small such that if

$$\|g\|_\infty \exp\left(C \cdot \left\|\frac{g}{r}\right\|_{L^{3,1}}\right) < \delta, \tag{4.3}$$

then

$$\max_{0 \le t \le 1} \|\omega^a(t,\cdot) - \omega(t,\cdot)\|_\infty < \epsilon.$$

Here in (4.3), $C > 0$ is the same absolute constant as in the inequality

$$\left\|\frac{u^r}{r}\right\|_\infty \le C\left\|\frac{\omega}{r}\right\|_{L^{3,1}}.$$

Proof of Lemma 4.1 In this proof our main "smallness" parameter is $\|g\|_\infty$. To simplify the notation, we shall denote $X = O(\delta)$ if the quantity X can be made arbitrarily small depending on $\|g\|_\infty$. For example we shall write $X = O(\delta)$ if X satisfies the inequality of the following sort:

$$|X| \lesssim_f \|g\|_\infty \exp\left(C\Big\|\frac{g}{r}\Big\|_{L^{3,1}}\right).$$

Here $C > 0$ is some absolute constant. As another example, if

$$|X| \lesssim_f \|g\|_\infty^{\frac{1}{2}},$$

we shall write $X = O(\delta)$ (the precise notation should be $X = O(\delta^{\frac{1}{2}})$, but we will not keep track of the various exponents and write $O(\delta^{\frac{1}{2}})$ as $O(\delta)$ for simplicity). Other inequalities similar to the above will all be denoted by the same notation $O(\delta)$ whenever there is no confusion. We shall denote $X = O(1)$ if

$$X \lesssim_f 1.$$

We first decompose the solution to (4.2) as

$$\omega = \omega^1 + \omega^2,$$

where ω^1 and ω^2 solve the *linear* systems

$$\begin{cases} \partial_t\left(\frac{\omega^1}{r}\right) + (u\cdot\nabla)\left(\frac{\omega^1}{r}\right) = 0, \\ \omega^1\Big|_{t=0} = f, \end{cases} \tag{4.4}$$

$$\begin{cases} \partial_t\left(\frac{\omega^2}{r}\right) + (u\cdot\nabla)\left(\frac{\omega^2}{r}\right) = 0, \\ \omega^2\Big|_{t=0} = g. \end{cases} \tag{4.5}$$

We first estimate $\|\frac{u^r}{r}\|_\infty$. By (4.2), we have

$$\begin{aligned} \Big\|\frac{\omega(t)}{r}\Big\|_{L^{3,1}} &= \Big\|\frac{\omega(0)}{r}\Big\|_{L^{3,1}} \\ &\le \Big\|\frac{f}{r}\Big\|_{L^{3,1}} + \Big\|\frac{g}{r}\Big\|_{L^{3,1}}, \quad \forall\, t \ge 0. \end{aligned}$$

Recalling $u = u^r e_r + u^z e_z$, we get

$$\begin{aligned} \left\| \frac{u^r(t)}{r} \right\|_\infty &\lesssim \left\| \frac{\omega(t)}{r} \right\|_{L^{3,1}} \\ &\lesssim \left\| \frac{f}{r} \right\|_{L^{3,1}} + \left\| \frac{g}{r} \right\|_{L^{3,1}}, \quad \forall\, t \geq 0. \end{aligned}$$

Now we consider (4.5). Rewrite (4.5) as

$$\partial_t \omega^2 + (u \cdot \nabla)\omega^2 = \frac{u^r}{r}\omega^2.$$

We obtain for some absolute constant $C > 0$,

$$\begin{aligned} \max_{0\leq t\leq 1} \|\omega^2(t)\|_\infty &\leq \|g\|_\infty \exp\left(C \max_{0\leq t\leq 1} \left\| \frac{u^r(t)}{r} \right\|_\infty \right) \\ &\leq \|g\|_\infty \exp\left(C \left(\left\| \frac{f}{r} \right\|_{L^{3,1}} + \left\| \frac{g}{r} \right\|_{L^{3,1}} \right) \right) \\ &= O(\delta). \end{aligned} \tag{4.6}$$

Thus, we only need to control $\|\omega^1 - \omega^a\|_\infty$.

Set $\eta = \omega^a - \omega^1$. Denote by u^1 and u^2 the velocity fields corresponding to ω^1 and ω^2, respectively. We first show that

$$\max_{0\leq t\leq 1} \|\eta(t,\cdot)\|_2 = O(\delta).$$

Rewrite (4.4) as

$$\partial_t \omega^1 + (u^1 \cdot \nabla)\omega^1 = (\omega^1 \cdot \nabla)u^1 - (u^2 \cdot \nabla)\omega^1 + (\omega^1 \cdot \nabla)u^2. \tag{4.7}$$

By (4.1), we have

$$\partial_t \omega^a + (u^a \cdot \nabla)\omega^a = (\omega^a \cdot \nabla)u^a.$$

Therefore, the equation for η takes the form

$$\begin{aligned} \partial_t \eta + ((u^a - u^1)\cdot\nabla)\omega^a + (u^1\cdot\nabla) &= (\eta\cdot\nabla)u^a + (\omega^1\cdot\nabla)(u^a - u^1) \\ &\quad + (u^2\cdot\nabla)\omega^a - (u^2\cdot\nabla)\eta - (\omega^1\cdot\nabla)u^2. \end{aligned} \tag{4.8}$$

Computing the L^2 norm then gives

$$\begin{aligned}\partial_t\big(\|\eta(t)\|_2^2\big) &\lesssim \|u^a-u^1\|_6\cdot\|D\omega^a\|_3\cdot\|\eta\|_2+\|\eta\|_2^2\cdot\|Du^a\|_\infty+\|\omega^1\|_\infty\cdot\|D(u^a-u^1)\|_2\cdot\|\eta\|_2\\&\quad+\|u^2\|_6\cdot\|D\omega^a\|_3\cdot\|\eta\|_2+\|\omega^1\|_\infty\cdot\|Du^2\|_2\cdot\|\eta\|_2\\&\lesssim \big(\|D\omega^a\|_3+\|Du^a\|_\infty+\|\omega^1\|_\infty\big)\|\eta\|_2^2+\big(\|u^2\|_6\|D\omega^a\|_3+\|\omega^1\|_\infty\|Du^2\|_2\big)\|\eta\|_2,\\&\lesssim \big(O(1)+\|\omega^1\|_\infty\big)\|\eta\|_2^2+\big(O(1)\cdot\|Du^2\|_2+\|\omega^1\|_\infty\|Du^2\|_2\big)\|\eta\|_2.\end{aligned}\tag{4.9}$$

By an estimate similar to (4.6), we have

$$\max_{0\le t\le 1}\big\|Du^2(t)\big\|_2\lesssim\max_{0\le t\le 1}\big\|\omega^2(t)\big\|_2=O(\delta).\tag{4.10}$$

Similarly, by Sobolev embedding,

$$\max_{0\le t\le 1}\|u^2(t)\|_2\lesssim\max_{0\le t\le 1}\|\omega^2(t)\|_{\frac{6}{5}}=O(\delta).\tag{4.11}$$

This together with (4.6) gives

$$\begin{aligned}\max_{0\le t\le 1}\|u^2(t)\|_\infty &\lesssim \max_{0\le t\le 1}\|u^2(t)\|_2+\max_{0\le t\le 1}\|\omega^2(t)\|_\infty\\&=O(\delta).\end{aligned}\tag{4.12}$$

By (4.4), we have

$$\begin{aligned}\left\|\frac{\omega^1(t)}{r}\right\|_{L^{3,1}}&=\left\|\frac{f}{r}\right\|_{L^{3,1}}=O(1),\quad\forall\, t\ge 0,\\\max_{0\le t\le 1}\left\|\frac{(u^1(t))^r}{r}\right\|_\infty&\lesssim\left\|\frac{\omega^1(t)}{r}\right\|_{L^{3,1}}=O(1),\\\max_{0\le t\le 1}\left\|\frac{\omega^1(t)}{r}\right\|_\infty&\le\left\|\frac{f}{r}\right\|_\infty=O(1).\end{aligned}\tag{4.13}$$

Here we write $u^1=(u^1)^r e_r+(u^1)^z e_z$.

Rewrite (4.7) as

$$\partial_t\omega^1+(u\cdot\nabla)\omega^1=\frac{(u^1)^r}{r}\omega^1+(u^2)^r\frac{\omega^1}{r}.$$

Using (4.12) and (4.13), we get

$$\max_{0\le t\le 1} \|\omega^1(t)\|_\infty = O(1). \tag{4.14}$$

Plugging (4.10) and (4.14) into (4.9), we obtain

$$\max_{0\le t\le 1} \|\eta(t,\cdot)\|_2 = O(\delta). \tag{4.15}$$

By (4.14)–(4.15) and Hölder, we get

$$\begin{aligned}\max_{0\le t\le 1} \|\omega^a(t)-\omega^1(t)\|_4 &\lesssim \max_{0\le t\le 1}\left(\|\eta(t)\|_2^{\frac12}\|\eta(t)\|_\infty^{\frac12}\right)\\ &= O(\delta). \end{aligned}\tag{4.16}$$

By L^2 conservation of velocity for (4.2), we have

$$\begin{aligned}\|u(t,\cdot)\|_2 = \|u(0)\|_2 &\lesssim \|f\|_{\dot H^{-1}} + \|g\|_{\dot H^{-1}}\\ &\lesssim \|f\|_1+\|f\|_\infty+\|g\|_1+\|g\|_\infty\\ &\lesssim \|f\|_\infty+\|g\|_\infty = O(1), \qquad \forall\, t\ge 0.\end{aligned}$$

Therefore by (4.11),

$$\begin{aligned}\max_{0\le t\le 1}\left\|u^a(t)-u^1(t)\right\|_2 &\lesssim \max_{0\le t\le 1}\left(\|u^a(t)\|_2+\|u^2(t)\|_2+\|u(t)\|_2\right)\\ &= O(1).\end{aligned}\tag{4.17}$$

By (4.16)–(4.17) and interpolation, we get

$$\begin{aligned}\max_{0\le t\le 1}\left\|u^a(t)-u^1(t)\right\|_\infty &\lesssim \max_{0\le t\le 1}\left\|u^a(t)-u^1(t)\right\|_2^{\frac17}\cdot\left\|\omega^a(t)-\omega^1(t)\right\|_4^{\frac67}\\ &= O(\delta).\end{aligned}\tag{4.18}$$

Here we should stress that we have abused the notation and denote $O(\delta^\beta)$ ($\beta>0$) simply as $O(\delta)$ (see the beginning part of this proof).

Now using (4.12), (4.13), and (4.18), we can rewrite (4.8) as

$$\partial_t \eta + (u\cdot\nabla)\eta = -((u^a-u^1)\cdot\nabla)\omega^a + (\eta\cdot\nabla)u^a + (u^a-u^1)^r\frac{\omega^1}{r} + (u^2\cdot\nabla)\omega^a - (u^2)^r\frac{\omega^1}{r}$$
$$= O(\delta) + O(1)\eta + O(\delta)\cdot O(1).$$

Obviously then

$$\max_{0\le t\le 1}\|\eta(t)\|_\infty = O(\delta).$$

The lemma is proved. ■

We now state a proposition that gives the solvability of the 3D axisymmetric without swirl Euler equation for a special class of initial data (vorticity). In particular we allow initial vorticity (denote it by ω_0) to carry infinite $\|\frac{\omega_0}{r}\|_{L^{3,1}}$ norm that is not covered by standard theory. The trade off here is that we need a precise control of L^∞ norm in the sense of Lemma 4.1.

Proposition 4.2. Suppose $\{g_i\}_{i=1}^\infty$ is a sequence of axisymmetric functions on $\mathbb{R}^3$ satisfying the following conditions:

- For each $i\ge 1$, $g_i(x) = g_i^\theta(r,z)e_\theta$, where g_i^θ is scalar valued and vanishes near $r=0$:

$$\operatorname{supp}(g_i^\theta)\subset\big\{(r,z): r>r_i\big\},\quad \text{for some } r_i>0.$$

- $g_i\in C_c^\infty(B(0,100))$ and $\|g_i\|_\infty < 2^{-i}$.
- For each $i\ge 2$, denote $f_i=\sum_{j=1}^{i-1} g_j$, then

$$\|g_i\|_\infty \exp\left(C\left\|\frac{g_i}{r}\right\|_{L^{3,1}}\right) < \delta_i,$$

  where $\delta_i=\delta(2^{-i}, f_i)$ as defined in (4.3).

Let

$$g=\sum_{i=1}^\infty g_i$$

and consider the system

$$\begin{cases} \partial_t\omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u, \quad 0 < t \le 1; \\ u = -\Delta^{-1}\nabla\times\omega, \\ \omega\Big|_{t=0} = g. \end{cases} \tag{4.19}$$

Then there exists a unique solution ω to (4.19) with the following properties:

(1) ω is compactly supported:

$$\mathrm{supp}(\omega(t,\cdot)) \subset B(0,R_0), \quad \forall\, 0 < t \le 1.$$

Here $R_0 > 0$ is an absolute constant.

(2) $\omega \in C_t^0 C_x^0([0,1]\times\overline{B(0,R_0)})$, $u \in C_t^0 L_x^2 \cap L_t^\infty L_x^\infty([0,1]\times\mathbb{R}^3)$. In fact $u \in C_t^0 C_x^\alpha([0,1]\times\mathbb{R}^3)$ for any $0 < \alpha < 1$.

Proof of Proposition 4.2 For each $l \ge 1$, let ω^l be the solution to the system

$$\begin{cases} \partial_t\big(\frac{\omega^l}{r}\big) + (u^l\cdot\nabla)\big(\frac{\omega^l}{r}\big) = 0, \quad 0 < t \le 1, \\ u^l = -\Delta^{-1}\nabla\times\omega^l, \\ \omega^l\Big|_{t=0} = \sum_{i=1}^l g_i. \end{cases}$$

By Lemma 4.1 and the assumptions on g_i, we have

$$\max_{0\le t\le 1}\left\|\omega^{l+1}(t) - \omega^l(t)\right\|_\infty < 2^{-l}. \tag{4.20}$$

Noting that

$$\begin{aligned}\max_{0\le t\le 1}\|\omega^1(t)\|_\infty &\lesssim \|g_1\|_\infty \exp\left(Const\cdot\left\|\frac{g_1}{r}\right\|_{L^{3,1}}\right) \\ &\lesssim 1,\end{aligned}$$

we obtain

$$\sup_{l\ge 1}\max_{0\le t\le 1}\left\|\omega^l(t)\right\|_\infty \lesssim 1.$$

By energy conservation, we have

$$\|u^l(t)\|_2 = \|u^l(0)\|_2 \lesssim \|\omega^l(0)\|_1 + \|\omega^l(0)\|_\infty \lesssim 1, \qquad \forall\, t \geq 0,\, l \geq 1.$$

Therefore,

$$\sup_{l\geq 1} \max_{0\leq t\leq 1} \|u^l(t)\|_\infty \lesssim 1.$$

This shows that for some absolute constant $R_0 > 0$, we have

$$\omega^l(t) \in C_c^\infty(B(0,R_0)), \quad \forall\, 0 < t \leq 1,\, l \geq 1.$$

By (4.20), the sequence ω^l is Cauchy in the Banach space $C_t^0 C_x^0([0,1] \times \overline{B(0,R_0)})$ and hence converges to the limit point ω in the same space. By interpolation and Sobolev embedding it is not difficult to check that u^l converges to $u \in C_t^0 L_x^2$. By Sobolev embedding we get $u \in L_t^\infty L_x^\infty \cap C_t^0 C_x^\alpha$ for any $\alpha < 1$. It is not difficult to check that ω is the desired solution. The proposition is proved. ∎

We now take a parameter $A \gg 1$ and define

$$\begin{aligned}\tilde{g}_A(x_1,x_2,z) &= \tilde{g}_A(r,z) \\ &= \frac{\sqrt{\log A}}{\sqrt{A}} \sum_{A\leq k\leq A+\sqrt{A}} \eta_k(r,z),\end{aligned} \tag{4.21}$$

where η_k is the same as in (3.12). Note the slight difference between $\tilde{g}_A$ and g_A defined in (3.11). The main reason of choosing $\tilde{g}_A$ is that in the perturbation theory later we need better control of higher Sobolev norms of the solution, that is, estimates like $\|\tilde{g}_A\|_{W^{1,q}} \lesssim 2^{A+}$, for all $3 < q \leq \infty$. In comparison $\|g_A\|_{W^{1,q}} \sim 2^{2A}$ since there we are summing η_k over $k \leq 2A$. This is why the modification is needed.

By a derivation similar to (3.14)–(3.18), it is easy to check

$$\|\tilde{g}_A e_\theta\|_{\dot{B}^{\frac{3}{2}}_{2,1}(\mathbb{R}^3)} + \left\|\frac{\tilde{g}_A}{r} e_\theta\right\|_{L^{3,1}(\mathbb{R}^3)} \lesssim \sqrt{\log A},$$

$$\|\tilde{g}_A\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} + \|\tilde{g}_A e_\theta\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} \lesssim \frac{\sqrt{\log A}}{A^{\frac{1}{4}}},$$

$$\|\tilde{g}_A\|_{L^p(\mathbb{R}^3)} + \|\tilde{g}_A e_\theta\|_{L^p(\mathbb{R}^3)} \lesssim \frac{\sqrt{\log A}}{\sqrt{A}} \cdot 2^{-\frac{3A}{p}}, \quad \forall\, 1 \le p \le \infty,$$

$$\|D(\tilde{g}_A e_\theta)\|_{L^p(\mathbb{R}^3)} \lesssim \sqrt{\log A} \cdot 2^{(1-\frac{3}{p})(A+\sqrt{A})}, \quad \forall\, 3 \le p \le \infty. \tag{4.22}$$

These estimates will be needed later.

Lemma 4.3. Let ω be a smooth solution to the following system (written in axisymmetric vorticity form)

$$\begin{cases} \partial_t\left(\frac{\omega}{r}\right) + (u + u^{\mathrm{ex}}) \cdot \nabla\left(\frac{\omega}{r}\right) = 0, \quad 0 < t \le 1, r = \sqrt{x_1^2 + x_2^2}, x = (x_1, x_2, z), \\ u = -\Delta^{-1}\nabla \times \omega, \\ \omega\Big|_{t=0} = \tilde{g}_A e_\theta, \end{cases}$$

where $\tilde{g}_A$ was the same as in (4.21) and u^{ex} is a given axisymmetric velocity field having the form (note that it is incompressible)

$$u^{\mathrm{ex}}(t, r, z) = a(t) r e_r - 2a(t) z e_z. \tag{4.23}$$

Assume for some constant $B_0 > 0$,

$$\sup_{0 \le t \le 1} |a(t)| \le B_0 < \infty. \tag{4.24}$$

Let $\phi = (\phi^r, \phi^z)$ be the forward characteristic lines associated with the velocity $u + u^{\mathrm{ex}}$ (see (3.2)) and let $\tilde{\phi}$ be the corresponding inverse map. Then there exists $A_0 = A_0(B_0) > 0$ such that if $A > A_0$, then

$$\max_{0 \le t \le \frac{1}{\log\log A}} \|(D\tilde{\phi})(t, \cdot)\|_\infty > \log\log A. \tag{4.25}$$

Proof of Lemma 4.3 We shall give a slightly simpler proof than that given in Proposition 3.9. The idea is to take full advantage of the symmetry assumption and the fact that the off-diagonal terms of Du vanishes completely at $(r,z)=(0,0)$. Assume (4.25) does not hold. By a derivation similar to (3.88)–(3.89), we have

$$\max_{0\le t\le \frac{1}{\log\log A}} \left(\|(D\tilde{\phi})(t,\cdot)\|_\infty + \|(D\phi)(t,\cdot)\|_\infty\right) \lesssim (\log\log A)^2. \tag{4.26}$$

By (4.21) and (3.13), observe that $\tilde{g}_A$ is an odd function of z. Denote $v(t,r,z)=u+u^{ex}$ and

$$u^{\mathrm{ex}} = (u^{\mathrm{ex}})^r e_r + (u^{\mathrm{ex}})^z e_z,$$
$$v = v^r e_r + v^z e_z.$$

It is easy to check that for any $t\ge 0$, $\omega(t)$ remains an odd function of z, and also

$$v^r(t,0,z) = v^z(t,r,0) = 0,$$
$$\phi^r(t,0,z) = \phi^z(t,r,0) = 0, \quad \forall\, r\ge 0,\ z\in\mathbb{R},\ t\ge 0.$$

Clearly then

$$(\partial_r v^z)(t,0,0) \equiv 0 \equiv (\partial_z v^r)(t,0,0),$$
$$(\partial_r \phi^z)(t,0,0) \equiv 0 \equiv (\partial_z \phi^r)(t,0,0), \quad \forall\, t\ge 0.$$

Since v is divergence free (see (4.23)), we have

$$2(\partial_r v^r)(t,0,0) + (\partial_z v^z)(t,0,0) \equiv 0, \quad \forall\, t\ge 0.$$

From the above identities, we then easily obtain

$$(\partial_r \phi^r)(t,0,0) = e^{\int_0^t (\partial_r v^r)(s,0,0)\,\mathrm{d}s},$$
$$(\partial_z \phi^z)(t,0,0) = e^{\int_0^t (\partial_z v^z)(s,0,0)\,\mathrm{d}s} = e^{-2\int_0^t (\partial_r v^r)(s,0,0)\,\mathrm{d}s}. \tag{4.27}$$

By (3.27) (easy to check that same estimate holds with g_A replaced by $\tilde{g}_A$), (4.26) and (4.24), we get

$$(\partial_r v^r)(t,0,0) = (\partial_r u^r)(t,0,0) + (\partial_r (u^{ex})^r)(t,0,0)$$
$$\gtrsim \sqrt{\log A}(\log\log A)^{-16} - B_0.$$

Plugging this into (4.27) then gives us

$$(\partial_r \phi^r)(t,0,0) \gtrsim e^{t\frac{\sqrt{\log A}}{(\log\log A)^{16}} - 2B_0}.$$

This obviously contradicts (4.26) for $t = \frac{1}{\log\log A}$ and A sufficiently large. ■

Lemma 4.4 (Control of the support). Let ω be a smooth solution to the following system (written in axisymmetric vorticity form)

$$\begin{cases} \partial_t\left(\frac{\omega}{r}\right) + (u + u_1 + u_2)\cdot\nabla\left(\frac{\omega}{r}\right) = 0, \quad 0 < t \le 1, r = \sqrt{x_1^2 + x_2^2}, x = (x_1, x_2, z), \\ u = -\Delta^{-1}\nabla\times\omega, \\ \omega\Big|_{t=0} = \tilde{g}_A e_\theta, \end{cases}$$

where the following conditions hold:

- $\tilde{g}_A$ is the same as in (4.21);
- u_1 and u_2 are given smooth incompressible axisymmetric vector fields having the form

$$u_1 = a(t) r e_r - 2a(t) z e_z,$$
$$u_2 = u_2^r e_r + u_2^z e_z,$$

  and for some constant $B > 0$,

$$\sup_{0\le t\le 1} |a(t)| \le B,$$
$$\sup_{0\le t\le 1} \left\| \frac{u_2^r(t)}{r} \right\|_\infty \le B,$$
$$|u_2(t,x)| \le B|x|^2, \quad \forall x \in \mathbb{R}^2, \quad 0 \le t \le 1. \tag{4.28}$$

Then there exists a constant $A_0 = A_0(B) > 0$ sufficiently large such that if $A > A_0$, then for any $0 \le t \le 1$,

$$\mathrm{supp}(\omega(t,\cdot)) \subset B(0,R), \qquad \text{with } R \le C_1 \cdot 2^{-A}, \tag{4.29}$$

where $C_1 > 0$ is a constant depending on B.

Also for any $0 \le t \le \frac{1}{\log\log A}$, we have

$$\mathrm{supp}(\omega(t,\cdot)) \subset B(0,R), \qquad \text{with } R \sim 2^{-A}, \tag{4.30}$$

where the implied constants (in $R \sim 2^{-A}$) are absolute constants.

Proof of Lemma 4.4 Denote $v = u + u_1 + u_2$ and write

$$v = v^r e_r + v^z e_z,$$
$$u = u^r e_r + u^z e_z.$$

By $L^{3,1}$ conservation of $\frac{\omega}{r}$ and (4.22), we have

$$\sup_{0\le t\le 1} \left\| \frac{u^r(t)}{r} \right\|_\infty \lesssim \left\| \frac{\omega(t=0,\cdot)}{r} \right\|_{L^{3,1}} \lesssim \sqrt{\log A}.$$

By (4.28), we get

$$\sup_{0\le t\le 1} \left\| \frac{v^r(t)}{r} \right\|_\infty \lesssim B + \sqrt{\log A}. \tag{4.31}$$

Rewrite the equation for ω as

$$\partial_t \omega + (v\cdot\nabla)\omega = \frac{v^r}{r}\omega.$$

By (4.31), a simple L^p estimate (note that v is incompressible) then gives

$$\sup_{0\le t\le 1} \|\omega(t)\|_2 \lesssim e^{C(B+\sqrt{\log A})} \|g_A\|_2,$$
$$\sup_{0\le t\le 1} \|\omega(t)\|_4 \lesssim e^{C(B+\sqrt{\log A})} \|g_A\|_4, \tag{4.32}$$

where $C > 0$ is an absolute constant.

Note that by (4.22), we have

$$\|\tilde g_A\|_2 \lesssim \frac{\sqrt{\log A}}{\sqrt{A}} 2^{-\frac{3}{2}A},$$
$$\|\tilde g_A\|_4 \lesssim \frac{\sqrt{\log A}}{\sqrt{A}} 2^{-\frac{3}{4}A}.$$

By (4.32) and interpolation, we then get

$$\begin{aligned}\sup_{0\le t\le 1}\|u(t)\|_\infty &\lesssim \sup_{0\le t\le 1}\|\omega(t)\|_2^{\frac{1}{3}}\cdot\|\omega(t)\|_4^{\frac{2}{3}}\\ &\lesssim e^{C(B+\sqrt{\log A})}\cdot\frac{\sqrt{\log A}}{A}\cdot 2^{-A}\\ &< A^{-\frac{1}{3}}2^{-A},\end{aligned}\tag{4.33}$$

where in the last inequality we need to take A sufficiently large.

Denote ϕ the (usual Euclidean) characteristic line associated with the velocity v. Then by (4.28) and (4.33), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(|\phi(t)|)\lesssim A^{-\frac{1}{3}}2^{-A}+B|\phi(t)|+B|\phi(t)|^2.$$

Since $|\phi(0)|\lesssim 2^{-A}$, obviously (4.29) and (4.30) follows (in the latter case since $t\le\frac{1}{\log\log A}$ we can take A sufficiently large to kill pre-factors). ■

Lemma 4.5. Let ω be a smooth solution to the following system (written in axisymmetric vorticity form)

$$\begin{cases}\partial_t\left(\frac{W}{r}\right)+(U+u_1+u_2)\cdot\nabla\left(\frac{W}{r}\right)=0, \quad 0<t\le 1, r=\sqrt{x_1^2+x_2^2}, x=(x_1,x_2,z),\\ U=-\Delta^{-1}\nabla\times W,\\ W\Big|_{t=0}=\tilde{g}_A e_\theta,\end{cases}$$

where the following conditions hold:

- $\tilde{g}_A$ is the same as in (4.21);
- u_1 and u_2 are given smooth incompressible axisymmetric vector fields having the form

$$u_1=a(t)re_r-2a(t)ze_z,$$

$$u_2=u_2^r e_r+u_2^z e_z,$$

and for some constant $B > 0$,

$$\sup_{0\le t\le 1} |a(t)| \le B,$$

$$|u_2(t,x)| \le B\cdot |x|^2,$$

$$|(Du_2)(t,x)| \le B\cdot |x|,$$

$$|(D^2u_2)(t,x)| \le B, \quad \forall x\in\mathbb{R}^3,\, 0\le t\le 1. \tag{4.34}$$

Let $\Phi = (\Phi^r, \Phi^z)$ be the characteristic line associated with the velocity field $U + u_1 + u_2$ (see (3.2)) and let $\tilde{\Phi}$ be the corresponding inverse map.

There exists a constant $A_0 = A_0(B) > 0$ sufficiently large such that if $A > A_0$, then either

$$\max_{0\le t\le \frac{1}{\log\log A}} \|W(t,\cdot)\|_{\dot{H}^{\frac{3}{2}}} > \log\log\log A, \tag{4.35}$$

or

$$\max_{0\le t\le \frac{1}{\log\log A}} \|(D\tilde{\Phi})(t,\cdot)\|_\infty > \log\log\log A. \tag{4.36}$$

Proof of Lemma 4.5 By Lemma 4.4 and (4.34), we have $\mathrm{supp}(W(t,\cdot)) \subset \{x : |x| \lesssim 2^{-A}\}$ and

$$\|u_2(t,\cdot)\|_{L^\infty(\mathrm{supp}(W(t,\cdot)))} \lesssim 4^{-A}, \quad \forall 0\le t\le 1,$$

$$\|(Du_2)(t,\cdot)\|_{L^\infty(\mathrm{supp}(W(t,\cdot)))} \lesssim 2^{-A}, \quad \forall 0\le t\le 1,$$

$$\|(D^2u_2)(t,\cdot)\|_{L^\infty(\mathrm{supp}(W(t,\cdot)))} \lesssim 1, \quad \forall 0\le t\le 1. \tag{4.37}$$

Throughout this proof we suppress the dependence of the implied constants on B since A will be taken sufficiently large.

Assume that both (4.35) and (4.36) do not hold, that is,

$$\max_{0\le t\le \frac{1}{\log\log A}} \|W(t,\cdot)\|_{\dot{H}^{\frac{3}{2}}} + \max_{0\le t\le \frac{1}{\log\log A}} \|(D\tilde{\Phi})(t,\cdot)\|_\infty \lesssim \log\log\log A. \tag{4.38}$$

Easy to check that

$$\max_{0\le t\le \frac{1}{\log\log A}} \|(D\Phi)(t,\cdot)\|_\infty \lesssim (\log\log\log A)^2. \tag{4.39}$$

We shall derive a contradiction. The idea is to compare W with the other solution ω to the following "unperturbed" system

$$\begin{cases} \partial_t\left(\frac{\omega}{r}\right) + (u+u_1)\cdot\nabla\left(\frac{\omega}{r}\right) = 0, \quad 0 < t \le 1, r = \sqrt{x_1^2+x_2^2}, x = (x_1, x_2, z), \\ u = -\Delta^{-1}\nabla\times\omega, \\ \omega\Big|_{t=0} = \tilde{g}_A e_\theta. \end{cases}$$

By using the conservation of $\|\frac{W}{r}\|_{L^{3,1}}$ and $\|\frac{\omega}{r}\|_{L^{3,1}}$, it is not difficult to check that

$$\begin{aligned} \sup_{0\le t\le 1}\left\|\frac{U^r(t)}{r}\right\|_\infty &\lesssim \sqrt{\log A}, \\ \sup_{0\le t\le 1}\left\|\frac{u^r(t)}{r}\right\|_\infty &\lesssim \sqrt{\log A}, \\ \sup_{0\le t\le 1}\|W(t)\|_q &\lesssim \frac{\sqrt{\log A}}{\sqrt{A}} 2^{-\frac{3A}{q}}, \quad \forall\, 1 < q \le \infty, \\ \sup_{0\le t\le 1}\|\omega(t)\|_q &\lesssim \frac{\sqrt{\log A}}{\sqrt{A}} 2^{-\frac{3A}{q}}, \quad \forall\, 1 < q \le \infty, \end{aligned} \tag{4.40}$$

where in the last two inequalities we have used (4.37).

We carry out the perturbation argument in several steps.

`Step 1`. Set $\eta = \omega - W$. We first show that

$$\|\eta(t,\cdot)\|_{B^0_{\infty,1}} \lesssim 2^{-\frac{A}{2}+}, \quad \forall\, 0 \le t \le \frac{1}{\log\log A}. \tag{4.41}$$

Here and below we use the notation $X+$ as in (2.1).

Rewrite the equations for ω and W as

$$\partial_t\omega + (u+u_1)\cdot\nabla\omega = (\omega\cdot\nabla)(u+u_1),$$

$$\partial_t W + (U+u_1+u_2)\cdot\nabla W = (W\cdot\nabla)(U+u_1+u_2).$$

Taking the difference, we have

$$\partial_t\eta + (u+u_1)\cdot\nabla\eta + (u-U-u_2)\cdot\nabla W = (\eta\cdot\nabla)(u+u_1) + (W\cdot\nabla)(u-U-u_2).$$

Let $1 < p < 3$. By (4.38) and (4.40), we have

$$\partial_t(\|\eta\|_p) \lesssim \|u-U\|_{(\frac{1}{p}-\frac{1}{3})^{-1}}\|DW\|_3 + \|DW\|_p \cdot 4^{-A} \cdot B + \|\eta\|_p \cdot \left(\left\|\frac{u^r}{r}\right\|_\infty + B\right)$$
$$+ \|D(u-U)\|_p \cdot \|W\|_\infty + \|W\|_p \cdot 2^{-A}$$
$$\lesssim \|\eta\|_p \cdot \log\log\log A + \log\log\log A \cdot 4^{-A} \cdot B + \|\eta\|_p \cdot (\sqrt{\log A} + B)$$
$$+ \|\eta\|_p \cdot \frac{\sqrt{\log A}}{\sqrt{A}} + 4^{-A}.$$

Set $\eta(0) = 0$, integrating in $t \le \frac{1}{\log\log A}$ then gives

$$\max_{0\le t\le \frac{1}{\log\log A}} \|\eta(t,\cdot)\|_p \lesssim 4^{-A+}, \quad \forall\, 1 < p < 3. \tag{4.42}$$

This estimate is particularly good for $p = 3-$.

Now for any $1 < q < \infty$, a standard energy estimate using (4.22), (4.37), and (4.40) (using $\|W\|_\infty$ and $\|\omega\|_\infty$) gives for any $0 \le t \le 1$,

$$\|W(t,\cdot)\|_{W^{1,q}} \lesssim 2^{A-},$$
$$\|\omega(t,\cdot)\|_{W^{1,q}} \lesssim 2^{A-},$$

and obviously

$$\max_{0\le t\le 1} \|\eta(t,\cdot)\|_{W^{1,q}} \lesssim 2^{A-}, \qquad \forall\, 1 < q < \infty.$$

Interpolating the above (set $q = \infty-$) with (4.42) (set $p = 3-$) then gives (4.41).

Step 2. Let $\phi = (\phi^r, \phi^z)$ be the characteristic line associated with the velocity field $u + u_1$. We show that

$$\max_{0\le t\le \frac{1}{\log\log A}} \|\phi(t,\cdot) - \Phi(t,\cdot)\|_\infty \lesssim 2^{-\frac{7}{6}A+}. \tag{4.43}$$

Set $Y(t) = \phi(t) - \Phi(t)$. By Lemma 4.4, we only need to consider the region $|x| \lesssim 2^{-A}$. By (4.34), we have the estimate

$$|u_2(t,\Phi(t,r,z))| \lesssim 4^{-A}, \qquad \forall\, 0 \le t \le \frac{1}{\log\log A}, \forall \sqrt{r^2+z^2} \lesssim 2^{-A}. \tag{4.44}$$

Let $Y(t) = \phi(t) - \Phi(t) = (Y^r(t), Y^z(t))$. In order to not confuse the notation, we denote

$$v = (u^r, u^z), \quad v_1 = \left(u_1^r, u_1^z\right),$$
$$V = (U^r, U^z), \quad v_2 = \left(u_2^r, u_2^z\right).$$

Then the equation for Y takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t} Y = (v + v_1)(\phi) - (v + v_1)(\Phi) + (v - V)(\Phi) - v_2(\Phi). \tag{4.45}$$

By (4.40) and a simple energy estimate, we have

$$\begin{aligned} \|Du\|_\infty &\lesssim \|\omega\|_2 + \|\omega\|_\infty \log(10 + \|\omega\|_{H^2}) \\ &\lesssim 1 + \frac{\sqrt{\log A}}{\sqrt{A}} \cdot A \lesssim \sqrt{A} \cdot \sqrt{\log A}. \end{aligned}$$

Since $u(t, 0, 0, z) = u^z e_z$, we have

$$\begin{aligned} \left| \frac{1}{r} u^r(t, r, z) \right| &= \left| \frac{\big(u(t, x_1, x_2, z) - u(t, 0, 0, z)\big) \cdot e_r}{r} \right| \\ &\lesssim \|Du\|_\infty \lesssim \sqrt{A} \cdot \sqrt{\log A}. \end{aligned} \tag{4.46}$$

By the incompressibility condition $\partial_r u^r = -\frac{1}{r} u^r - \partial_z u^z$, we obtain

$$\|\partial_r u^r\|_\infty \lesssim \|Du\|_\infty \lesssim \sqrt{A} \cdot \sqrt{\log A}.$$

Similarly we have the estimate for $\|\partial_z u^r\|_\infty$, $\|\partial_r u^z\|_\infty$, $\|\partial_z u^z\|_\infty$, and hence

$$\|Dv\|_\infty \lesssim \sqrt{A} \cdot \sqrt{\log A}. \tag{4.47}$$

By (4.40), (4.41), and interpolation, we have

$$\begin{aligned} \max_{0 \le t \le \frac{1}{\log\log A}} \|u(t) - U(t)\|_\infty &\lesssim \max_{0 \le t \le \frac{1}{\log\log A}} \big(\|\omega(t)\|_2 + \|W(t)\|_2\big)^{\frac{2}{3}} \|\omega(t) - W(t)\|_\infty^{\frac{1}{3}} \\ &\lesssim 2^{-A+} \cdot 2^{-\frac{A}{6}+} \\ &\lesssim 2^{-\frac{7}{6}A+}. \end{aligned}$$

Here we denote by $2^{\text{const}\,A+}$ the bound $2^{(\text{const}+\epsilon)A}$ for some sufficiently small constant $\epsilon > 0$. Similar conventions will be used below.

Therefore,

$$\max_{0\le t\le \frac{1}{\log\log A}} \|v(t) - V(t)\|_\infty \lesssim 2^{-\frac{7}{6}A+}. \tag{4.48}$$

Plugging the estimates (4.47)–(4.48) into (4.45) and using (4.44), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(|Y(t)|) \lesssim \sqrt{A}\cdot\sqrt{\log A}\cdot |Y(t)| + 2^{-\frac{7}{6}A+} + 4^{-A}.$$

Integrating in time, we get

$$\begin{aligned}\max_{0\le t\le \frac{1}{\log\log A}} |Y(t)| &\lesssim \int_0^{\frac{1}{\log\log A}} e^{\frac{1}{\log\log A}\sqrt{A}\cdot\sqrt{\log A}}\left(2^{-\frac{7}{6}A+} + 4^{-A}\right)\mathrm{d}s\\ &\lesssim 2^{-\frac{7}{6}A+}.\end{aligned}$$

Therefore, (4.43) is proved.

Step 3. We show that

$$\begin{aligned}&\|\partial_{rr}u^r(t)\|_\infty + \|\partial_{rz}u^r(t)\|_\infty + \|\partial_{zz}u^r(t)\|_\infty + \|\partial_{rr}u^z(t)\|_\infty + \|\partial_{rz}u^z(t)\|_\infty\\ &\quad + \|\partial_{zz}u^z(t)\|_\infty \lesssim 2^{A+}, \quad \forall\, 0\le t\le 1.\end{aligned} \tag{4.49}$$

and

$$\begin{aligned}&\|\partial_{rr}U^r(t)\|_\infty + \|\partial_{rz}U^r(t)\|_\infty + \|\partial_{zz}U^r(t)\|_\infty + \|\partial_{rr}U^z(t)\|_\infty + \|\partial_{rz}U^z(t)\|_\infty\\ &\quad + \|\partial_{zz}U^z(t)\|_\infty \lesssim 2^{A+}, \quad \forall\, 0\le t\le 1.\end{aligned} \tag{4.50}$$

We shall only prove (4.49) since the proof for (4.50) is essentially the same.

By a simple energy estimate, we have

$$\max_{0\le t\le 1}\left(\|D^2u(t)\|_\infty + \|D\omega(t)\|_\infty\right) \lesssim 2^{A+}. \tag{4.51}$$

Write

$$u = (u^1, u^2, u^z).$$

Obviously

$$u^1(t,x_1,x_2,z)=\frac{1}{r}u^r x_1, \qquad u^2=\frac{1}{r}u^r x_2.$$

Since u is axisymmetric, easy to check that

$$\Big(u^1(t,x_1,x_2,z),\, u^2(t,x_1,x_2,z)\Big)=\alpha(t,z)(x_1,\, x_2)+O(r^2),$$

where $\alpha(t,z)$ is a constant depending only on (t,z). From this and (4.51), it is not difficult to show that

$$\begin{aligned}\big|(\partial_2 u^1)(t,0,x_2,z)\big| &\lesssim \|D^2u\|_\infty\cdot|x_2|\\ &\lesssim 2^{A+}|x_2|, \quad \forall\, 0\le t\le 1,\ x_2\in\mathbb{R}.\end{aligned}$$

By the fundamental theorem of calculus, we then have for any (x_1,x_2,z) $(r=\sqrt{x_1^2+x_2^2})$,

$$\begin{aligned}\left|\frac{(\partial_2 u^1)(t,x_1,x_2,z)}{r}\right| &\lesssim \left|\frac{(\partial_2 u^1)(t,x_1,x_2,z)-(\partial_2 u^1)(t,0,x_2,z)}{r}\right|+\left|\frac{(\partial_2 u^1)(t,0,x_2,z)}{r}\right|\\ &\lesssim 2^{A+}.\end{aligned} \tag{4.52}$$

Denote $g=\frac{1}{r}u^r$. Since

$$u^1(t,x_1,x_2,z)=g(t,r,z)x_1, \quad r=\sqrt{x_1^2+x_2^2},$$

differentiating in x_2 then gives us

$$\partial_2 u^1=(\partial_r g)\cdot\frac{x_2x_1}{r}.$$

Therefore choosing $|x_1|\sim|x_2|\sim r$ (observe that $\partial_r g$ is still a function of (r,z)) and using (4.52), we obtain

$$\left\|\partial_r\left(\frac{1}{r}u^r(t)\right)\right\|_\infty=\|\partial_r g\|_\infty\lesssim 2^{A+}, \qquad \forall\, 0\le t\le 1. \tag{4.53}$$

By (3.36), we have

$$\partial_{rr}u^r=-\partial_r\left(\frac{1}{r}u^r\right)-\partial_{zz}u^r-\partial_z\omega^\theta.$$

By (4.51),

$$\|\partial_{zz}u^r\|_\infty = \big\|(\partial_{zz}u)\cdot e_r\big\|_\infty \lesssim \|D^2u\|_\infty \lesssim 2^{A+},$$
$$\|\partial_z\omega^\theta\|_\infty = \big\|(\partial_z\omega)\cdot e_\theta\big\|_\infty \lesssim 2^{A+}. \tag{4.54}$$

Therefore by (4.53) and (4.54), we get

$$\max_{0\le t\le 1}\big\|\partial_{rr}u^r(t)\big\|_\infty \lesssim 2^{A+}.$$

Similar to (4.46), we have

$$\begin{aligned}\left|\frac{1}{r}(\partial_z u^r)(t,r,z)\right| &= \left|\frac{\big((\partial_z u)(t,x_1,x_2,z)-(\partial_z u)(t,0,0,z)\big)\cdot e_r}{r}\right| \\ &\lesssim \|D^2u\|_\infty \lesssim 2^{A+}.\end{aligned}$$

Since $\omega^\theta = \partial_r u^z - \partial_z u^r$, we get

$$\begin{aligned}\left\|\frac{1}{r}\partial_r u^z\right\|_\infty &\lesssim \left\|\frac{\omega^\theta}{r}\right\|_\infty + \left\|\frac{1}{r}\partial_z u^r\right\|_\infty \\ &\lesssim 2^{A+}.\end{aligned}$$

We then get

$$\begin{aligned}\|\partial_{rr}u^z\|_\infty &\lesssim \|\Delta u^z\|_\infty + \left\|\frac{1}{r}\partial_r u^z\right\|_\infty + \|\partial_{zz}u^z\|_\infty \\ &\lesssim 2^{A+}, \qquad \forall\, 0\le t\le 1.\end{aligned}$$

We have proved that $\|\partial_{rr}u^r\|_\infty$ and $\|\partial_{rr}u^z\|_\infty$ are both under control. The rest of the terms in (4.49) are similarly estimated. We omit further details.

Step 4. Set $e(t) = (D\phi)(t)$, $E(t) = (D\Phi)(t)$, then obviously

$$\partial_t E = (DV)(\Phi)E + (Dv_1)(\Phi)E + (Dv_2)(\Phi)E,$$
$$\partial_t e = (Dv)(\phi)e + (Dv_1)(\phi)e.$$

Observe that

$$Dv_1 = a(t)\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Set $q = E - e$. By Lemma 4.4, we only need to control q in the region $|x| \lesssim 2^{-A}$. In this region we have

$$\left\|Dv_2(\Phi)\right\|_\infty \lesssim 2^{-A}.$$

The equation for q takes the form

$$\begin{aligned}\partial_t q = ((DV)(\Phi) - (DV)(\phi))E + ((DV)(\phi) - (Dv)(\phi))E + (Dv)(\phi)q \\ + a(t)\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} q + (Dv_2)(\Phi)E.\end{aligned}$$

By (4.48), (4.49), (4.50), and interpolation, we have

$$\max_{0 \le t \le \frac{1}{\log\log A}} \|D(v - V)\|_\infty \lesssim 2^{-\frac{1}{12}A+}. \tag{4.55}$$

By (4.50),(4.43), (4.39), (4.47), and (4.55), we have

$$\begin{aligned}\partial_t(|q|) &\lesssim \|D^2 V\|_\infty \cdot |\phi - \Phi| \cdot \|E\|_\infty + \|D(V - v)\|_\infty \cdot \|E\|_\infty \\ &\quad + \|Dv\|_\infty \cdot |q| + B \cdot |q| + \|Dv_2(\Phi)\|_\infty \cdot \|E\|_\infty \\ &\lesssim 2^{A+} \cdot 2^{-\frac{7}{6}A+} \cdot (\log\log\log A)^2 + 2^{-\frac{1}{12}A+} \cdot (\log\log\log A)^2 \\ &\quad + (\sqrt{A} \cdot \sqrt{\log A} + B)|q| + 2^{-A} \cdot (\log\log\log A)^2.\end{aligned}$$

Integrating in time $t \le \frac{1}{\log\log A}$, we then obtain

$$\max_{0 \le t \le \frac{1}{\log\log A}} \|q(t)\|_\infty \lesssim 1.$$

But this obviously contradicts (4.25). ∎

The next proposition is the key to our construction in the 3D compactly supported data case. It is written in the same style as in the 2D case in [2]. The overall statement of the proposition is a bit long and over-stretched due to some additional technical conditions pertaining to the 3D situation. Nevertheless, the structure of the proposition is the same as that in the 2D case. In short summary the main body of the proposition should read as "Let ω_{-1} satisfy...Then for any $0 < \epsilon < \epsilon_0$, one can find ω_0 with the properties... and δ_0 such that for any ω_j with the properties..., the following hold true:...".

Proposition 4.6. Let $\omega_{-1} \in C_c^\infty(B(0,100))$ be a given axisymmetric function such that $\omega_{-1} = \omega_{-1}^\theta e_\theta$, $\omega_{-1}^\theta = \omega_{-1}^\theta(r,z)$ is scalar valued and for some $r_{-1} > 0$, $0 < R_0 < \frac{1}{100}$,

$$\operatorname{supp}(\omega_{-1}^\theta) \subset \left\{(r,z):\ r > r_{-1},\ z \le -4R_0\right\}.$$

Denote $u_{-1} = -\Delta^{-1}\nabla \times \omega_{-1}$ and

$$u_{-1}^* = \|u_{-1}\|_2.$$

Then for any $0 < \epsilon \le \epsilon_0$ with $\epsilon_0 = \epsilon_0(\omega_{-1}) \ll R_0$ sufficiently small, we can find a smooth axisymmetric function $\omega_0 = \omega_0^\theta e_\theta$ (depending only on (ϵ, ω_{-1})) with the properties:

- $\omega_0 \in C_c^\infty(B(0,100))$ and for some $r_0 > 0$,

$$\operatorname{supp}(\omega_0^\theta(r,z)) \subset \left\{(r,z):\ r_0 < r < \epsilon,\ -\epsilon < z < \epsilon\right\}. \tag{4.56}$$

- $\|\omega_0\|_\infty \exp(C\|\frac{\omega_0}{r}\|_{L^{3,1}}) < \delta(\epsilon^2, \omega_{-1})$ (see (4.3));
- denote $u_0 = -\Delta^{-1}\nabla \times \omega_0$, then

$$\|u_0\|_2 < \epsilon u_{-1}^* < \frac{1}{4}u_{-1}^*, \tag{4.57}$$

and $\delta_0 = \delta_0(\omega_{-1}, \omega_0) \ll \epsilon$ sufficiently small such that for any smooth axisymmetric functions $\omega_j = \omega_j^\theta e_\theta$, $1 \le j \le N$ (here $N \ge 1$ is arbitrary but finite) with the properties:

- $\omega_j \in C_c^\infty(B(0,100))$ and $\operatorname{supp}(\omega_j^\theta) \subset \{(r,z):\ r > r_j, z > 2R_0\}$ for some $r_j > 0$.

- for each $j \geq 1$, denote $f_j = \omega_{-1}^\theta + \omega_0^\theta + \sum_{i=1}^{j-1} \omega_i^\theta$, then

$$\|\omega_j^\theta\|_\infty \cdot \exp\left(C \left\| \frac{\omega_j^\theta}{r} \right\|_{L^{3,1}}\right) < \delta_j,$$

where $\delta_j = \delta(2^{-3j}\delta_0, f_j)$ as defined in (4.3);

- denote $u_j = -\Delta^{-1}\nabla \times \omega_j$, then

$$\|u_j\|_2 < \frac{\epsilon}{2^{j+1}} u_{-1}^*;$$

the following hold true:

Let ω be the smooth solution to the axisymmetric system

$$\begin{cases} \partial_t \left(\frac{\omega}{r}\right) + (u \cdot \nabla)\left(\frac{\omega}{r}\right) = 0, \quad 0 < t \leq 1, \\ u = -\Delta^{-1}\nabla \times \omega, \\ \omega\Big|_{t=0} = \left(\omega_{-1}^\theta + \omega_0^\theta + \sum_{j=1}^N \omega_j^\theta\right) e_\theta, \end{cases}$$

then

(1) for any $0 \leq t \leq \epsilon$, we have the decomposition

$$\omega(t) = \omega_A(t) + \omega_B(t) + \omega_C(t), \tag{4.58}$$

where

$$\text{supp}(\omega_A(t)) \subset \left\{(r,z) : z \leq -4R_0 + \sqrt{\epsilon}\right\};$$

$$\text{supp}(\omega_B(t)) \subset \left\{(r,z) : |z| \leq \sqrt{\epsilon}\right\};$$

$$\text{supp}(\omega_C(t)) \subset \left\{(r,z) : z \geq 2R_0 - \sqrt{\epsilon}\right\};$$

and $\omega_A(t=0) = \omega_{-1}$, $\omega_B(t=0) = \omega_0^\theta e_\theta$, $\omega_C(t=0) = (\sum_{j=1}^N \omega_j^\theta) e_\theta$.

(2) the L^∞ norm of ω_B and ω_C is uniformly small on the interval $[0,1]$:

$$\max_{0 \leq t \leq 1} \left(\|\omega_B(t)\|_{L^\infty} + \|\omega_C(t)\|_{L^\infty}\right) \leq \epsilon. \tag{4.59}$$

(3) the $\dot{H}^{\frac{3}{2}}$ norm of ω_B is inflated rapidly on the time interval $[0,\epsilon]$: there exists $0 < t_0^1 = t_0^1(\epsilon, \omega_{-1}, \omega_0) < \epsilon$, $0 < t_0^2 = t_0^2(\epsilon, \omega_{-1}, \omega_0) < \epsilon$, such that

$$\begin{aligned} \left\|\omega_B(t=0)\right\|_{\dot{H}^{\frac{3}{2}}} &< \epsilon, \\ \left\|\omega_B(t)\right\|_{\dot{H}^{\frac{3}{2}}} &> \frac{1}{\epsilon}, \quad \text{for any } \; t_0^1 \le t \le t_0^2. \end{aligned} \tag{4.60}$$

(4) all H^k, $k \ge 2$ norms of ω_B can be bounded purely in terms of initial data ω_0 on the time interval $[0,\epsilon]$: for any $k \ge 2$,

$$\max_{0 \le t \le \epsilon} \|\omega_B(t)\|_{H^k} \le C\big(k, R_0, u_{-1}^*\big)\|\omega_0\|_{H^k}. \tag{4.61}$$

Note here the bound of $\|\omega_B\|_{H^k}$ is "almost local" in the sense that it depends only on u_{-1}^* but not on other higher Sobolev norms of ω_A or ω_C. Similarly we have

$$\max_{0 \le t \le \epsilon} \|\omega_A(t)\|_{H^k} \le C\big(k, R_0, u_{-1}^*\big)\|\omega_{-1}\|_{H^k}, \quad \forall\, k \ge 2. \tag{4.62}$$

Proof of Proposition 4.6. The nontrivial point is to find ω_0 such that (4.60) is achieved. We first show that a generic ω_0 (i.e., satisfying the properties specified in (4.56)–(4.57)) is enough to make (4.58), (4.59), and (4.61) hold.

By conservation of $\|u(t)\|_2$, we have

$$\begin{aligned} \|u(t)\|_2 = \|u(0)\|_2 &\le u_{-1}^* + \sum_{j=1}^{\infty} u_{-1}^* \cdot 2^{-j} \\ &\le 2u_{-1}^*, \quad \forall\, t \ge 0. \end{aligned} \tag{4.63}$$

Let ω_L be the smooth solution to the axisymmetric system

$$\begin{cases} \partial_t\big(\frac{\omega_L}{r}\big) + (u_L \cdot \nabla)\big(\frac{\omega_L}{r}\big) = 0, \quad 0 < t \le 1, \\ u_L = -\Delta^{-1}\nabla \times \omega_L, \\ \omega_L(t=0) = \omega_{-1}. \end{cases}$$

Obviously

$$\max_{0\le t\le 1} \|\omega_L(t)\|_\infty \lesssim_{\omega_{-1}} 1.$$

By Lemma 4.1, we have

$$\max_{0\le t\le 1} \|\omega(t)-\omega_L(t)\|_\infty \ll \epsilon$$

and clearly

$$\max_{0\le t\le 1} \|\omega(t)\|_\infty \lesssim_{\omega_{-1}} 1.$$

Interpolating the above with (4.63) then gives

$$\max_{0\le t\le 1} \|u(t)\|_\infty \le c_1,$$

where $c_1 > 0$ is a constant depending only on ω_{-1}.

This shows that the support of $\omega(t)$ moves at a speed at most c_1. Since we can always choose ϵ sufficiently small such that $c_1\epsilon \ll \sqrt{\epsilon}$, the decomposition (4.58) then obviously follows.

The inequality (4.59) is a simple consequence of Lemma 4.1. To show (4.61), we note that for $0 \le t \le \epsilon$, $\omega_B = \omega_B(t)$ solves the equation

$$\partial_t\omega_B + \big((u_B+u_{ex})\cdot\nabla\big)\omega_B = (\omega_B\cdot\nabla)(u_B+u_{ex}),$$

where

$$u_B(t) = -\Delta^{-1}\nabla\times\omega_B(t),$$

$$u_{ex}(t) = -\Delta^{-1}\nabla\times(\omega_A(t)+\omega_C(t)).$$

Since for $0 \le t \le \epsilon$ and ϵ sufficiently small,

$$d(\mathrm{supp}(\omega_A(t)),\mathrm{supp}(\omega_B(t))) \ge R_0,$$

$$d(\mathrm{supp}(\omega_C(t)),\mathrm{supp}(\omega_B(t))) \ge R_0,$$

we can then write for $x \in \mathrm{supp}(\omega_B(t))$,

$$u_{ex}(t,x) = \int_{R^3} K(x-y)(\omega_A(t,y) + \omega_C(t,y))\,\mathrm{d}y, \tag{4.64}$$

where the modified kernel $K(\cdot)$ satisfies

$$\left|(\partial^\alpha K)(x)\right| \lesssim_{R_0,\alpha} (1+|x|^2)^{-\frac{2+|\alpha|}{2}}, \quad \forall x \in \mathbb{R}^3,\ |\alpha| \geq 0.$$

Since $\omega(t) = \nabla \times u(t)$, we can rewrite (4.64) as

$$\begin{aligned} u_{ex}(t,x) &= \int_{\mathbb{R}^3} K(x-y)\omega(t,y)\,\mathrm{d}y - \int_{\mathbb{R}^3} K(x-y)\omega_B(t,y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^3} K(x-y)\nabla \times u(t,y)\,\mathrm{d}y - \int_{\mathbb{R}^3} K(x-y)\omega_B(t,y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^3} \tilde{K}(x-y)u(t,y)\,\mathrm{d}y - \int_{\mathbb{R}^3} K(x-y)\omega_B(t,y)\,\mathrm{d}y \\ &=: u_{ex}^{(1)}(t,x) + u_{ex}^{(2)}(t,x). \end{aligned}$$

Obviously we only need to bound $u_{ex}^{(1)}$. Since $|(\partial^\alpha \tilde{K})(x)| \lesssim_{R_0,\alpha} (1+|x|^2)^{-\frac{3+|\alpha|}{2}}$, we have

$$\left\|u_{ex}^{(1)}(t,\cdot)\right\|_{H^k} \lesssim_{k,R_0} \|u(t)\|_2 \lesssim_{k,R_0} u_{-1}^*, \quad \text{for any } k \geq 0.$$

The inequality (4.61) then easily follows from this and a simple energy estimate. Similarly one can prove (4.62).

It remains for us to show the existence of ω_0 such that (4.60) hold. First we show that it suffices to consider the following reduced system

$$\begin{cases} \partial_t\left(\frac{W}{r}\right) + (U\cdot\nabla)\left(\frac{W}{r}\right) = 0, \quad 0 < t \leq 1, \\ U = -\Delta^{-1}\nabla \times W, \\ W\Big|_{t=0} = \omega_{-1} + \omega_0. \end{cases}$$

By Lemma 4.1 and our assumptions on ω_j, we have

$$\max_{0\leq t\leq 1} \|\omega(t) - W(t)\|_\infty \lesssim \delta_0 \ll \epsilon. \tag{4.65}$$

Since $\max_{0\le t\le 1}\|u(t)\|_\infty \lesssim_{\omega_{-1}} 1$ and $\max_{0\le t\le 1}\|U(t)\|_\infty \lesssim_{\omega_{-1}} 1$, we have for some $c_2 = c_2(\omega_{-1}) > 0$,

$$\mathrm{supp}(W(t)) \subset B(0, 100 + c_2), \quad \forall\, 0 \le t \le 1,$$
$$\mathrm{supp}(\omega(t)) \subset B(0, 100 + c_2), \quad \forall\, 0 \le t \le 1.$$

Therefore, by (4.65) and Hölder, we get

$$\max_{0\le t\le 1}\|\omega(t) - W(t)\|_2 \lesssim_{\omega_{-1}} \delta_0^{\frac{1}{2}}. \tag{4.66}$$

For $0 \le t \le \epsilon$, we write the decomposition of W as

$$W(t) = W_L(t) + W_R(t),$$

where

$$\mathrm{supp}(W_L(t)) \subset \left\{(r,z):\ z \le -4R_0 + \sqrt{\epsilon}\right\};$$
$$\mathrm{supp}(W_R(t)) \subset \left\{(r,z):\ |z| \le \sqrt{\epsilon}\right\};$$

and $W_L(t=0) = \omega_{-1}$, $W_R(t=0) = \omega_0$.

By (4.61) and a similar bound for $W_R(t)$, we have

$$\max_{0\le t\le\epsilon}\left\|\omega_B(t) - W_R(t)\right\|_{H^3} \lesssim_{\omega_{-1},R_0,u^*_{-1}} \|\omega_0\|_{H^3}.$$

Interpolating the above with (4.66) and choosing δ_0 sufficiently small, we then get

$$\max_{0\le t\le\epsilon}\left\|\omega_B(t) - W_R(t)\right\|_{H^2} \le \epsilon.$$

This shows that it suffices for us to inflate the $\|W_R(t)\|_{\dot H^{\frac{3}{2}}}$ norm.

To this end, let W^1 be the smooth solution to

$$\begin{cases} \partial_t\left(\frac{W^1}{r}\right) + (U^1\cdot\nabla)\left(\frac{W^1}{r}\right) = 0, \quad 0 < t \le 1, \\ U^1 = -\Delta^{-1}\nabla\times W^1, \\ W^1\Big|_{t=0} = \omega_{-1} + \tilde g_A e_\theta, \end{cases}$$

where $\tilde{g}_A$ is the same as defined in (4.21), and we shall take A to be sufficiently large without too much explicit mentioning. Eventually we shall take ω_0 to be a suitable perturbation of $\tilde{g}_A$ and let W be the corresponding solution.

For $0 \le t \le \frac{1}{\log\log A}$, we can decompose the solution W^1 as

$$W^1(t) = W^1_L(t) + W^1_R(t),$$

where

$$\begin{aligned}\operatorname{supp}\left(W^1_L(t)\right) &\subset \left\{(r,z):\ z \le -4R_0 + \frac{1}{\sqrt{\log\log A}}\right\},\\ \operatorname{supp}\left(W^1_R(t)\right) &\subset \left\{(r,z):\ |z| \le \frac{1}{\sqrt{\log\log A}}\right\}.\end{aligned}$$

The equation for W^1_R takes the form

$$\begin{cases}\partial_t\left(\frac{W^1_R}{r}\right) + \left((U^1_R + U^1_L)\cdot\nabla\right)\left(\frac{W^1_R}{r}\right) = 0,\\ U^1_R = -\Delta^{-1}\nabla\times W^1_R,\\ U^1_L = -\Delta^{-1}\nabla\times W^1_L,\\ W^1_R(t=0) = \tilde{g}_A e_\theta.\end{cases}$$

Write

$$U^1_L = U^r_L e_r + U^z_L e_z.$$

Let $\xi(t)$ solves the ODE

$$\begin{cases}\frac{d}{dt}\xi(t) = U^z_L(0,0,\xi(t)),\\ \xi(0) = 0.\end{cases}$$

We can expand $U^1_L(t)$ near the point $(0,0,\xi(t))$ to get

$$U^1_L(t,x_1,x_2,z+\xi(t)) = U^z_L(t,0,0,\xi(t))e_z + \underbrace{a(t)re_r - 2a(t)ze_z}_{=:u_1(t,x_1,x_2,z)} + u_2(t,x_1,x_2,z),$$

where for any $0 \le t \le \frac{1}{\log\log A}$,

$$\begin{aligned} \left|a(t)\right| &\lesssim_{\omega_{-1},R_0} 1, \\ \left|u_2(t,x)\right| &\lesssim_{\omega_{-1},R_0} |x|^2, \\ \left|(Du_2)(t,x)\right| &\lesssim_{\omega_{-1},R_0} |x|, \\ \left|(D^2u_2)(t,x)\right| &\lesssim_{\omega_{-1},R_0} 1, \qquad \forall\, x \in \mathbb{R}^3. \end{aligned}$$

Note that u_2 is axisymmetric without swirl, that is, $u_2 = u_2^r e_r + u_2^z e_z$.

Introduce $\Omega(t) = \Omega(t, x_1, x_2, z)$ such that

$$\begin{aligned} \Omega(t, x_1, x_2, z) &:= W_R^1\big(t, x_1, x_2, z + \xi(t)\big), \\ U_\Omega(t) &:= -\Delta^{-1}\nabla \times \Omega(t). \end{aligned}$$

It is then not difficult to check that the equation for Ω takes the form

$$\partial_t \left(\frac{\Omega}{r}\right) + \big((U_\Omega + u_1 + u_2)\cdot\nabla\big)\left(\frac{\Omega}{r}\right) = 0.$$

Let $\Phi_\Omega = (\Phi_\Omega^r, \Phi_\Omega^z)$ be the characteristic line associated with $U_\Omega + u_1 + u_2$ and let $\tilde{\Phi}_\Omega$ be the corresponding inverse map. By Lemma 4.5, for A sufficiently large, we have either

$$\max_{0 \le t \le \frac{1}{\log\log A}} \left\|\Omega(t)\right\|_{\dot{H}^{\frac{3}{2}}} > \log\log\log A, \tag{4.67}$$

or

$$\max_{0 \le t \le \frac{1}{\log\log A}} \left\|(D\tilde{\Phi}_\Omega)(t)\right\|_\infty > \log\log\log A \tag{4.68}$$

must hold.

Now discuss two cases.

`Case 1`: (4.67) hold. In this it is case easy to check that

$$\max_{0 \le t \le \frac{1}{\log\log A}} \left\|W_R^1(t)\right\|_{\dot{H}^{\frac{3}{2}}} \gtrsim \log\log\log A.$$

Therefore, we can just let $W(t) = W^1(t)$ with $\omega_0 = \tilde{g}_A$.

Case 2: (4.68) hold. In this case we just need to apply a perturbation argument similar to that in the proof of Proposition 3.11. It is easy to check that this case is also OK.

Concluding from the above two cases, the proposition is proved. ■

We are now ready to complete the following.

Proof of Theorem 1.5. We shall only sketch the proof for $\omega_0^{(g)} \equiv 0$. The construction of $\omega_0^{(p)}$ for the general nonzero $\omega_0^{(g)}$ is a simple modification of the proof presented below. For example, one can just take the 1st patch as $\omega_0^{(g)}$ and start the perturbation for $j \geq 2$.

We now begin the proof for $\omega_0^{(g)} \equiv 0$. For each integer $j \geq 1$, define $x_*^j = (0, 0, \sum_{k=1}^{j} \frac{1}{2^k})$. Obviously for any $j \geq 2$, we have

$$\left|x_*^{j+1} - x_*^j\right| = \frac{1}{2^{j+1}},$$
$$\left|x_*^j - x_*^{j-1}\right| = \frac{1}{2^j}.$$

We shall choose x_*^j to be the center of the jth patch. So the distance between the nearest patches is about 2^{-j}. Define

$$x_* = \lim_{j\to\infty} x_*^j = (0, 0, 1).$$

Our constructed solution will exhibit some additional regularity away from the limit point x_*.

Let W^1 be a smooth axisymmetric solution to the Euler equation (in vorticity form)

$$\begin{cases} \partial_t\left(\frac{W^1}{r}\right) + (U^1 \cdot \nabla)\left(\frac{W^1}{r}\right) = 0, \quad 0 < t \leq 1, x = (x_1, x_2, z), r = \sqrt{x_1^2 + x_2^2}, \\ U^1 = -\Delta^{-1}\nabla \times W^1, \\ W^1\Big|_{t=0} = W_0^1 = W_0^{1,\theta} e_\theta, \end{cases}$$

such that $W_0^{1,\theta} = W_0^{1,\theta}(r, z)$ is scalar valued, $\mathrm{supp}(W_0^{1,\theta}) \subset \{(r, z) : r > r_0\}$ for some $r_0 > 0$, $W^1(t) \in C_c^\infty(B(x_*^1, \frac{1}{2^{10}}))$ for any $0 \leq t \leq 1$ and

$$\left\|U^1(0, \cdot)\right\|_{H^{\frac{5}{2}}} + \max_{0\leq t\leq 1} \left\|W^1(t, \cdot)\right\|_\infty \leq \frac{1}{2^{100}}. \tag{4.69}$$

In view of the scaling symmetry ($\omega \to \omega_\lambda(t,x) = \lambda\omega(\lambda t, x)$) and translation symmetry (in the axisymmetric case we just shift only along the z axis so as to keep axisymmetry) of the Euler equation, we can always find a nonzero W^1 satisfying the aforementioned conditions by transforming an arbitrary compactly supported solution.

By repeated applying Proposition 4.6 (one needs to shift along the z axis if necessary), we can find a sequence of smooth solutions W^j, $j \geq 2$, solving the equations

$$\begin{cases} \partial_t\left(\frac{W^j}{r}\right) + (U^j \cdot \nabla)\left(\frac{W^j}{r}\right) = 0, \quad 0 < t \leq 1, \\ U^j = -\Delta^{-1}\nabla \times W^j, \\ W^j\Big|_{t=0} = W_0^j = W_0^{j,\theta} e_\theta, \end{cases}$$

such that the following hold:

- $W_0^j = (\sum_{k=1}^j f_k) e_\theta$, where $f_1 = W_0^{1,\theta}$, and for $k \geq 2$, $\mathrm{supp}(f_k) \subset \{(r,z) : r > r_k\}$ for some $r_k > 0$.
- Define $F_k = f_k e_\theta$. Then for each $k \geq 1$, $F_k \in C_c^\infty(B(x_*^k, \frac{1}{2^{10k}}))$. Furthermore,

$$\left\| \Delta^{-1}\nabla \times F_k \right\|_{H^{\frac{5}{2}}} \leq 2^{-100k}, \quad \forall k \geq 1. \tag{4.70}$$

- For any $j \geq 2$,

$$\max_{0 \leq t \leq 1} \left\| W^j(t,\cdot) - W^{j-1}(t,\cdot) \right\|_\infty \leq 2^{-100j}. \tag{4.71}$$

- For each $j_0 \geq 2$, there exists $t_{j_0}^1$, $t_{j_0}^2$ with $0 < t_{j_0}^1 < t_{j_0}^2 < 2^{-j_0}$, such that for any $j \geq j_0 + 2$, we have the decomposition:

$$W^j(t,x) = W^j_{<j_0}(t,x) + W^j_{j_0}(t,x) + W^j_{>j_0}(t,x), \quad \forall t \leq t_{j_0}^2, \tag{4.72}$$

where $W^j_{<j_0} \in C_c^\infty(\mathbb{R}^3)$, $W^j_{j_0} \in C_c^\infty(\mathbb{R}^3)$ and $W^j_{>j_0} \in C_c^\infty(\mathbb{R}^3)$ satisfy

$$\mathrm{supp}\left(W^j_{<j_0}\right) \subset \left\{ x = (x_1, x_2, z) : \ z \leq \sum_{k=1}^{j_0 - 1} 2^{-k} + \frac{1}{8} \cdot 2^{-j_0} \right\},$$

$$\mathrm{supp}\left(W^j_{j_0}\right) \subset \left\{ x = (x_1, x_2, z) : \ \sum_{k=1}^{j_0} 2^{-k} - \frac{1}{8} \cdot 2^{-j_0} < z < \sum_{k=1}^{j_0} 2^{-k} + \frac{1}{8} \cdot 2^{-j_0} \right\},$$

$$\mathrm{supp}\left(W^j_{>j_0}\right) \subset \left\{ x = (x_1, x_2, z) : \ z > \sum_{k=1}^{j_0} 2^{-k} + \frac{1}{4} 2^{-j_0} \right\}.$$

Here

$$W^j_{<j_0}(t=0)=\sum_{k=1}^{j_0-1}F_k,\quad W^j_{j_0}(t=0)=F_{j_0},$$

$$W^j_{>j_0}(t=0)=\sum_{k=j_0+1}^{j}F_k.$$

Furthermore

$$\|W^j_{j_0}(t,\cdot)\|_{\dot H^{\frac{3}{2}}(\mathbb{R}^3)}>j_0,\quad \forall\, t\in[t^1_{j_0},t^2_{j_0}];\tag{4.73}$$

$$\|W^j_{j_0}(t,\cdot)\|_{L^2(\mathbb{R}^3)}\le 2^{-100j_0},\quad \forall\, t\le t^2_{j_0}.\tag{4.74}$$

$$\max_{0\le t\le t^2_{j_0}}\left(\|W^j_{j_0}(t,\cdot)\|_{H^k(\mathbb{R}^3)}+\|W^j_{<j_0}(t,\cdot)\|_{H^k(\mathbb{R}^3)}\right)\le C_{j_0,k}<\infty,\quad \forall\, k\ge 2,\tag{4.75}$$

where $C_{j_0,k}$ is a constant depending only on k and $(F_1,F_2,\cdots,F_{j_0})$.

We now show the existence of the solution ω as the limit of W^j, $j\to\infty$. By L^2 conservation of velocity and (4.70), we have

$$\max_{0\le t\le 1}\|U^j(t,\cdot)\|_2=\|U^j(0,\cdot)\|_2$$
$$\le\sum_{k=1}^{\infty}2^{-100k}\le 2^{-99},\quad \forall\, j\ge 1.\tag{4.76}$$

By (4.69) and (4.71),

$$\max_{0\le t\le 1}\|W^j(t,\cdot)\|_\infty\le\sum_{k=1}^{j}2^{-100k}\le 2^{-99},\quad \forall\, j\ge 1.\tag{4.77}$$

By (4.76), (4.77), and interpolation, we then get

$$\sup_{j\ge 1}\max_{0\le t\le 1}\|U^j(t,\cdot)\|_\infty\lesssim 1.\tag{4.78}$$

Since $\operatorname{supp}(W^j(t,\cdot))\subset B(0,2)$, (4.78) then implies that

$$\operatorname{supp}(W^j(t,\cdot))\subset B(0,C_1),\quad \forall\, 0\le t\le 1,\, j\ge 1,\tag{4.79}$$

where $C_1 > 0$ is an absolute constant. By (4.71) and (4.79), the sequence W^j is Cauchy in the space $C_t^0 C_x^0([0,1] \times \overline{B(0,C_1)})$ and hence converges to the limit solution w in the same space. By Sobolev embedding and interpolation, it is not difficult to check that U^j also converges to $u = -\Delta^{-1}\nabla \times \omega \in C_t^0 L_x^2 \cap C_t^0 C_x^\alpha([0,1] \times \mathbb{R}^3)$ for any $0 < \alpha < 1$. It follows easily that ω is the desired solution satisfying the 1st two statements in Theorem 1.5.

It remains for us to check the last two properties of ω in Theorem 1.5.

Fix any $j_0 \geq 2$. By (4.72), (4.75), and taking the limit $j \to \infty$, we get the decomposition of $\omega(t,x)$ for $t \leq t_{j_0}^2$ as

$$\omega(t,x) = \omega_{<j_0}(t,x) + \omega_{j_0}(t,x) + \omega_{>j_0}(t,x), \tag{4.80}$$

where $\omega_{<j_0}(t) \in C_c^\infty(\mathbb{R}^3)$, $\omega_{j_0}(t) \in C_c^\infty(\mathbb{R}^3)$, $\omega_{>j_0}(t) \in C_c^0(\mathbb{R}^3)$ for $t \leq t_{j_0}^2$, and

$$\operatorname{supp}(\omega_{<j_0}) \subset \left\{ x = (x_1,x_2,z) : \; z \leq \sum_{k=1}^{j_0-1} 2^{-k} + \frac{1}{8}\cdot 2^{-j_0} \right\},$$

$$\operatorname{supp}(\omega_{j_0}) \subset \left\{ x = (x_1,x_2,z) : \; \sum_{k=1}^{j_0} 2^{-k} - \frac{1}{8}\cdot 2^{-j_0} < z < \sum_{k=1}^{j_0} 2^{-k} + \frac{1}{8}\cdot 2^{-j_0} \right\};$$

$$\operatorname{supp}(\omega_{>j_0}) \subset \left\{ x = (x_1,x_2,z) : \; z > \sum_{k=1}^{j_0} 2^{-k} + \frac{1}{4} 2^{-j_0} \right\}.$$

Furthermore,

$$\|\omega_{j_0}(t,\cdot)\|_{\dot H^{\frac{3}{2}}(\mathbb{R}^3)} \geq j_0, \quad \forall\, t \in [t_{j_0}^1, t_{j_0}^2]; \tag{4.81}$$

$$\|\omega_{j_0}(t,\cdot)\|_{L^2(\mathbb{R}^3)} \leq 2^{-100 j_0}, \quad \forall\, t \leq t_{j_0}^2. \tag{4.82}$$

$$\max_{0 \leq t \leq t_{j_0}^2} \left(\|\omega_{j_0}(t,\cdot)\|_{H^k(\mathbb{R}^3)} + \|\omega_{<j_0}(t,\cdot)\|_{H^k(\mathbb{R}^3)} \right) \leq C_{j_0,k} < \infty, \quad \forall\, k \geq 2, \tag{4.83}$$

where $C_{j_0,k}$ is a constant depending only on k and $(F_1, F_2, \cdots, F_{j_0})$. Now for any $y = (y_1, y_2, y_3) \neq x_* = (0,0,1)$, consider three cases. If $y_3 \geq 1$, then in this case by our choice of initial data and finite transport speed, we can find a small neighborhood N_y of y and $0 < t_y < 1$ such that $\omega(t,x) = 0$ for any $x \in N_y$ and $0 \leq t \leq t_y$. If $y_3 < 1$, then we can choose j_0 sufficiently large such that $y \in \{x = (x_1,x_2,z) : \; z < \sum_{k=1}^{j_0-1} 2^{-k} + \frac{1}{16}\cdot 2^{-j_0}\}$. In this case we can just choose $t_y = t^2_{j_0}$ and N_y to be a small open neighborhood contained in $\{x = (x_1, x_2, z):\ z < \sum_{k=1}^{j_0-1} 2^{-k} + \frac{1}{16} \cdot 2^{-j_0}\}$. By (4.83) $\omega(t) \in C^\infty(N_y)$ for any $0 \le t \le t_y$. Therefore statement (3) in Theorem 1.5 is proved.

Finally we prove statement (4) in Theorem 1.5. For each integer $n \ge 1$, we shall take j_n to be sufficiently large and decompose ω according to (4.80) with j_0 replaced by j_n. By a slight abuse of notation we denote $t^1_n = t^1_{j_n}$, $t^2_n = t^2_{j_n}$, and $\omega_n = \omega_{j_n}$. Define

$$\begin{aligned}
K_n &= \overline{\{x \in \mathbb{R}^3:\ \omega_n(t,x) \neq 0 \text{ for some } 0 \le t \le t^2_n\}},\\
\Omega^1_n &= \left\{x \in \mathbb{R}^3 : \operatorname{dist}(x, K_n) < \frac{1}{2^{100 j_n}}\right\},\\
\Omega^2_n &= \left\{x \in \mathbb{R}^3 : \operatorname{dist}(x, K_n) < \frac{1}{1000} \cdot \frac{1}{2^{j_n}}\right\}.
\end{aligned}$$

The inequality (1.7) follows from (4.81). To show (1.8), we note that if $x \in \mathbb{R}^3 \setminus \Omega^2_n$ and $y \in K_n$, then

$$|x - y| \gtrsim 2^{-j_n}.$$

We can then write for $x \in \mathbb{R}^3 \setminus \Omega^2_n$,

$$\begin{aligned}
\big(|\nabla|^3 \omega_n\big)(t,x) &= \big(|\nabla|^{-1}(-\Delta)^2 \omega_n\big)(t,x)\\
&= \int_{\mathbb{R}^3} K(x-y)\chi_{|x-y| \gtrsim 2^{-j_n}} \big((-\Delta)\omega_n\big)(t,y)\, dy\\
&= \int_{\mathbb{R}^3} \tilde{K}(x-y)\omega_n(t,y)\, dy,
\end{aligned}$$

where χ is a smooth cut-off function and we have integrated by parts in the last step. The modified kernel $\tilde{K}(\cdot)$ is smooth and obeys the bound

$$|\tilde{K}(x)| \lesssim 2^{10 j_n}(1 + |x|^2)^{-1}, \quad \forall\, x \in \mathbb{R}^3.$$

Thus by (4.82) and taking j_n sufficiently large, we obtain

$$\max_{0 \le t \le t^2_n} \|(|\nabla|^3 \omega_n)(t,x)\|_{L^2(\mathbb{R}^3 \setminus \Omega^2_n)} \lesssim 2^{10 j_n} 2^{-100 j_n} \le 1.$$

Theorem 1.5 is now proved. ■

5 Ill-posedness in Besov spaces

In this last section we settle the ill-posedness in the Besov case.

Theorem 5.1. For any $\omega_0^{(g)} \in C_c^\infty(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$, any $\epsilon > 0$, and any $1 < p < \infty$, $1 < q \leq \infty$, we can find a C^∞ perturbation $\omega_0^{(p)} : \mathbb{R}^2 \to \mathbb{R}$ such that the following hold true:

(1) The perturbation is very small:

$$\|\omega_0^{(p)}\|_{L^1(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{L^\infty(\mathbb{R}^2)} + \|\omega^{(p)}\|_{\dot{H}^{-1}(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{B^{\frac{2}{p}}_{p,q}(\mathbb{R}^2)} < \epsilon.$$

(2) Let $\omega_0 = \omega_0^{(g)} + \omega_0^{(p)}$. The initial velocity $u_0 = \Delta^{-1}\nabla^\perp\omega_0$ has regularity $u_0 \in L^2(\mathbb{R}^2) \cap B^{1+\frac{2}{p}}_{p,q}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$.

(3) There exists a unique classical solution $\omega = \omega(t)$ to the 2D Euler equation (in vorticity form)

$$\begin{cases} \partial_t\omega + (\Delta^{-1}\nabla^\perp\omega \cdot \nabla)\omega = 0, & 0 < t \leq 1,\ x \in \mathbb{R}^2, \\ \omega\Big|_{t=0} = \omega_0, \end{cases}$$

satisfying $\omega(t) \in L^1 \cap L^\infty \cap C^\infty \cap \dot{H}^{-1}$, $u = \Delta^{-1}\nabla^\perp\omega \in C^\infty \cap L^2 \cap L^\infty$ for each $0 \leq t \leq 1$.

(4) For any $0 < t_0 \leq 1$, we have

$$\text{ess-sup}_{0<t\leq t_0} \|\omega(t,\cdot)\|_{\dot{B}^{\frac{2}{p}}_{p,\infty}} = +\infty. \tag{5.1}$$

Proof of Theorem 5.1. Again with out loss of generality we assume $\omega_0^{(g)} \equiv 0$. We shall sketch the details and point out the important changes (as compared to the proof of 2D H^1 non-compact case). The 1st crucial step is the local construction. Since $B^{\frac{2}{p}}_{p,q_1} \hookrightarrow B^{\frac{2}{p}}_{p,q_2}$ whenever $q_1 < q_2$, it suffices for us to consider the case $B^{\frac{2}{p}}_{p,q}$ with $1 < q < p$. Fix such p and q. We will prove the following:

Claim: For any small $\delta > 0$, there exists a smooth initial data $\omega_0^\delta \in C_c^\infty(B(0,\delta))$ and $t_\delta \in (0,\delta)$ such that if ω^δ is the smooth solution to

$$\begin{cases} \partial_t\omega^\delta + (\Delta^{-1}\nabla^\perp\omega^\delta \cdot \nabla)\omega^\delta = 0, & 0 < t \leq 1,\ x \in \mathbb{R}^2, \\ \omega^\delta\Big|_{t=0} = \omega_0^\delta, \end{cases}$$

then the following hold:

- $\|\omega_0^\delta\|_{B^{\frac{2}{p}}_{p,q}} + \|\omega_0^\delta\|_{L^\infty} + \|\omega_0^\delta\|_{\dot H^{-1}} \le \delta.$
- $\mathrm{supp}(\omega^\delta(t,\cdot)) \subset B(0,\delta)$ for any $0 \le t \le \delta$.
- $\|\omega^\delta(t_\delta,\cdot)\|_{\dot B^{\frac{2}{p}}_{p,\infty}} > \frac{1}{\delta}$.

To prove the claim, we first take $\phi_0 \in C_c^\infty(\mathbb{R}^2)$ to be a radial bump function such that $\mathrm{supp}(\phi_0) \subset B(0,1)$ and $0 \le \phi_0 \le 1$. Define

$$\eta_0(x_1,x_2) = \sum_{a_1,a_2=\pm 1} a_1 a_2 \phi_0\left(\frac{(x_1-a_1,x_2-a_2)}{2^{-10}}\right).$$

Take $A \gg 1$ and define one parameter of functions

$$g_A^0 = \frac{1}{\log A} \sum_{A<k<A+\log A} \eta_0(2^k x).$$

Easy to check that

$$\|g_A^0\|_{\dot B^{\frac{2}{r}}_{r,1}} \lesssim 1, \quad \forall\, 1 \le r \le \infty,$$

$$\|g_A^0\|_{\dot B^0_{\infty,\infty}} \lesssim \|g_A^0\|_\infty \lesssim \frac{1}{\log A}.$$

Therefore, by interpolation (choosing $r = p/q$),

$$\|g_A^0\|_{\dot B^{\frac{2}{p}}_{p,q}} \lesssim \frac{1}{(\log A)^{2\epsilon_1}},$$

where the exponent $\epsilon_1 = \frac{1}{2}(1-\frac{1}{q}) \in (0,\frac{1}{2})$. Now take

$$h_A = (\log A)^{\epsilon_1} g_A^0. \tag{5.2}$$

Obviously we have

$$\|h_A\|_\infty \lesssim \frac{1}{(\log A)^{1-\epsilon_1}},$$

$$\|h_A\|_1 + \|h_A\|_{\dot H^{-1}} \lesssim 2^{-2A},$$

$$\|h_A\|_{\dot B^{\frac{2}{p}}_{p,q}} \lesssim \frac{1}{(\log A)^{\epsilon_1}}. \tag{5.3}$$

Let W_A be the smooth solution to the system

$$\begin{cases} \partial_t W_A + (\Delta^{-1}\nabla^\perp W_A \cdot \nabla) W_A = 0, \quad t > 0, x \in \mathbb{R}^2, \\ W_A\Big|_{t=0} = h_A. \end{cases} \tag{5.4}$$

Define the forward characteristics ϕ_A such that

$$\begin{cases} \partial_t \phi_A(t,x) = (\Delta^{-1}\nabla^\perp W_A)(t, \phi_A(t,x)), \\ \phi_A(t=0,x) = x \in \mathbb{R}^2. \end{cases} \tag{5.5}$$

We then have for A sufficiently large,

$$M_A := \max_{0 \le t \le \frac{1}{\log\log A}} \|(D\phi_A)(t,\cdot)\|_\infty \ge \log\log A. \tag{5.6}$$

Clearly we can find $0 < t_A < \frac{1}{\log\log A}$ and x_A such that

$$\|(D\phi_A)(t_A, x_A)\|_\infty > \frac{4}{5} M_A.$$

Denote $\phi_A(t,x_1,x_2) = (\phi_A^1(t,x_1,x_2), \phi_A^2(t,x_1,x_2))$. Without loss of generality we can assume

$$\left|(\partial_2 \phi_A^2)(t_A, x_A)\right| > \frac{4}{5} M_A.$$

By continuity we can find a small neighborhood $O_A = B(x_A, r_A)$ of x_A such that

$$\left|(\partial_2 \phi_A^2)(t_A, x)\right| > \frac{4}{5} M_A, \quad \forall x \in O_A. \tag{5.7}$$

Depending on the location of x_A, we need to shrink $0 < r_A < 1$ slightly further and define an even function $b \in C_c^\infty(\mathbb{R}^2)$ as follows. Fix a smooth radial cut-off function $\Phi_0 \in C_c^\infty(\mathbb{R}^2)$ such that $\Phi_0(x) = 1$ for $|x| \le \frac{1}{2}$ and $\Phi_0(x) = 0$ for $|x| > 1$. If $x_A = (0,0)$, we just define $b(x) = r_A^{-\frac{2}{p}} \Phi_0(\frac{x}{r_A})$. If $x_A = (a_*, 0)$ for some $a_* \neq 0$, then we choose $r_A > 0$ such that $r_A \ll |x_A|$. In this case we choose b as an even function of x_1 and x_2 that takes the form

$$b(x) = r_A^{-\frac{2}{p}} \Big(\Phi_0\Big(\frac{x - x_A}{r_A}\Big) + \Phi_0\Big(\frac{x + x_A}{r_A}\Big)\Big).$$

The case $x_A = (0, c_*)$ is similar. Now if $x_A = (a_*, c_*)$ with $a_* \neq 0$ and $c_* \neq 0$, then we choose $r_A \ll \min\{|a_*|, |c_*|\}$ and define

$$b(x) = r_A^{-\frac{2}{p}} \sum_{\epsilon_1=\pm 1, \epsilon_2=\pm 1} \Phi_0\left(\frac{x - (\epsilon_1 a_*, \epsilon_2 c_*)}{r_A}\right).$$

Easy to check that b is an even function of x_1 and x_2.

Therefore, in all situations we can choose an even function $b \in C_c^\infty(\mathbb{R}^2)$ such that

$$\begin{aligned} \|b\|_{L^p(O_A)} &\sim 1, \\ \|b\|_{L^p(\mathbb{R}^2)} &\sim 1, \\ \|b\|_{\dot{B}^0_{p,1}(\mathbb{R}^2)} &\lesssim 1. \end{aligned} \tag{5.8}$$

In the above the implied constants depend only on the definition of the smooth cut-off function Φ_0 and thus can be made as absolute constants. To simplify later notations and discussions, we shall still denote by O_A the support of b, which are unions of even reflections of O_A on the plane. The last inequality in (5.8) is due to the fact that translation in the real domain is equivalent to phase modulation in the frequency domain and hence

$$\|b\|_{\dot{B}^0_{p,1}} \lesssim \left\| r_A^{-\frac{2}{p}} \Phi_0\left(\frac{\cdot}{r_A}\right)\right\|_{\dot{B}^0_{p,1}} \lesssim \|\Phi_0\|_{\dot{B}^0_{p,1}} \lesssim 1. \tag{5.9}$$

Now we consider two cases.

Case 1: $\max_{0\le t\le \frac{1}{\log\log A}} \|W_A(t,\cdot)\|_{\dot{B}^{\frac{2}{p}}_{p,\infty}} \ge \log\log\log\log A$. In this case we set $\omega_0^\delta = W_A$ with $A = A(\delta)$ chosen sufficiently large. No particular work is needed in this case.

Case 2:

$$\max_{0\le t\le \frac{1}{\log\log A}} \|W_A(t,\cdot)\|_{\dot{B}^{\frac{2}{p}}_{p,\infty}} < \log\log\log\log A. \tag{5.10}$$

In this case we consider

$$\tilde{h}_A = h_A + \underbrace{\frac{1}{\log\log\log A} \cdot k^{-\frac{2}{p}} \sin(kx_1) \cdot b(x)}_{:=\beta(x)},$$

where $b(x)$ was chosen as in (5.8). Once again we shall take the parameter k sufficiently large. Consider first $N \ll k$. Write

$$\sin(kx_1)b(x) = -\frac{1}{k}\partial_{x_1}(\cos(kx_1)b(x)) + \frac{1}{k}\cos(kx_1)\partial_{x_1}b(x).$$

Of course a natural idea is to consider cutting off the high frequencies of b, say replacing b by $b_1 := P_{<N_1}b$ for some sufficiently large N_1. This will simplify the computation of $\dot{B}^{\frac{2}{p}}_{p,1}$ norm of the perturbation $\sin(kx_1)b_1(x)$ since for large $k \gg N_1$ the function $\sin(kx_1)b_1(x)$ will have frequency localized to $\{\xi : |\xi| \sim k\}$. However, the disadvantage of doing this is that b_1 is not compactly supported in the x-space. This will bring some more unnecessary technical complications in the gluing of patch solutions later.

By Bernstein and the above identity, we get

$$\begin{aligned} N^{\frac{2}{p}}\|P_N(\sin(kx_1)b(x))\|_p &\lesssim \frac{N^{1+\frac{2}{p}}}{k}\|b\|_p + \frac{1}{k}N^{\frac{2}{p}}\|P_N(\cos(kx_1)\partial_{x_1}b)\|_p \\ &\lesssim \frac{N^{1+\frac{2}{p}}}{k}\|b\|_p + \frac{1}{k}N^{\frac{2}{p}}\|\partial_{x_1}b\|_p. \end{aligned}$$

Summing over dyadic $N \ll k$ and letting k be sufficiently large, we obtain

$$\frac{1}{k^{\frac{2}{p}}}\sum_{N\ll k} N^{\frac{2}{p}}\left\|P_N(\sin(kx_1)b(x))\right\|_p \lesssim 1.$$

Next consider $N \gg k$. By frequency localization, observe

$$P_N(\sin(kx_1)b) = P_N(\sin(kx_1)\tilde{P}_N b),$$

where $\tilde{P}_N$ is a fattened frequency projector adapted to the regime $|\xi| \sim N$. Clearly by taking k sufficiently large, we have

$$\frac{1}{k^{\frac{2}{p}}}\sum_{N\gg k} N^{\frac{2}{p}}\left\|P_N(\sin(kx_1)b(x))\right\|_p \lesssim \frac{1}{k^{\frac{2}{p}}}\sum_{N\gg k} N^{\frac{2}{p}}\|\tilde{P}_N b\|_p \lesssim k^{-\frac{2}{p}}\|b\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \lesssim 1.$$

In the intermediate regime $N \sim k$, there are finitely many such dyadic N and we have

$$\frac{1}{k^{\frac{2}{p}}}\sum_{N\sim k} N^{\frac{2}{p}}\left\|P_N(\sin(kx_1)b(x))\right\|_p \lesssim \|b\|_p \lesssim 1.$$

Summing over all cases, we have proved

$$\frac{1}{k^{\frac{2}{p}}}\big\|b(x)\sin(kx_1)\big\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \lesssim 1.$$

Therefore,

$$\|\beta\|_{\dot{B}^{\frac{2}{p}}_{p,1}} \lesssim \frac{1}{\log\log\log A}.$$

By a similar analysis, we also have

$$\|\beta\|_{B^{\frac{2}{p}}_{r,1}} = O(1), \quad \forall\, p \le r \le \infty. \tag{5.11}$$

Denote $e_1 = (1,0)$. Then

$$\begin{aligned}\||\nabla|^{-1}\beta\|_2^2 &\lesssim \frac{1}{k^{\frac{4}{p}}}\int_{\mathbb{R}^2}\frac{1}{|\xi|^2}\big|\hat{b}(\xi+ke_1)-\hat{b}(\xi-ke_1)\big|^2\,\mathrm{d}\xi\\ &\lesssim k^{-\frac{4}{p}}\big(\|xb(x)\|_1^2+\|b\|_2^2\big)\\ &= O(k^{-\frac{4}{p}}).\end{aligned}$$

Therefore,

$$\|\beta\|_{\dot{H}^{-1}} = O\big(k^{-\frac{2}{p}}\big).$$

By (5.3) and choosing k sufficiently large, we then have

$$\begin{aligned}\|\tilde{h}_A\|_\infty &\lesssim \frac{1}{(\log A)^{1-\epsilon_1}},\\ \|\tilde{h}_A\|_1+\|\tilde{h}_A\|_{\dot{H}^{-1}} &\lesssim 2^{-2A},\\ \|\tilde{h}_A\|_{\dot{B}^{\frac{2}{p}}_{p,q}} &\lesssim \frac{1}{(\log A)^{\epsilon_1}}+\frac{1}{\log\log\log A}.\end{aligned}$$

Let W_A^1 be the smooth solution to the equation

$$\begin{cases}\partial_t W_A^1 + \big(\Delta^{-1}\nabla^\perp W_A^1\cdot\nabla\big)W_A^1 = 0, \quad 0 < t \le 1,\\ W_A^1\Big|_{t=0} = \tilde{h}_A.\end{cases}$$

Let $\eta = W_A^1 - W_A$ where W_A is the solution to (5.4). Then η satisfies

$$\begin{cases} \partial_t \eta + (\Delta^{-1}\nabla^\perp \eta \cdot \nabla) W_A^1 + (\Delta^{-1}\nabla^\perp W_A \cdot \nabla)\eta = 0, \\ \eta\Big|_{t=0} = \beta. \end{cases}$$

Now

$$\begin{aligned} \partial_t(\||\nabla|^{-1}\eta\|_2^2) &\lesssim \|\Delta^{-1}\nabla^\perp \eta\|_2 \cdot \|W_A^1\|_\infty \cdot \||\nabla|^{-1}\eta\|_2 + \left| \int (\Delta^{-1}\nabla^\perp W_A \cdot \nabla)(\Delta\Delta^{-1}\eta) \cdot \Delta^{-1}\eta \,\mathrm{d}x \right| \\ &\lesssim \||\nabla|^{-1}\eta\|_2^2 \cdot \big(\|W_A^1\|_\infty + \|\mathcal{R}_{ij} W_A\|_\infty\big). \end{aligned}$$

Hence,

$$\max_{0\le t\le 1} \big\|(|\nabla|^{-1}\eta)(t,\cdot)\big\|_2 = O\big(k^{-\frac{2}{p}}\big). \tag{5.12}$$

Take $r \in (p,\infty)$. By (5.11) and local well-posedness in $B^{\frac{2}{p}}_{r,1}$,

$$\max_{0\le t\le 1} \left(\|W_A^1(t,\cdot)\|_{B^{\frac{2}{p}}_{r,1}} + \|W_A(t,\cdot)\|_{B^{\frac{2}{p}}_{r,1}} \right) = O(1). \tag{5.13}$$

Interpolating the above with (5.12), we get

$$\max_{0\le t\le 1} \|\eta(t,\cdot)\|_{B^0_{\infty,1}} = O(k^{-\alpha}). \tag{5.14}$$

Here and below we denote by the general notation $X = O(k^{-\alpha})$ if the quantity $X \lesssim k^{-\alpha}$ for some $\alpha > 0$. The value of α does not play much role in the analysis as long as $\alpha > 0$.

Let Φ_A be the characteristic line associated with W_A^1, that is,

$$\begin{cases} \partial_t \Phi_A(t,x) = \big(\Delta^{-1}\nabla^\perp W_A^1\big)\big(t, \Phi_A(t,x)\big), \\ \Phi_A(0,x) = x \in \mathbb{R}^2. \end{cases}$$

Set $J(t) = (D\Phi_A)(t) - (D\phi_A)(t)$. Then

$$\partial_t J = (\mathcal{R} W_A^1)(\Phi_A) J + \Big(\mathcal{R}(W_A^1 - W_A)\Big)(\Phi_A) D\phi_A + \Big((\mathcal{R} W_A)(\Phi_A) - (\mathcal{R} W_A)(\phi_A)\Big) D\phi_A.$$

Using (5.14) and the above equation, it is easy to check

$$\max_{0\le t\le 1}\left(\|(D\Phi_A)(t,\cdot)-(D\phi_A)(t,\cdot)\|_\infty+\|\Phi_A(t,\cdot)-\phi_A(t,\cdot)\|_\infty\right)=O(k^{-\alpha}). \tag{5.15}$$

Let W_A^2 and W_A^3 be the smooth solutions to the *linear* equations

$$\begin{cases}\partial_t W_A^2+(V_A\cdot\nabla)W_A^2=0, & t>0,\\ W_A^2\Big|_{t=0}=g_A,\end{cases}$$

$$\begin{cases}\partial_t W_A^3+(V_A\cdot\nabla)W_A^3=0, & t>0,\\ W_A^3\Big|_{t=0}=\beta,\end{cases}$$

where $V_A(t,x)=(\Delta^{-1}\nabla^\perp W_A^1)(t,x)$. Obviously,

$$W_A^1=W_A^2+W_A^3.$$

We first show that

$$\max_{0\le t\le 1}\|W_A(t,\cdot)-W_A^2(t,\cdot)\|_{\dot B^{\frac{2}{p}}_{p,\infty}}=O(k^{-\alpha}). \tag{5.16}$$

By (5.13), it is easy to check

$$\begin{aligned}\max_{0\le t\le 1}\|D^2W_A^2(t,\cdot)\|_p&=O(1), \quad \text{if } 1<p\le 2,\\ \max_{0\le t\le 1}\|DW_A^2(t,\cdot)\|_p&=O(1), \quad \text{if } 2<p<\infty.\end{aligned} \tag{5.17}$$

On the other hand, by the fundamental theorem of calculus, we have

$$\begin{aligned}W_A^2-W_A&=g_A(\tilde\Phi_A)-g_A(\tilde\phi_A)\\ &=\int_0^1(Dg_A)(\tilde\phi_A+s(\tilde\Phi_A-\tilde\phi_A))\,\mathrm{d}s(\tilde\Phi_A-\tilde\phi_A).\end{aligned}$$

Here $\tilde\Phi_A$ and $\tilde\phi_A$ denote the inverse map of Φ_A and ϕ_A, respectively. By (5.15) and Hölder, we then get

$$\max_{0\le t\le 1}\|W_A^2(t,\cdot)-W_A(t,\cdot)\|_p\lesssim\max_{0\le t\le 1}\|W_A^2(t,\cdot)-W_A(t,\cdot)\|_\infty=O(k^{-\alpha}).$$

Alternatively, one can derive the estimate in an "Eulerian" way, that is, directly derive an L^p estimate from the equation.

Interpolating this with (5.17) then yields (5.16).

By (5.16) and (5.10), we only need to show

$$\|W_A^3(t_A,\cdot)\|_{\dot{B}^{\frac{2}{p}}_{p,\infty}} \gg \log\log\log A.$$

For this we need to introduce W_A^4, which solves the linear equation

$$\begin{cases} \partial_t W_A^4 + (U_A\cdot\nabla)W_A^4 = 0, \quad 0 < t \le 1, \\ W_A^4\Big|_{t=0} = \sin(kx_1)b(x) =: W_{4,0}, \end{cases}$$

where $U_A = \Delta^{-1}\nabla^\perp W_A$.

We shall *not* directly compare W_A^3 with $k^{-\frac{2}{p}}\frac{1}{\log\log\log A}W_A^4$ and run a perturbation argument in $\dot{B}^{\frac{2}{p}}_{p,\infty}$. Instead we will carry out an indirect argument as follows.

We first analyze the structure of W_A^4. Write $\tilde{\phi}_A = (\tilde{\phi}_A^1, \tilde{\phi}_A^2)$ and

$$\begin{aligned} W_A^4(t_A,x) &= W_{4,0}(\tilde{\phi}_A(t_A,x)) \\ &= \sin(k\tilde{\phi}_A^1(t_A,x))\cdot b(\tilde{\phi}_A(t_A,x)). \end{aligned}$$

Now consider

$$F(\xi) = \int_{\mathbb{R}^2} \sin(k\tilde{\phi}_A^1(t_A,x))b(\tilde{\phi}_A(t_A,x))e^{-ix\cdot\xi}\,\mathrm{d}x. \tag{5.18}$$

By a change of variable $x \to \phi_A(t_A,x)$ in (5.18) (and recall that the map is volume preserving), we get

$$F(\xi) = \frac{1}{2i}\int_{\mathbb{R}^2} b(x)\cdot e^{-i\phi_A(t_A,x)\cdot\xi + ikx_1}\,\mathrm{d}x - \frac{1}{2i}\int_{\mathbb{R}^2} b(x)\cdot e^{-i\phi_A(t_A,x)\cdot\xi - ikx_1}\,\mathrm{d}x.$$

Consider the phase $\phi_A(t_A,x)\cdot\xi + kx_1$. Write

$$(D\phi_A)(t_A,x)\xi + k\begin{pmatrix}1\\0\end{pmatrix} = (D\phi_A)(t_A,x)\left(\xi + k\big((D\phi_A)(t_A,x)\big)^{-1}\begin{pmatrix}1\\0\end{pmatrix}\right).$$

Since $\big((D\phi_A)(t_A,x)\big)^{-1} = \mathrm{adj}((D\phi_A)(t_A,x))$, by (5.6) and (5.7), we get

$$\frac{1}{C_1} \le \frac{1}{M_A}\left|\left((D\phi_A)(t_A,x)\right)^{-1}\begin{pmatrix}1\\0\end{pmatrix}\right| \le C_1, \quad \forall x \in O_A,$$

where $C_1 > 0$ is an absolute constant. Now if $|\xi| \ge 2C_1 \cdot kM_A$ and $x \in O_A$, then

$$\begin{aligned}\left|(D\phi_A)(t_A,x)\xi + k\begin{pmatrix}1\\0\end{pmatrix}\right| &\gtrsim \left|\big((D\phi_A)(t_A,x)\big)^{-1}\right|^{-1} \cdot \left|\xi + k\big((D\phi_A)(t_A,x)\big)^{-1}\begin{pmatrix}1\\0\end{pmatrix}\right| \\ &\gtrsim M_A^{-1} \cdot |\xi|. \end{aligned} \tag{5.19}$$

Similarly, if $|\xi| \le \frac{1}{2C_1}kM_A$ and $x \in O_A$, then

$$\left|(D\phi_A)(t_A,x)\xi + k\begin{pmatrix}1\\0\end{pmatrix}\right| \gtrsim M_A^{-1} \cdot kM_A \gtrsim k. \tag{5.20}$$

This shows that $F(\xi)$ is essentially localized to $|\xi| \sim kM_A$. By (5.19), (5.20), and a stationary phase argument (note that the derivatives of ϕ_A are independent of k!), we have

$$\|P_{\gg kM_A} W_A^4(t_A,\cdot)\|_p + \|P_{\ll kM_A} W_A^4(t_A,\cdot)\|_p = O(k^{-\alpha}).$$

Consequently,

$$\begin{aligned}\|P_{\sim kM_A} W_A^4(t_A,\cdot)\|_p &\ge \|W_A^4(t_A,\cdot)\|_p - \|P_{\gg kM_A} W_A^4(t_A,\cdot)\|_p - \|P_{\ll kM_A} W_A^4(t_A,\cdot)\|_p \\ &\ge \|\sin(kx_1)b(x)\|_p - O(k^{-\alpha}),\end{aligned}$$

where in the last step we have used L^p conservation. Take an integer m such that $10p > 2m > p$, obviously for k sufficiently large,

$$\begin{aligned}\|\sin(kx_1)b(x)\|_p^p &\ge \int_{\mathbb{R}^2} (\sin(kx_1))^{2m}|b(x)|^p\,\mathrm{d}x \\ &\gtrsim_p \int_{\mathbb{R}^2} (1-\cos(2kx_1))^m|b(x)|^p\,\mathrm{d}x \\ &\gtrsim_p \|b\|_p^p + \sum_{1\le j\le m}(-1)^j\binom{m}{j}\int_{\mathbb{R}^2}(\cos(2kx_1))^j|b(x)|^p\,\mathrm{d}x \\ &\gtrsim_p \|b\|_p^p + O(k^{-\alpha}) \gtrsim \|b\|_p^p \gtrsim 1.\end{aligned}$$

Therefore,

$$\|P_{\sim kM_A} W_A^4(t_A, \cdot)\|_p \gtrsim 1. \tag{5.21}$$

Now set

$$\eta_1 = W_A^4 - k^{\frac{2}{p}} \cdot (\log\log\log A) \cdot W_A^3.$$

Clearly,

$$\begin{cases} \partial_t \eta_1 + ((U_A - V_A) \cdot \nabla) W_A^4 + (V_A \cdot \nabla)\eta_1 = 0, \\ \eta_1 \Big|_{t=0} = 0. \end{cases}$$

By (5.14) (to control $U_A - V_A$) and a similar argument as in the derivation of (5.12), we have

$$\max_{0 \le t \le 1} \||\nabla|^{-1} \eta_1(t, \cdot)\|_2 = O(k^{-\alpha}).$$

Since $\|\eta_1\|_1 + \|\eta_1\|_\infty = O(1)$, interpolation then gives

$$\max_{0 \le t \le 1} \|\eta_1(t, \cdot)\|_p = O(k^{-\alpha}).$$

By (5.21), we then obtain

$$k^{\frac{2}{p}} \|P_{\sim kM_A} W_A^3(t_A, \cdot)\|_p \gtrsim \frac{1}{\log\log\log A}.$$

Clearly,

$$\begin{aligned} \|W_A^3(t_A, \cdot)\|_{\dot{B}^{\frac{2}{p}}_{p,\infty}} &\gtrsim (kM_A)^{\frac{2}{p}} \|P_{\sim kM_A} W_A^3(t_A, \cdot)\|_p \\ &\gtrsim \frac{M_A^{\frac{2}{p}}}{\log\log\log A} \\ &\gtrsim \frac{(\log\log A)^{\frac{2}{p}}}{\log\log\log A} \gg \log\log\log A. \end{aligned}$$

This settles Case 2.

We have proved the claim in the local construction step.

To finish the proof of Theorem 5.1 we then “glue” the local construction and build a solution in the form

$$\omega(t,x)=\sum_{j=1}^{\infty}\omega_j(t,x),$$

where each ω_j has compact support and $\mathrm{dist}(\mathrm{supp}(\omega_j),\mathrm{supp}(\omega_k)) =: R_{jk} \gg 1$ for $j\neq k$. Furthermore for any $n>1$ we can find $0<t_n<\frac{1}{n}$ and j_n such that

$$\|\omega_{j_n}(t_n,\cdot)\|_{\dot B^{\frac{2}{p}}_{p,\infty}} > n. \tag{5.22}$$

Due to the nonlocal nature of the Besov norm $\|\cdot\|_{\dot B^{\frac{2}{p}}_{p,\infty}}$, we have to control the interactions of the patches ω_j. For this we will need to use the convexity inequality: if $1<r<\infty$ and $x,y\in\mathbb{C}^d$, then

$$|x+y|^r \geq |x|^r + r|x|^{r-2}\bar{x}\cdot y, \qquad \forall\, x,\ y\in\mathbb{C}^d. \tag{5.23}$$

Now fix any dyadic $N\geq 2$. By the convexity inequality above, we have for any j,

$$\begin{aligned}\|P_N\omega\|_p^p &= \int_{\mathbb{R}^2} |P_N\omega_j+\sum_{k\neq j}P_N\omega_k|^p\,\mathrm{d}x\\ &\geq \|P_N\omega_j\|_p^p + p\sum_{k\neq j}\int_{\mathbb{R}^2}|P_N\omega_j|^{p-2}(P_N\omega_j)\cdot P_N\omega_k\,\mathrm{d}x.\end{aligned}$$

Observe that for any $m\geq 1$, $N\geq 2$,

$$\|P_N\omega_m\|_{L^p(\{x:\,\mathrm{dist}(x,\mathrm{supp}(\omega_m))>2\})} \lesssim N^{-100}\|P_N\omega_m\|_p.$$

By this and the fact $R_{jk}\gg 1$ for $j\neq k$, we have

$$\sum_{k\neq j}\left|\int_{\mathbb{R}^2}|P_N\omega_j|^{p-2}(P_N\omega_j)\cdot P_N\omega_k\,\mathrm{d}x\right| \lesssim \sum_{k\neq j} N^{-100}\|\omega_j\|_p^{p-1}\cdot\|\omega_k\|_p \lesssim N^{-100},$$

where we need to use the fact $\sum_{k=1}^{\infty}\|\omega_k\|_p\lesssim 1$, which can be easily accommodated into the construction. Clearly for any j,

$$\|P_N\omega\|_p^p \geq \|P_N\omega_j\|_p^p - \mathrm{const}\cdot N^{-100}.$$

From this and (5.22), it is then easy to check (5.1) holds. ■

Theorem 5.2. For any $\omega_0^{(g)} \in C_c^\infty(\mathbb{R}^2) \cap \dot{H}^{-1}(\mathbb{R}^2)$ that is odd in x_1, any $\epsilon > 0$, and any $1 < p < \infty$, $1 < q \le \infty$, we can find a perturbation $\omega_0^{(p)} : \mathbb{R}^2 \to \mathbb{R}$ such that the following hold true:

(1) $\omega_0^{(p)}$ is compactly supported, continuous, and

$$\|\omega_0^{(p)}\|_{L^\infty(\mathbb{R}^2)} + \|\omega^{(p)}\|_{\dot{H}^{-1}(\mathbb{R}^2)} + \|\omega_0^{(p)}\|_{B^{\frac{2}{p}}_{p,q}(\mathbb{R}^2)} < \epsilon.$$

(2) Let $\omega_0 = \omega_0^{(g)} + \omega_0^{(p)}$. Corresponding to ω_0, there exists a unique time-global solution $\omega = \omega(t)$ to the 2D Euler equation satisfying $\omega(t) \in L^\infty \cap \dot{H}^{-1}$. Furthermore, $\omega \in C_t^0 C_x^0$ and $u = \Delta^{-1}\nabla^\perp \omega \in C_t^0 L_x^2 \cap C_t^0 C_x^\alpha$ for any $0 < \alpha < 1$.

(3) $\omega(t)$ has additional local regularity in the following sense: there exists $x_* \in \mathbb{R}^2$ such that for any $x \ne x_*$, there exists a neighborhood $N_x \ni x$, $t_x > 0$ such that $w(t,\cdot) \in C^\infty(N_x)$ for any $0 \le t \le t_x$.

(4) For any $0 < t_0 \le 1$, we have

$$\text{ess-sup}_{0<t\le t_0} \|\omega(t,\cdot)\|_{\dot{B}^{\frac{2}{p}}_{p,\infty}} = +\infty. \tag{5.24}$$

More precisely, there exist $0 < t_n^1 < t_n^2 < \frac{1}{n}$, open precompact sets Ω_n^1, Ω_n^2 with $\Omega_n^1 \subset \overline{\Omega_n^1} \subset \Omega_n^2$, $n = 1, 2, 3, \cdots$ such that

- $\omega(t) \in C^\infty(\Omega_n^2)$ for all $0 \le t \le t_n^2$;
- $\omega(t,x) \equiv 0$ for any $x \in \Omega_n^2 \setminus \Omega_n^1$, $0 \le t \le t_n^2$;
- Define $\omega_n(t,x) = \omega(t,x)$ for $x \in \Omega_n^1$, and $\omega_n(t,x) = 0$ otherwise. Then $\omega_n(t) \in C_c^\infty(\mathbb{R}^2)$, and for some dyadic $N_n \ge 2$,

$$N_n^{\frac{2}{p}} \|(P_{N_n}\omega_n)(t,\cdot)\|_{L^p(\mathbb{R}^2)} > n, \quad \forall\, t_n^1 \le t \le t_n^2, \tag{5.25}$$

and

$$\|(P_{N_n}\omega_n)(t,\cdot)\|_{L^p(x\in\mathbb{R}^2\setminus\Omega_n^2)} + \|\big(P_{N_n}(\omega - \omega_n)\big)\|_{L^p(\Omega_n^2)} \le \frac{1}{N_n^{100}}, \quad \forall\, 0 \le t \le t_n^2. \tag{5.26}$$

Proof of Theorem 5.2. Again WLOG assume $\omega_0^{(g)} \equiv 0$. We shall only sketch the needed modifications (compared to the proof of 2D H^1 compact case and repeating some of the steps in Theorem 5.1). In the local construction step, we take same h_A as in (5.2). We then prove local large Lagrangian deformation as

$$\max_{0\le t\le \frac{1}{\log\log\log A}} \|(D\Phi)(t,\cdot)\|_\infty > \log\log\log\log A.$$

The next step in the construction is to obtain a local patching lemma in $\dot{B}^{\frac{2}{p}}_{p,q}$ ($q=1+$) norm and show that (below ω_0 is the same notation as in (6.20) of Lemma 6.4 in [2], where we decompose the solution accordingly)

$$\max_{0\le t\le t_0} \|P_{>\epsilon^{-\frac{1}{10}}}\omega_0(t,\cdot)\|_{\dot{B}^{\frac{2}{p}}_{p,\infty}} > \frac{1}{\epsilon}. \tag{5.27}$$

Also it should be noted that we need to choose $\epsilon \ll R_0$. One can then easily check

$$\max_{0\le t\le t_0}\Big(\|P_N\omega_0(t,\cdot)\|_{L^p(x:\,\mathrm{dist}(x,\mathrm{supp}(\omega_0))\gtrsim R_0)} + \|P_N(\omega-\omega_0)(t,\cdot)\|_{L^p(x:\,\mathrm{dist}(x,\mathrm{supp}(\omega_0))\lesssim R_0)}\Big)$$
$$\lesssim_m N^{-m}, \quad \forall N > \epsilon^{-\frac{1}{10}}. \tag{5.28}$$

The last step is to glue the patch solutions. This is essentially the same as the proof of 2D case in [2]. The inequalities (5.27) and (5.28) then imply (5.25) and (5.26), respectively. To show (5.24) from (5.25)–(5.26), we just decompose ω as

$$\omega(t,x) = \omega_n(t,x) + g_n(t,x).$$

By (5.23),

$$\|P_{N_n}\omega\|_p^p \ge \|P_{N_n}\omega_n\|_p^p - p\left|\int_{\mathbb{R}^2} |P_{N_n}\omega_n|^{p-2}(P_{N_n}\omega_n)\cdot P_{N_n}g_n\,\mathrm{d}x\right|.$$

By construction, we have for some $R_n > 0$, $\mathrm{dist}(\mathrm{supp}(\omega_n),\mathrm{supp}(g_n)) > 3R_n$, and

$$\|P_{N_n}g_n\|_{L^p(x\in\mathbb{R}^2:\,\mathrm{dist}(x,\mathrm{supp}(\omega_n))\le R_n)} \lesssim \frac{1}{N_n^{100}},$$
$$\|P_{N_n}\omega_n\|_{L^p(x\in\mathbb{R}^2:\,\mathrm{dist}(x,\mathrm{supp}(\omega_n))> R_n)} \lesssim \frac{1}{N_n^{100}}.$$

Clearly,

$$\|P_{N_n}\omega\|_p^p \ge \|P_{N_n}\omega_n\|_p^p - \frac{\mathrm{const}}{N_n^{100}}.$$

Thus (5.24) follows. ∎

The last two theorems are on the ill-posedness of 3D Euler in Besov spaces. We omit the proof since it mimics the ones made in preceding sections.

Theorem 5.3. Consider the 3D incompressible Euler equation in vorticity form:

$$\begin{cases} \partial_t \omega + (u\cdot\nabla)\omega = (\omega\cdot\nabla)u, \quad 0 < t \le 1,\ x = (x_1, x_2, z) \in \mathbb{R}^3; \\ u = -\Delta^{-1}\nabla\times\omega, \\ \omega\Big|_{t=0} = \omega_0. \end{cases} \tag{5.29}$$

For any axisymmetric vorticity $\omega_0^{(g)} \in C_c^\infty(\mathbb{R}^3)$, any $\epsilon > 0$, and any $1 < p < \infty$, $1 < q \le \infty$, we can find a C^∞ perturbation $\omega_0^{(p)} : \mathbb{R}^3 \to \mathbb{R}^3$ such that the following hold true:

(1) The perturbation is very small:

$$\|\omega_0^{(p)}\|_{L^1(\mathbb{R}^3)} + \|\omega_0^{(p)}\|_{L^\infty(\mathbb{R}^3)} + \|\omega_0^{(p)}\|_{B^{\frac{3}{p}}_{p,q}(\mathbb{R}^3)} < \epsilon.$$

(2) Let $\omega_0 = \omega_0^{(g)} + \omega_0^{(p)}$. Let u_0 be the velocity corresponding to the initial vorticity ω_0. We have $u_0 \in B^{\frac{3}{p}+1}_{p,q}(\mathbb{R}^3) \cap C^\infty(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.

(3) Corresponding to ω_0, there exists a unique solution $\omega = \omega(t)$ to (5.29) on the whole time interval $[0, 1]$ such that

$$\sup_{0\le t\le 1} (\|\omega(t,\cdot)\|_{L^\infty} + \|\omega(t,\cdot)\|_{L^1}) < \infty.$$

Moreover, $\omega \in C^\infty$ and $u \in C^\infty$ so that the solution is actually classical.

(4) For any $0 < t_0 \le 1$, we have

$$\text{ess-sup}_{0<t\le t_0} \|\omega(t,\cdot)\|_{\dot{B}^{\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)} = +\infty.$$

Theorem 5.4. For any axisymmetric vorticity $\omega_0^{(g)} \in C_c^\infty(\mathbb{R}^3)$, any $\epsilon > 0$, and any $1 < p < \infty$, $1 < q \le \infty$, we can find a perturbation $\omega_0^{(p)} : \mathbb{R}^3 \to \mathbb{R}^3$ such that the following hold true:

(1) $\omega_0^{(p)}$ is compactly supported, continuous, and

$$\|\omega_0^{(p)}\|_{L^\infty(\mathbb{R}^3)} + \|\omega_0^{(p)}\|_{B^{\frac{3}{p}}_{p,q}(\mathbb{R}^3)} < \epsilon.$$

(2) Let $\omega_0 = \omega_0^{(g)} + \omega_0^{(p)}$. Corresponding to ω_0 there exists a unique solution $\omega = \omega(t,x)$ to the Euler equation (1.3) on the time interval $[0,1]$ satisfying

$$\sup_{0\le t\le 1} \|\omega(t,\cdot)\|_\infty < \infty,$$
$$\text{supp}(\omega(t,\cdot)) \subset \{x,\ |x| < R\}, \quad \forall\, 0 \le t \le 1, \tag{5.30}$$

where $R > 0$ is some constant. Furthermore, $\omega \in C_t^0 C_x^0$ and $u \in C_t^0 L_x^2 \cap C_t^0 C_x^\alpha$ for any $\alpha < 1$.

(3) $\omega(t)$ has additional local regularity in the following sense: there exists $x_* \in \mathbb{R}^3$ such that for any $x \ne x_*$; there exists a neighborhood $N_x \ni x$, $t_x > 0$ such that $w(t) \in C^\infty(N_x)$ for any $0 \le t \le t_x$.

(4) For any $0 < t_0 \le 1$, we have

$$\text{ess-sup}_{0<t\le t_0} \|\omega(t,\cdot)\|_{\dot{B}^{\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)} = +\infty. \tag{5.31}$$

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