



Sharp One Component Regularity for Navier–Stokes

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Abstract

We consider the conditional regularity of mild solution v to the incompressible Navier–Stokes equations in three dimensions. Let $e \in \mathbb{S}^2$ and $0 < T^* < \infty$. CHEMIN and ZHANG (Ann Sci Éc Norm Supér 49:131–167, 2016) proved the regularity of v on $(0, T^*]$ if there exists $p \in (4, 6)$ such that

$$\int_0^{T^*} \|v \cdot e\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p dt < \infty.$$

CHEMIN ET AL. (Arch Ration Mech Anal 224(3):871–905, 2017) extended the range of p to $(4, \infty)$. In this article we settle the case $p \in [2, 4]$. Our proof also works for the case $p \in (4, \infty)$.

1. Introduction

Consider the Cauchy problem of the three-dimensional incompressible Navier–Stokes equations on \mathbb{R}^3

$$\begin{cases} \partial_t v + v \cdot \nabla v - \Delta v + \nabla P = 0, & x \in \mathbb{R}^3, t > 0, \\ \operatorname{div} v = 0, & x \in \mathbb{R}^3, t > 0, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^3. \end{cases} \quad (1.1)$$

Here $v : [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ represents the velocity field of the fluid flow and $P : [0, \infty) \times \mathbb{R}^3 \rightarrow \mathbb{R}$ denotes the pressure. The first two terms represent Newton’s acceleration law in Eulerian coordinates whilst the term $-\nabla P$ corresponds to the fluid stress. For the dissipation term we have set the kinematic viscosity to be 1 for simplicity. Since universal physical laws should be independent of the underlying units (dimension), equation (1.1) remains invariant under natural scaling transformations. If (v, P) is a solution to (1.1), then for any $\lambda > 0$,

$$v_\lambda(t, x) = \lambda v(\lambda^2 t, \lambda x), \quad P_\lambda(t, x) = \lambda^2 P(\lambda^2 t, \lambda x)$$

is also a solution corresponding to rescaled initial data $v_{0,\lambda}(x) = \lambda v_0(\lambda x)$. Such scaling transformation determines the critical space (norm) for Navier–Stokes and plays a fundamental role in the wellposedness theory.

The existence of global weak solutions to (1.1) is known since the famous work of Leray [11] (see also Hopf [9] for the bounded domain case) for initial data $v_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} v_0 = 0$. The uniqueness and global regularity of Leray–Hopf weak solutions is still one of the most challenging open problems. On the other hand, there exists a vast literature on finite time blowup or non-blowup criterions for local strong solutions. For instance, the Prodi–Serrin–Ladyzhenskaya criterion says that if

$$\int_0^{T^*} \|v(t, \cdot)\|_{L^q}^p dt < \infty, \quad \frac{2}{p} + \frac{3}{q} = 1$$

for some $3 \leq q \leq \infty$, then v is still regular at time $T^* < \infty$, based on a series of important works [5, 7, 10, 15–17]. We point out that the quantity involved is a dimensionless one with respect to the natural scaling of the Navier–Stokes equations. Later on, many efforts have been made on weakening the above criterion by imposing constraints only on partial components or directional derivatives of velocity field; see, for instance, [1, 2, 6, 8, 13, 14, 18] and the references therein.

In a recent work [3], CHEMIN and ZHANG initiated a programme to prove the regularity of solutions by only imposing the following assumption:

$$I_p(v \cdot e) \triangleq \int_0^{T^*} \|v \cdot e\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p dt < \infty.$$

Here $e \in \mathbb{S}^2$ and $2 \leq p < \infty$. The remarkable feature of the quantity $I_p(v \cdot e)$ lies in the fact that it is a *dimensionless* quantity which only involves *one component* of the velocity field. As an important step towards this line of research, CHEMIN and ZHANG [3] succeeded in the case of $4 < p < 6$, which was subsequently extended by CHEMIN ET AL. [4] to $4 < p < \infty$. In this article, we give a streamlined proof for all $2 \leq p < \infty$. More precisely, we prove the following theorem:

Theorem 1.1. *Let $v_0 \in \dot{H}^{\frac{1}{2}}$ with $\nabla \cdot v_0 = 0$ and $\Omega_0 = \nabla \times v_0 \in L^{r_0}$ for some $1 < r_0 < 2$. Let $0 < T^* < \infty$ and let*

$$v \in C\left([0, T^*]; \dot{H}^{\frac{1}{2}}\right) \cap L^2\left([0, T^*]; \dot{H}^{\frac{3}{2}}\right)$$

be the unique local mild solution to the three-dimensional Navier–Stokes equations (1.1) with initial data v_0 . If $I_p(v \cdot e) < \infty$ for some $p \in [2, \infty)$ and $e \in \mathbb{S}^2$, then $v \in C([0, T^]; \dot{H}^{\frac{1}{2}}) \cap L^2([0, T^*]; \dot{H}^{\frac{3}{2}})$ and must be regular up to time T^* ; more precisely*

$$\max_{0 \leq t \leq T^*} \left(\|v(t)\|_{\dot{H}^{\frac{1}{2}}} + \|\Omega(t)\|_{L^{r_0}} \right) < \infty,$$

and for any $0 < t_0 < T^$,*

$$\max_{t_0 \leq t \leq T^*} \left(\|v(t)\|_{\dot{H}^1} + \|\nabla \Omega(t)\|_{L^{r_0}} \right) < \infty.$$

Remark 1.2. By standard smoothing estimates, the solution v enjoys higher regularity: $v \in \dot{H}^m$, $\Omega \in W^{m,r_0}$ for any $m \geq 1$ and any $0 < t \leq T^*$.

Remark 1.3. In order to simplify the presentation we did not try to lower down the regularity requirement on initial data although this can certainly be optimised by a more refined analysis. We will pursue this interesting issue elsewhere. In view of the two-dimensional Biot–Savart law it is of some importance that $\Omega \in L^{r_0}$ for some $1 < r_0 < 2$. The bulk of our analysis will focus on the case $2 \leq p \leq 4$ which was previously open. The case $4 < p < \infty$ can also be treated by our analysis and is included in a later section. It should be noted that in [4] the case $4 < p < \infty$ is treated under the assumption that the initial vorticity $\Omega_0 \in L^{\frac{3}{2}} \cap L^2$. By Sobolev embedding the condition $\Omega_0 \in L^{\frac{3}{2}}$ implies that the initial velocity $v_0 \in \dot{H}^{\frac{1}{2}}$. In comparison with [4] our analysis in the regime $4 < p < \infty$ offers a slight relaxation since we only require $v_0 \in \dot{H}^{\frac{1}{2}}$ with $\Omega_0 \in L^{r_0}$ for *some* $r_0 \in (1, 2)$.

We now give a brief overview of the proof and explain some main steps. Without loss of generality, we assume $e = (0, 0, 1)$ throughout this paper and thus the dimensionless quantity $I_p(v \cdot e)$ in the above theorem becomes

$$I_p(v^3) = \int_0^{T^*} \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p dt.$$

Step 1 Reduction to the two-dimensional vorticity $\omega = -\partial_2 v^1 + \partial_1 v^2$.

For given initial data $v_0 \in \dot{H}^{\frac{1}{2}}$, thanks to the smoothing estimates, we have $v(t) \in \dot{H}^s$ for any $s \geq 1/2$ immediately on the short time interval $(0, \eta_0]$ for some $\eta_0 > 0$ sufficiently small. Therefore by a shift of the time origin if necessary we may assume without loss of generality that $v_0 \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^1$. By a similar reasoning we may also assume $\Omega_0 \in W^{4,r_0}$. As a first step, we show that (see Proposition 3.1) for any $T > 0$,

$$\max_{0 \leq t \leq T} \|v\|_{\dot{H}^1} + \|\nabla v\|_{L^2((0,T], \dot{H}^1)} \leq 2\|v_0\|_{\dot{H}^1} \cdot e^{\text{const} \cdot M(T)},$$

where

$$M(T) = \int_0^T \|\omega\|_{\dot{H}^{-\frac{1}{2} + \frac{2}{p}}}^p dt + \int_0^T \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p dt.$$

Whilst the controlling quantity $M(T)$ works for the full range $p \in [2, \infty)$, it should be noted that for $4 < p < \infty$, $-\frac{1}{2} + \frac{2}{p} < 0$ and the controlling norm for ω is a negative Sobolev norm which is not convenient to use (due to low frequencies) in later computations. For this reason we also prove in Proposition 3.1 (see Remark 3.3) that for $4 < p < \infty$ the quantity $M(T)$ can be replaced by

$$\tilde{M}(T) = T \cdot \left(1 + \sup_{0 \leq t \leq T} \|\omega(t)\|_{\tilde{r}} \right)^p + \int_0^T \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p dt,$$

where \tilde{r} can be any number satisfying $\frac{1}{2} < \frac{1}{\tilde{r}} < \frac{2}{3} \left(1 - \frac{1}{p} \right)$.

The preceding argument then establishes a sharp non-blowup criterion: v is regular on $[0, T^*]$ if $M(T^*) < \infty$ (resp. $\tilde{M}(T^*) < \infty$ for $4 < p < \infty$). In view of the assumption on I_p in Theorem 1.1, it then suffices for us to prove

$$\tilde{I}_p(\omega) \triangleq \int_0^{T^*} \|\omega\|_{\dot{H}^{-\frac{1}{2} + \frac{2}{p}}}^p dt < \infty.$$

For $p \in (4, \infty)$, it suffices to control

$$\sup_{0 \leq t \leq T^*} \|\omega(t)\|_{\tilde{r}}$$

for some $\frac{1}{2} < \frac{1}{\tilde{r}} < \frac{2}{3}(1 - \frac{1}{p})$. We also note that the propagation of regularity of Ω in W^{4,r_0} is not a problem thanks to the control of $\|v\|_{\dot{H}^{\frac{1}{2}} \cap \dot{H}^1}$ (see Proposition 3.1).

Step 2 Anisotropic decomposition of the velocity.

A remarkable idea introduced in CHEMIN–ZHANG in [3] is to use the decomposition of the velocity field along horizontal and vertical directions and use the two-dimensional vorticity ω and v^3 as governing unknowns. Denote

$$\nabla_h = (\partial_1, \partial_2), \quad \nabla_h^\perp = (-\partial_2, \partial_1) \quad \text{and} \quad \Delta_h = \partial_1^2 + \partial_2^2.$$

Then, by using Biot–Savart’s law in the horizontal variables, we have

$$\begin{aligned} v_{\text{curl}}^h &= \nabla_h^\perp \Delta_h^{-1} \omega, \quad v_{\text{div}}^h = \nabla_h \Delta_h^{-1} \partial_3 v^3, \\ v^h &= v_{\text{curl}}^h - v_{\text{div}}^h \\ &= \begin{pmatrix} -\partial_2 \Delta_h^{-1} \omega - \partial_1 \Delta_h^{-1} \partial_3 v^3 \\ \partial_1 \Delta_h^{-1} \omega - \partial_2 \Delta_h^{-1} \partial_3 v^3 \end{pmatrix}, \end{aligned} \tag{1.2}$$

where

$$\omega = \partial_1 v^2 - \partial_2 v^1.$$

It is easy to check that

$$\partial_t \omega + v \cdot \nabla \omega - \Delta \omega = \partial_3 v^3 \omega + \partial_2 v^3 \partial_3 v^1 - \partial_1 v^3 \partial_3 v^2; \tag{1.3}$$

$$\begin{aligned} \partial_t \partial_k v^3 + v \cdot \nabla \partial_k v^3 - \Delta \partial_k v^3 &= -\partial_k v \cdot \nabla v^3 \\ &+ \partial_3 \partial_k \Delta^{-1} \left(\sum_{i,j=1}^3 \partial_j v^i \partial_i v^j \right), \quad k = 1, 2, 3. \end{aligned} \tag{1.4}$$

¹ In [3], CHEMIN–ZHANG considered $(\omega, \partial_3 v^3)$ as the governing unknowns which is very natural in view of the physical picture that v^3 should be slowly changing in the vertical direction. In order to control horizontal derivatives Chemin–Zhang used anisotropic spaces carrying positive and negative fractional derivatives in horizontal and vertical directions respectively. In this paper we found it more convenient to work with the full gradient ∇v^3 in order to trade off fractional derivatives in the vertical direction.

Thanks to Biot–Savart’s law, the above system written for (ω, v^3) is equivalent to the original Navier–Stokes system for $v = (v^1, v^2, v^3)$.

Step 3 Estimate of (ω, v^3) . This is the main part of our analysis. For fixed $2 \leq p < \infty$, we shall choose $1 < r < 2$, r sufficiently close to 2, and work with the norms

$$\|\omega\|_{L^r(\mathbb{R}^3)}, \quad \||\nabla_h|^{-\delta}\nabla v^3\|_2,$$

where

$$\delta = \delta(r) = \frac{3}{r} - \frac{3}{2}.$$

It is not difficult to check that the above two norms have the same scaling as $\|v\|_{\dot{H}^{1-\frac{3}{r}+\frac{3}{2}}} \sim \|v\|_{\dot{H}^{1-}}$ for $r = 2-$.² These norms are certainly well-defined since for fixed r_0 (recall the initial vorticity $\Omega_0 \in L^{r_0}$ by assumption)

$$\begin{aligned} \|\omega\|_r + \||\nabla_h|^{-\delta}\nabla v\|_2 &\lesssim \|\Omega\|_r + \||\nabla_h|^{-\delta}\Omega\|_2 \\ &\lesssim \|\Omega\|_{W^{4,r_0}}, \end{aligned}$$

if we take r sufficiently close to 2.

There are several reasons why we choose the norm $\|\omega\|_{2-}$. Firstly it is natural to choose $\|\omega\|_p$ norm for some p since in (1.3) the convection term $v \cdot \nabla\omega$ will not enter the estimates due to incompressibility. Secondly, if we compute the time derivative of $\|\omega\|_2^2$, then by using (1.3), we need to treat the nonlinear terms such as

$$\int_{\mathbb{R}^3} \partial_2 v^3 \partial_3 v^1 \omega dx = - \int_{\mathbb{R}^3} \partial_2 v^3 \partial_3 \partial_2 \Delta_h^{-1} \omega \cdot \omega dx - \int_{\mathbb{R}^3} \partial_2 v^3 \partial_3 \partial_1 \Delta_h^{-1} \partial_3 v^3 \cdot \omega dx.$$

Note that the term $\partial_3 \partial_2 \Delta_h^{-1} \omega$ scales as $|\nabla_h|^{-1} \partial_3 \omega$ for which two-dimensional L^∞ embedding cannot map back to L^2 . For this reason one must resort to $\|\omega\|_r$ for some $r < 2$. By a similar reasoning, for v^3 some negative regularity is needed in the horizontal direction. This is the one of the reasons for choosing the governing norm as $\||\nabla_h|^{-\delta}\nabla v^3\|_2$.

There are a myriad of technical issues in connection with the aforementioned borderline situations. To get a glimpse into this, take for example $p = 2$ for which $I_p(v^3)$ becomes

$$I_2(v^3) = \int_0^{T^*} \|v^3\|_{\dot{H}^{\frac{3}{2}}}^2 dt.$$

When computing the time evolution of $\|\omega\|_r$ -norm, we need to estimate a term such as (see Section 4 for more details)

$$I_{22} := \int \partial_2 v^3 \partial_2 \Delta_h^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx.$$

² For any quantity X when there is no ambiguity we shall use the notation $X+$ to denote $X + \varepsilon$ with sufficiently small ε . The notation $X-$ is similarly defined.

The only control we have on ω is $\|\omega\|_r$ and $\|\nabla\omega\|_r$ (from the diffusion term). Therefore by using Sobolev embedding and Hölder it is quite natural to bound the above as

$$|I_{22}| \lesssim \|\partial_2 v^3\|_{L^2_{x_1, x_2} L^\infty_{x_3}} \|\nabla\omega\|_r \cdot \|\omega\|_r^{r-1}.$$

However, even though the quantity $\|\partial_2 v^3\|_{L^2_{x_1, x_2} L^\infty_{x_3}}$ scales the same way as $\|v^3\|_{\dot{H}^{\frac{3}{2}}}$, it cannot be bounded by it due to the lack of embedding of $\dot{H}^{\frac{1}{2}}$ into L^∞ in 1D. To get around this problem we perform a refined Littlewood–Paley decomposition in the vertical direction and manage to obtain a logarithmic inequality of the form

$$\begin{aligned} & \int \partial_2 v^3 |\nabla_h|^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx \\ & \lesssim \sqrt{\log(10 + \|\nabla_h|^{-\delta} \partial v^3\|_{L^2} + \|\omega\|_{L^r})} (\|v^3\|_{\dot{H}^{\frac{3}{2}}} + 1) \|\nabla\omega\|_{L^r} \|\omega\|_{L^r}^{r-1} \\ & \quad + \left(\|\nabla_h|^{-\delta} \partial^2 v^3\|_{L^2}^{\frac{1}{2} + \delta + \varepsilon_1} + \|\nabla_h|^{-\delta} \partial^2 v^3\|_{L^2}^{\frac{1}{2} + \delta - \varepsilon_1} \right) \\ & \quad \cdot \frac{1}{1 + \|\omega\|_{L^r}^{100}} \|\nabla\omega\|_{L^r} \|\omega\|_{L^r}^{\frac{r}{2} - 1}. \end{aligned}$$

Such estimates turn out to be crucial for the Gronwall argument to work. There are many other technical issues which cannot be mentioned in this short introduction. In any case by a very involved analysis on the time evolution of these norms and taking advantage of the a priori finiteness of $I_p(v^3)$, we obtain uniform control of $\|\omega\|_r + \|\nabla_h|^{-\delta} \nabla v^3\|_2$ on the time interval $[0, T^*]$.

Step 4 Estimate of $\tilde{I}_p(\omega)$ for $2 \leq p \leq 4$. Thanks to the estimate of $\|\omega\|_r$ in Step 3, the case $4 < p < \infty$ is already proven with the help of Proposition 3.1 and Remark 3.3. To finish the proof of the main theorem it remains to estimate $\tilde{I}_p(\omega)$ for $2 \leq p \leq 4$. Our strategy is to first take r sufficiently close to 2 for each fixed $2 \leq p \leq 4$, and then use the finiteness of the scaling-above-critical quantity $\|\omega\|_{L_t^\infty L_x^r}$ together with $\|\nabla(|\omega|^{\frac{r}{2}})\|_{L_t^2 L_x^2}$ obtained in Step 3 to bound the critical (dimension-less) quantity $\tilde{I}_p(\omega)$. Such a bound is certainly expected from a scaling heuristic since both $\|\omega\|_{L_t^\infty L_x^{2-}}$ and $\|\nabla(|\omega|^{1-})\|_{L_t^2 L_x^2}$ carries almost \dot{H}^1 scaling of velocity. This then concludes the proof of the main theorem.

The rest of this paper is organised as follows. In Section 2 we set up some notation and collect a few useful lemmas. In Section 3 we prove Proposition 3.1 which reduces matters to the control of the horizontal vorticity ω . In Section 4 and Section 5, we obtain a priori estimates of $\|\omega\|_r$ and $\|\nabla_h|^{-\delta} \nabla v^3\|_2$ for the case $2 \leq p \leq 4$. In Section 5.1 we explain how to do the case $4 < p < \infty$. The final section is devoted to the proof of the main theorem.

2. Notation and Preliminaries

Let us first recall some Sobolev type inequalities which are relevant to the L^2 estimate for $|f|^{\frac{r}{2}}$ and $\nabla|f|^{\frac{r}{2}}$. The following Lemma will often be used without explicit mentioning:

Lemma 2.1. *Let the dimension $n \geq 1$. Fix $k \in \{1, \dots, n\}$. Let $1 < r < \infty$. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies $\partial_k f \in C^0$ and $f, \partial_{kk} f \in L^r(\mathbb{R}^n)$. Then $\partial_k(|f|^{\frac{r}{2}}) \in L^2(\mathbb{R}^n)$ and*

$$\begin{aligned} - \int_{\mathbb{R}^3} \partial_{kk} f \cdot |f|^{r-2} f dx &= (r - 1) \int_{f \neq 0} |\partial_k f|^2 |f|^{r-2} dx \\ &= \frac{4(r - 1)}{r^2} \|\partial_k(|f|^{\frac{r}{2}})\|_{L^2(f \neq 0)}^2 \\ &= \frac{4(r - 1)}{r^2} \|\partial_k(|f|^{\frac{r}{2}})\|_{L^2}^2. \end{aligned}$$

Furthermore for $1 < r \leq 2$,

$$\|\partial_k f\|_r \leq \frac{2}{r} \|\partial_k(|f|^{\frac{r}{2}})\|_2 \cdot \|f\|_r^{1-\frac{r}{2}}. \tag{2.1}$$

For the first group of equalities we also have the following vector-valued version. Suppose $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies $\partial_k g \in C^0$ and $g, \partial_{kk} g \in L^r(\mathbb{R}^n)$. Then $\partial_k(|g|^{\frac{r}{2}}) \in L^2(\mathbb{R}^n)$ and

$$- \int \partial_{kk} g \cdot |g|^{r-2} g dx \geq \frac{4(r - 1)}{r^2} \|\partial_k(|g|^{\frac{r}{2}})\|_{L^2}^2.$$

Remark 2.2. Dividing both sides of the first group of equalities by the factor $(r - 1)$ and taking a suitable limit $r \rightarrow 1$ (under some natural assumptions on f), one can derive the analogue of the above for the end-point $r = 1$ as

$$-\frac{1}{4} \int \partial_{kk} f \operatorname{sgn}(f) \log |f| dx = \|\partial_k(|f|^{\frac{1}{2}})\|_2^2.$$

For a positive function f , this exactly corresponds to the flux (Fisher information) of the entropy functional $\int(-f \log f)$. One should note that in this spirit the entropy is a natural limit of dissipation law for $|f|^{\frac{r}{2}}$ as $r \rightarrow 1$. This gives another explanation as to why $-f \log f$ should appear as natural monotone quantities.

Proof. It is the regime $1 < r < 2$ which merits a careful analysis. The first equality follows by a careful integration by parts (using smooth spatial cut-offs and regularising $|f|$ by $(|f|^2 + \varepsilon^2)^{\frac{1}{2}}$) and the fact that $\{x : f(x) = 0, \partial_k f(x) \neq 0\}$ has Lebesgue measure zero. The second equality is trivial on the set $f \neq 0$. For the third equality, observe for $\varepsilon \rightarrow 0+$,

$$\begin{aligned} f_\varepsilon &= \left(|f|^2 + \varepsilon^2\right)^{\frac{r}{4}} \rightarrow |f|^{\frac{r}{2}}, \quad \text{a.e. in } \mathbb{R}^n, \\ \partial_k f_\varepsilon &= \frac{r}{2} \left(|f|^2 + \varepsilon^2\right)^{\frac{r}{4}-1} f \partial_k f, \\ \|\partial_k f_\varepsilon\|_2 &\leq \frac{r}{2} \|\partial_k f \cdot |f|^{\frac{r}{2}-1}\|_{L^2(f \neq 0)} = \|\partial_k(|f|^{\frac{r}{2}})\|_{L^2(f \neq 0)}. \end{aligned}$$

It follows easily that $\partial_k(|f|^{\frac{r}{2}}) \in L^2$, and

$$\|\partial_k(|f|^{\frac{r}{2}})\|_2 \leq \|\partial_k(|f|^{\frac{r}{2}})\|_{L^2(f \neq 0)}.$$

Hence the equality holds.

For the inequality (2.1), one recalls that the set $\{x : f(x) = 0, \partial_k f(x) \neq 0\}$ has Lebesgue measure zero, and hence

$$\begin{aligned} \int |\partial_k f|^r &= \int_{f \neq 0} |\partial_k f|^r \cdot |f|^{\frac{r(r-2)}{2}} \cdot |f|^{\frac{r(2-r)}{2}} \\ &\leq \left(\int_{f \neq 0} |\partial_k f|^2 |f|^{r-2} \right)^{\frac{r}{2}} \cdot \left(\int |f|^r dx \right)^{\frac{2-r}{2}}. \end{aligned}$$

For the last inequality (WLOG again assume $1 < r < 2$), one first notes that $\partial_k(|g^j|^{\frac{r}{2}}) \in L^2$ for each component g^j . Thus $\partial_k(|g|^{\frac{r}{2}}) = \partial_k((\sum_{j=1}^n |g^j|^{\frac{r}{2}})^{\frac{4}{r}})^{\frac{r}{4}} \in L^2$ by using the chain rule. The desired inequality then follows by an argument similar to the scalar case. We omit the details. \square

For any $1 \leq p \leq \infty$ and measurable $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we will use $\|f\|_{L^p(\mathbb{R}^n)}$, $\|f\|_{L^p}$ or simply $\|f\|_p$ to denote the usual L^p norm. For a vector valued function $f = (f^1, \dots, f^m)$, we still denote $\|f\|_p := \sum_{j=1}^m \|f^j\|_p$.

For any $0 < T < \infty$ and any Banach space \mathbb{B} with norm $\|\cdot\|_{\mathbb{B}}$, we will use the notation $C([0, T], \mathbb{B})$ or $C_t^0 \mathbb{B}$ to denote the space of continuous \mathbb{B} -valued functions endowed with the norm

$$\|f\|_{C([0, T], \mathbb{B})} := \max_{0 \leq t \leq T} \|f(t)\|_{\mathbb{B}}.$$

Also, for $1 \leq p \leq \infty$, we define

$$\|f\|_{L_t^p \mathbb{B}([0, T])} := \|\|f(t)\|_{\mathbb{B}}\|_{L_t^p([0, T])}.$$

We shall adopt the following convention for the Fourier transform:

$$\begin{aligned} \hat{f}(\xi) &= \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx; \\ f(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix \cdot \xi} d\xi. \end{aligned}$$

For $s \in \mathbb{R}$, the fractional Laplacian $|\nabla|^s$ then corresponds to the Fourier multiplier $|\xi|^s$ defined as

$$\widehat{|\nabla|^s f}(\xi) = |\xi|^s \hat{f}(\xi),$$

whenever it is well-defined. For $s \geq 0, 1 \leq p < \infty$, we define the semi-norm and norms:

$$\begin{aligned} \|f\|_{\dot{W}^{s,p}} &= \| |\nabla|^s f \|_p, \\ \|f\|_{W^{s,p}} &= \| |\nabla|^s f \|_p + \|f\|_p. \end{aligned}$$

When $p = 2$ we denote $\dot{H}^s = \dot{W}^{s,2}$ and $H^s = W^{s,2}$ in accordance with the usual notation.

For any two quantities X and Y , we denote $X \lesssim Y$ if $X \leq CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \geq CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$

and $Y \lesssim X$. The dependence of the constant C on other parameters or constants are usually clear from the context and we will often suppress this dependence. We shall denote $X \lesssim_{Z_1, Z_2, \dots, Z_k} Y$ if $X \leq CY$ and the constant C depends on the quantities Z_1, \dots, Z_k .

For any two quantities X and Y , we shall denote $X \ll Y$ if $X \leq cY$ for some sufficiently small constant c . The smallness of the constant c is usually clear from the context. The notation $X \gg Y$ is similarly defined. Note that our use of \ll and \gg here is *different* from the usual Vinogradov notation in number theory or asymptotic analysis.

We will need to use the Littlewood–Paley (LP) frequency projection operators. To fix the notation, let ϕ_0 be a radial function in $C_c^\infty(\mathbb{R}^n)$ and satisfy

$$0 \leq \phi_0 \leq 1, \quad \phi_0(\xi) = 1 \text{ for } |\xi| \leq 1, \quad \phi_0(\xi) = 0 \text{ for } |\xi| \geq 7/6.$$

Let $\phi(\xi) := \phi_0(\xi) - \phi_0(2\xi)$ which is supported in $\frac{1}{2} \leq |\xi| \leq \frac{7}{6}$. For any $f \in \mathcal{S}(\mathbb{R}^n)$, $j \in \mathbb{Z}$, define

$$\begin{aligned} \widehat{P_{\leq j} f}(\xi) &= \phi_0(2^{-j}\xi) \hat{f}(\xi), \\ \widehat{P_j f}(\xi) &= \phi(2^{-j}\xi) \hat{f}(\xi), \quad \xi \in \mathbb{R}^n. \end{aligned}$$

We will denote $P_{>j} = I - P_{\leq j}$ (I is the identity operator) and for any $-\infty < a < b < \infty$, denote $P_{[a,b]} = \sum_{a \leq j \leq b} P_j$. Sometimes for simplicity of notation (and when there is no obvious confusion) we will write $f_j = P_j f$, $f_{\leq j} = P_{\leq j} f$ and $f_{a \leq \cdot \leq b} = \sum_{j=a}^b f_j$. By using the support property of ϕ , we have $P_j P_{j'} = 0$ whenever $|j - j'| > 1$.

Sometimes it is convenient to use “fattened” Littlewood–Paley projection operators \widetilde{P}_j and $\widetilde{P}_{\ll j}$ defined by

$$\begin{aligned} \widehat{\widetilde{P}_j f}(\xi) &= \phi_1(2^{-j}\xi) \hat{f}(\xi), \\ \widehat{\widetilde{P}_{\ll j} f}(\xi) &= \phi_2(2^{-j}\xi) \hat{f}(\xi), \end{aligned}$$

where $\phi_1, \phi_2 \in C_c^\infty$ has support in $\{|\xi| \sim 1\}$ and $\{|\xi| \ll 1\}$ respectively. As a model case one can consider $\text{supp}(\phi_1) \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$ whereas $\text{supp}(\phi_2) \subset \{|\xi| \leq \frac{1}{4}\}$. The precise numerology does not play much role in the following computations and estimates as long as their supports stay well separated.

In section 5 we will use a simple (yet powerful) lemma which gives trilinear para-product decomposition of product of functions. To simplify the notation we shall write $\int_{\mathbb{R}^n} (\cdot) dx$ simply as $\int (\cdot)$.

Lemma 2.3. (Trilinear paraproduct decomposition) *For any $f, g, h \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\begin{aligned} \int fgh &= \sum_j \int (f_j g_{[j-3, j+3]} h_{[j-10, j+5]} + f_j g_{[j-3, j+3]} h_{<j-10} \\ &\quad + f_j g_{<j-3} h_{[j-2, j+2]} + f_{<j-3} g_j h_{[j-2, j+2]}). \end{aligned}$$

To simplify the notation, we write the above as

$$\int fgh = \sum_j \int (\tilde{P}_j f \tilde{P}_j g \tilde{P}_j h + \tilde{P}_j f \tilde{P}_j g \tilde{P}_{\ll j} h + \tilde{P}_j f \tilde{P}_{\ll j} g \tilde{P}_j h + \tilde{P}_{\ll j} f \tilde{P}_j g \tilde{P}_j h),$$

where \tilde{P}_j and $\tilde{P}_{\ll j}$ have frequency localized to $\{|\xi| \sim 2^j\}$ and $\{|\xi| \ll 2^j\}$, respectively.

Proof. By frequency localization, we have

$$\begin{aligned} \int fgh &= \sum_j \left(\int f_j g_{[j-3, j+3]} h + \int f_j g_{< j-3} h + \int f_j g_{> j+3} h \right) \\ &= \sum_j \int f_j g_{[j-3, j+3]} h_{[j-10, j+5]} + \sum_j \int f_j g_{[j-3, j+3]} h_{< j-10} \\ &\quad + \sum_j \int f_j g_{< j-3} h_{[j-2, j+2]} + \sum_k \int f_{< k-3} g_k h. \end{aligned}$$

Writing the last term as

$$\sum_j \int f_{< j-3} g_j h_{[j-2, j+2]}$$

then yields the result. \square

3. Reduction to ω

In this section we establish a non-blowup criterion involving only v^3 and the horizontal vorticity $\omega = -\partial_2 v^1 + \partial_1 v^2$.

Proposition 3.1. *Let $0 < T < \infty$ and $v \in C_t^0 \dot{H}^{\frac{1}{2}}([0, T]) \cap L_t^2 \dot{H}^{\frac{3}{2}}([0, T])$ be a local mild solution to system (1.1) with $v_0 \in \dot{H}^{\frac{1}{2}}$. Let $p \in [2, \infty)$. Assume that*

$$M(T) = \int_0^T \|\omega\|_{\dot{H}^{-\frac{1}{2} + \frac{2}{p}}}^p dt + \int_0^T \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p dt < \infty.$$

Then the local solution v can be continued past T and remains regular on $(0, T + \delta]$ for some $\delta > 0$. For any $0 < t_0 < T$,

$$\max_{t_0 \leq t \leq T} \|v\|_{\dot{H}^1} + \|\nabla v\|_{L^2([0, T], \dot{H}^1)} \leq 2\|v(t_0)\|_{\dot{H}^1} \cdot e^{\text{const} \cdot M(T)} < \infty.$$

Furthermore if $\Omega_0 = \nabla \times v_0 \in L^{r_0}$ for some $r_0 \in (1, 2]$, then $\Omega \in C([0, T], L_x^{r_0})$, and for any $0 < t_0 < T$,

$$\sup_{t_0 \leq t \leq T} (\|\nabla \Omega(t)\|_{L^{r_0}} + \|\nabla^4 \Omega(t)\|_{L^{r_0}}) < \infty.$$

Remark 3.2. For $p \in [2, 4]$, one can replace $\|\omega\|_{\dot{H}^{-\frac{1}{2}+\frac{2}{p}}}$ by the weaker norm $\| |\nabla_h|^{-\frac{1}{2}+\frac{2}{p}} \omega \|_{L^2}$.

Remark 3.3. For $p \in (4, \infty)$, one can replace the quantity $M(T)$ by

$$\tilde{M}(T) = T \cdot \left(1 + \sup_{0 \leq t \leq T} \|\omega(t)\|_r \right)^p + \int_0^T \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p dt,$$

where r satisfies $\frac{1}{2} < \frac{1}{r} < \frac{2}{3}(1 - \frac{1}{p})$. The implied constants in the Gronwall will also depend on r but we shall suppress this dependence. The proof is a simple modification of the corresponding argument for $M(T)$. By examining the estimate of K_3 in the proof below, it is clear that

$$\begin{aligned} |K_3| &\lesssim \|P_{<1} \mathcal{R}_2(\omega)\|_r \|\tilde{P}_{<1}(\partial v \cdot \partial v)\|_{r-1} + \| |\nabla|^{-\frac{1}{2}+\frac{2}{p}} P_{\geq 1} \mathcal{R}_2(\omega) \|_2 \\ &\quad \cdot \| |\nabla|^{\frac{1}{2}-\frac{2}{p}}(\partial v \cdot \partial v) \|_2 \\ &\lesssim \|\omega\|_r \cdot \|\nabla v\|_2^2 + \|\omega\|_r \cdot \|\Delta v\|_2^{2-\frac{2}{p}} \cdot \|\nabla v\|_2^{\frac{2}{p}} \\ &\leq \frac{1}{100} \|\Delta v\|_2^2 + C \cdot (\|\omega\|_r + \|\omega\|_r^p) \cdot \|\nabla v\|_2^2. \end{aligned}$$

A Gronwall argument then concludes the estimates.

Proof. By using smoothing estimates we may assume without loss of generality that $t_0 = 0$ and $v_0 \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^1$. We first control $\|v\|_{\dot{H}^1}$. Applying the spatial derivative ∇ to the Navier–Stokes equations (1.1), and then taking the L^2 inner product of the resulting equations with ∇v , we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 &= - \int_{\mathbb{R}^3} (\partial_i v \cdot \nabla) v \cdot \partial_i v dx \\ &= - \int_{\mathbb{R}^3} \partial_i v^3 \partial_3 v \cdot \partial_i v dx - \int_{\mathbb{R}^3} \partial_i v^h \partial_h v^3 \partial_i v^3 dx - \int_{\mathbb{R}^3} \partial_i v^h \partial_h v^{\tilde{h}} \partial_i v^{\tilde{h}} dx \\ &= K_1 + K_2 + K_3. \end{aligned}$$

Here we used Einstein’s convention over repeated indices. We emphasize that throughout this paper, the summation over i is always from 1 to 3, but the summation over h and \tilde{h} are always from 1 to 2.

We first estimate K_1 and K_2 . Clearly, for $2 \leq p \leq 4$ (note that $-\frac{1}{2} + \frac{2}{p} \geq 0$),

$$\begin{aligned} |K_1| + |K_2| &\lesssim \|\nabla v^3\|_{\frac{3p}{2p-2}} \|\nabla v\|_{\frac{6p}{p+2}}^2 \\ &\lesssim \|\nabla v^3\|_{\dot{H}^{-\frac{1}{2}+\frac{2}{p}}} \cdot \| |\nabla|^{1-\frac{1}{p}} \nabla v \|_2^2 \\ &\lesssim \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla v\|_2^{\frac{2}{p}} \cdot \|\Delta v\|_2^{2-\frac{2}{p}} \\ &\leq \frac{1}{16} \|\Delta v\|_{L^2}^2 + C \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p \|\nabla v\|_{L^2}^2. \end{aligned}$$

Here and below, C represents a constant whose value may change from line to line. On the other hand for $4 < p < \infty$, noting that $\frac{1}{2} - \frac{2}{p} > 0$, we have

$$\begin{aligned} |K_1| + |K_2| &\lesssim \|\ |\nabla|^{-\frac{1}{2} + \frac{2}{p}} \partial v^3\|_2 \cdot \|\ |\nabla|^{\frac{1}{2} - \frac{2}{p}} (\partial v \cdot \partial v)\|_2 \\ &\lesssim \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \cdot \|\ |\nabla|^{\frac{1}{2} - \frac{2}{p}} \partial v\|_3 \cdot \|\partial v\|_6 \\ &\lesssim \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \cdot \|\Delta v\|_2 \cdot \|\ |\nabla|^{1 - \frac{2}{p}} \partial v\|_2 \\ &\lesssim \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \cdot \|\Delta v\|_2^{2 - \frac{2}{p}} \cdot \|\nabla v\|_2^{\frac{2}{p}}. \end{aligned}$$

For K_3 , one observes that

$$\partial_h v^{\tilde{h}} = \mathcal{R}_2(\omega) + \mathcal{R}_2(\partial_3 v^3),$$

where \mathcal{R}_2 is a two-dimensional Riesz transform. These terms can be estimated in a similar way as in K_1 and K_2 by using Sobolev norm $\|\cdot\|_{\frac{3p}{2p-2}}$ for $2 \leq p \leq 4$ and fractional operator $|\nabla|^{-\frac{1}{2} + \frac{2}{p}}$ for $4 < p < \infty$.

Collecting the estimates, we obtain

$$\frac{d}{dt} \|\nabla v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 \leq C \|\omega\|_{\dot{H}^{-\frac{1}{2} + \frac{2}{p}}}^p \|\nabla v\|_{L^2}^2 + C \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p \|\nabla v\|_{L^2}^2.$$

Then, the Gronwall inequality gives, for any $0 < T_1 < T$,

$$\begin{aligned} \|\nabla v(T_1)\|_{L^2}^2 + \int_0^{T_1} \|\Delta v(t)\|_{L^2}^2 dt \\ \leq \|\nabla v_0\|_{L^2}^2 \exp\left(C \int_0^T \|\omega\|_{\dot{H}^{-\frac{1}{2} + \frac{2}{p}}}^p dt + C \int_0^T \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p dt\right). \end{aligned}$$

On the other hand, for the $\|v\|_{\dot{H}^{\frac{1}{2}}}$ -norm, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|v\|_{\dot{H}^{\frac{1}{2}}}^2) + \|\ |\nabla| \nabla|^{\frac{1}{2}} v\|_2^2 &\leq \|v\|_6 \|\nabla v\|_3 \|\nabla v\|_2 \\ &\lesssim \|\nabla v\|_2^2 \|v\|_{\dot{H}^{\frac{3}{2}}} \\ &\leq \frac{1}{8} \|\ |\nabla| \nabla|^{\frac{1}{2}} v\|_2^2 + C \|\nabla v\|_2^4. \end{aligned}$$

This then easily yields the control of $\|v\|_{\dot{H}^{\frac{1}{2}}}$. Since we have uniform estimates on $\|v\|_{\dot{H}^{\frac{1}{2}}} + \|v\|_{\dot{H}^1}$ on the time interval $[0, T)$, the solution v can be continued past T .

Finally we show continuity of $\Omega = \nabla \times v$ in L^{r_0} norm. First we show $\Omega \in L_t^\infty L_x^{r_0}([0, T])$. Consider the vorticity equation

$$\partial_t \Omega + (v \cdot \nabla) \Omega = \Delta \Omega + (\Omega \cdot \nabla) v.$$

Clearly

$$\begin{aligned} \frac{1}{r_0} \frac{d}{dt} (\|\Omega\|_{r_0}^{r_0}) + \frac{4(r_0 - 1)}{r_0^2} \|\nabla(|\Omega|^{\frac{r_0}{2}})\|_2^2 &\lesssim \|\nabla v\|_3 \cdot \|\Omega\|_{r_0}^{\frac{3}{2}} \\ &\lesssim \|v\|_{\dot{H}^{\frac{3}{2}}} \|\Omega\|_{r_0}^{\frac{7}{2}} \|\nabla(|\Omega|^{\frac{r_0}{2}})\|_2 \\ &\leq C \|v\|_{\dot{H}^{\frac{3}{2}}}^2 \|\Omega\|_{r_0}^{r_0} + \frac{r_0 - 1}{r_0^2} \|\nabla(|\Omega|^{\frac{r_0}{2}})\|_2^2. \end{aligned}$$

Since we have shown $\|v\|_{L_t^2 \dot{H}^{\frac{3}{2}}(0, T)} < \infty$, the Gronwall inequality then easily yields

$$\|\Omega\|_{L_t^\infty L_x^{r_0}(0, T)} + \|\nabla(|\Omega|^{\frac{r_0}{2}})\|_{L_t^2 L_x^2(0, T)} < \infty.$$

Now to show continuity in L^{r_0} norm we shall only check the (right) continuity at $t_0 = 0$. The continuity at each positive time $t_0 \in (0, T]$ is easier (and omitted) thanks to the usual smoothing effect. For the continuity at $t_0 = 0$ we only need to examine the integrals

$$\int_0^t e^{(t-s)\Delta} (\Omega \cdot \nabla v)(s) ds, \quad \text{and} \quad \int_0^t e^{(t-s)\Delta} (v \cdot \nabla \Omega)(s) ds.$$

In terms of the first integral, we discuss two cases.

Case 1: $\frac{3}{2} \leq r_0 \leq 2$. Clearly

$$\begin{aligned} \left\| \int_0^t e^{(t-s)\Delta} (\Omega \cdot \nabla v)(s) ds \right\|_{r_0} &\lesssim \int_0^t \|\Omega(s)\|_{2r_0}^2 ds \lesssim \int_0^t \|\Omega\|_{r_0}^{\frac{r_0}{2}} \|\Omega\|_{r_0}^{\frac{r_0}{4}} ds \\ &\lesssim \int_0^t \|\Omega\|_{r_0}^{\frac{r_0}{2}} \|\nabla(|\Omega|^{\frac{r_0}{2}})\|_2^{\frac{3}{2}} ds. \end{aligned}$$

Since $3/r_0 \leq 2$, the above clearly tends to zero as $t \rightarrow 0+$.

Case 2: $1 < r_0 < \frac{3}{2}$. We have

$$\left\| \int_0^t e^{(t-s)\Delta} (\Omega \cdot \nabla v)(s) ds \right\|_{r_0} \lesssim \int_0^t \|\Omega(s)\|_{2r_0}^2 ds \lesssim \int_0^t \|\nabla(|\Omega|^{\frac{5}{2} - \frac{3}{2r_0}} v(s))\|_2^2 ds.$$

Since $1 < \frac{5}{2} - \frac{3}{2r_0} < \frac{3}{2}$ and $v \in C_t^0 \dot{H}^{\frac{1}{2}} \cap L_t^2 \dot{H}^{\frac{3}{2}}$, the last integral above also tends to zero as $t \rightarrow 0+$.

Now we consider the integral $\int_0^t e^{(t-s)\Delta} (v \cdot \nabla \Omega)(s) ds$. By using the property of the mild solution v , namely $\lim_{s \rightarrow 0+} s^{\frac{1}{2}} \|v(s)\|_{L_x^\infty} = 0$, we have (below we also used $\nabla \cdot v = 0$)

$$\begin{aligned} \left\| \int_0^t e^{(t-s)\Delta} (v \cdot \nabla \Omega)(s) ds \right\|_{r_0} &\lesssim \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} \cdot s^{\frac{1}{2}} \|v(s)\|_\infty \|\Omega(s)\|_{r_0} ds \end{aligned}$$

$$\begin{aligned}
 & + \int_{\frac{t}{2}}^t s^{-\frac{1}{2}} \cdot s^{\frac{1}{2}} \|v(s)\|_{\infty} \|\nabla\Omega(s)\|_{r_0} ds \\
 & \lesssim \|s^{\frac{1}{2}} v(s)\|_{L_s^{\infty} L_x^{\infty}([0,t])} \cdot \left(\|\Omega(s)\|_{L_s^{\infty} L_x^{r_0}([0,t])} + \|\nabla\Omega(s)\|_{L_s^2 L_x^{r_0}([0,t])} \right) \rightarrow 0,
 \end{aligned}$$

as t tends to $0+$. Note that here in the last step we used

$$\|\nabla\Omega\|_{r_0} \lesssim \|\nabla(|\Omega|^{\frac{r_0}{2}})\|_2 \cdot \|\Omega\|_{r_0}^{1-\frac{r_0}{2}} \lesssim \|\nabla(|\Omega|^{\frac{r_0}{2}})\|_2,$$

which is L^2 integrable in time for $1 < r_0 \leq 2$. This finishes the proof of $\Omega \in C([0, T], L_x^{r_0})$.

Finally we note that the estimate for $\nabla\Omega$ is trivial in view of the smoothing effect. We omit the details. \square

4. Estimate of ω : case $2 \leq p \leq 4$

In this section we first give the estimate of the horizontal vorticity ω for the case $2 \leq p \leq 4$. Recall that ω satisfies the equation

$$\partial_t \omega + (v \cdot \nabla)\omega - \Delta\omega = \partial_3 v^3 \omega + \partial_2 v^3 \partial_3 v^1 - \partial_1 v^3 \partial_3 v^2. \tag{4.1}$$

Taking the L^2 inner product of equation (4.1) with $|\omega|^{r-2}$, one has

$$\begin{aligned}
 & \frac{1}{r} \frac{d}{dt} \|\omega\|_{L^r}^r + \frac{4(r-1)}{r^2} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^2 \\
 & = \int \partial_3 v^3 |\omega|^r dx + \int \partial_2 v^3 \partial_3 v^1 |\omega|^{r-2} dx - \int \partial_1 v^3 \partial_3 v^2 |\omega|^{r-2} dx \tag{4.2} \\
 & =: I_1 + I_2 + I_3.
 \end{aligned}$$

Let us first estimate the term I_1 . According to Hölder and interpolation inequalities, we have

$$\begin{aligned}
 I_1 & = \int \partial_3 v^3 |\omega|^r dx \\
 & \leq \|\partial_3 v^3\|_{L^{\frac{3p}{2(p-1)}}} \|\omega\|_{L^{\frac{3p}{p+2}}}^r \\
 & \lesssim \|\partial_3 v^3\|_{\dot{H}^{-\frac{1}{2}+\frac{2}{p}}} \|\omega\|_{L^{\frac{6p}{p+2}}}^{\frac{r}{2}} \|\omega\|_{L^2}^{2-\frac{2}{p}} \\
 & \lesssim \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\omega\|_{L^2}^{\frac{r}{2}} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^{2-\frac{2}{p}}.
 \end{aligned}$$

The estimate of I_3 is similar to I_2 and therefore will be omitted. In what follows, we will focus on the estimate of I_2 . Using the decomposition of v^h which is introduced in the introduction (1.2), I_2 can be rewritten as

$$\begin{aligned}
 I_2 & = \int_{\mathbb{R}^3} \partial_2 v^3 \partial_3 v^1 |\omega|^{r-2} dx \\
 & = - \int_{\mathbb{R}^3} \partial_2 v^3 \partial_1 \Delta_h^{-1} \partial_3^2 v^3 |\omega|^{r-2} dx - \int_{\mathbb{R}^3} \partial_2 v^3 \partial_2 \Delta_h^{-1} \partial_3 \omega |\omega|^{r-2} dx \\
 & =: I_{21} + I_{22}.
 \end{aligned}$$

Before continuing the estimates, we collect below some useful notation and conventions.

Notation

- For each fixed $2 \leq p < \infty$, we shall take $r < 2$ sufficiently close to 2. The explicit requirement on r can be worked out but for simplicity we shall often suppress it. We denote

$$\delta = \delta(r) = \frac{3}{r} - \frac{3}{2} > 0.$$

- For a scalar function $f = f(x_1, x_2, x_3)$ and $1 \leq p, q \leq \infty$ we use the mixed norm notation

$$\|f\|_{L_v^q L_h^p} := \left\| \|f(x_h, x_3)\|_{L_{x_h}^p(\mathbb{R}^2)} \right\|_{L_{x_3}^q(\mathbb{R})}.$$

The notation $\|f\|_{L_h^p L_v^q}$ is similarly defined.

- We use ∇ or $\partial = (\partial_1, \partial_2, \partial_3)$ to denote the usual gradient operator. Occasionally we also use ∂^2 to denote the whole collection of second order operators $(\partial_i \partial_j)_{1 \leq i, j \leq 3}$. By Fourier transform, it is easy to check that

$$\begin{aligned} \|\nabla^{1-\delta} f\|_2 &\leq \|\nabla_h^{-\delta} |\nabla| f\|_2 \sim \|\nabla_h^{-\delta} \partial f\|_2 \sim \sum_{j=1}^3 \|\nabla_h^{-\delta} \partial_j f\|_2, \\ \|\nabla^{2-\delta} f\|_2 &\leq \|\nabla_h^{-\delta} \Delta f\|_2 \sim \|\nabla_h^{-\delta} \partial^2 f\|_2 \sim \sum_{i,j=1}^3 \|\nabla_h^{-\delta} \partial_i \partial_j f\|_2. \end{aligned}$$

We will often use these inequalities without explicit mentioning.

- In various interpolation inequalities we shall use the letter ε to denote a sufficiently small positive constant whose smallness is clear from the context. Such notation is quite useful in handling certain end-point situations. For example instead of estimating $\|v^3\|_{L_h^2 L_v^\infty}$ we can estimate the scaling-equivalent quantity $\|v^3\|_{L_h^{\frac{2}{1-\varepsilon}} L_v^{\frac{1}{\varepsilon}}}$. The latter can be easily controlled by $\|v^3\|_{\dot{H}^{\frac{1}{2}}}$, thanks to Sobolev embedding.
- The relation of the parameters p, r and ε is as follows. First we fix $p \in [2, \infty)$. After that we will choose $r < 2$ (depending on p) sufficiently close to 2. After r is chosen, we will choose ε sufficiently small in the interpolation inequalities to get around borderline situations.
- By a slight abuse of notation, we will sometimes write operators such as $\partial_1(-\Delta_h)^{-1}$ or $\partial_2(-\Delta_h)^{-1}$ simply as $|\nabla_h|^{-1}$ as all estimates below will hold the same for both operators.

We now continue the estimates. For I_{21} , when $p \in [2, 4)$, we take r sufficiently close to 2 and satisfy

$$\max \left\{ \frac{p}{2}, \frac{3}{2} \right\} < r < 2.$$

Applying Hölder and Sobolev, one can deduce that

$$\begin{aligned}
 I_{21} &= \int \partial_2 v^3 \partial_1 \Delta_h^{-1} \partial_3^2 v^3 \omega |\omega|^{r-2} dx \\
 &\leq \|\partial_2 v^3\|_{L_h^{\left(\frac{1}{r}-\frac{\delta}{2}\right)^{-1}} L_v^{\left(\frac{1}{2}+\frac{1}{r}-\frac{2}{p}\right)^{-1}}} \|\nabla_h|^{-1} \partial_3^2 v^3\|_{L_h^{\frac{2}{\delta}} L_v^2} \|\omega|\omega|^{r-2}\|_{L_h^{\frac{r}{r-1}} L_v^{\left(\frac{2}{p}-\frac{1}{r}\right)^{-1}}} \\
 &\lesssim \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla_h|^{-\delta} \partial_3^2 v^3\|_{L^2} \|\omega\|_{L^2}^{\frac{r}{2}} \left\| \omega \right\|_{L_h^2 L_v^{\frac{2p(r-1)}{2r-p}}}^{\frac{2(r-1)}{r}} \\
 &\lesssim \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla_h|^{-\delta} \partial_3^2 v^3\|_{L^2} \|\omega\|_{L^2}^{\frac{r}{2}} \|\nabla|\omega|\|_{L^2}^{\frac{1-\frac{2}{r}+\frac{2}{p}}{2}} \|\omega\|_{L^2}^{1-\frac{2}{p}},
 \end{aligned}$$

where we recall $\delta = 3\left(\frac{1}{r} - \frac{1}{2}\right)$.

When $p = 4$, we take r sufficiently close to 2. Then

$$\begin{aligned}
 I_{21} &= \int \partial_2 v^3 \partial_1 \Delta_h^{-1} \partial_3^2 v^3 \omega |\omega|^{r-2} dx \\
 &\leq \|\partial_2 v^3\|_{L_v^2 L_h^2} \|\nabla_h|^{-1} \partial_3^2 v^3\|_{L_v^{\frac{2}{\delta}} L_h^2} \|\omega|\omega|^{r-2}\|_{L_v^\infty L_h^{\frac{2}{1-\delta}}} \\
 &\lesssim \|v^3\|_{\dot{H}^1} \|\nabla_h|^{-\delta} \partial_3^2 v^3\|_{L^2} \|\omega\|_{L^2}^{\frac{r}{2}} \left\| \omega \right\|_{L_v^\infty L_h^{\frac{2}{1-\delta}}}^{\frac{2(r-1)}{r}} \\
 &\lesssim \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\nabla_h|^{-\delta} \partial_3^2 v^3\|_{L^2} \|\omega\|_{L^2}^{\frac{r}{2}} \|\nabla|\omega|\|_{L^2}^{\frac{3-\frac{2}{r}}{2}} \|\omega\|_{L^2}^{\frac{1}{2}}.
 \end{aligned}$$

Here for $\|\omega\|_{L_v^\infty L_h^{\frac{2}{1-\delta}}}^{\frac{2(r-1)}{r}}$, we have used interpolation inequalities to get

$$\begin{aligned}
 &\left\| \|\omega\|_{L_h^{\frac{2}{1-\delta}}}^{\frac{2(r-1)}{r}} \right\|_{L_v^\infty} \lesssim \left\| \|\nabla_h|^{-\frac{2-r}{4(r-1)}} |\omega|\|_{L_v^\infty}^{\frac{r}{2}} \right\|_{L_h^2} \\
 &\lesssim \left\| \|\nabla_h|^{-\frac{2-r}{4(r-1)}} |\omega|\|_{L_v^2}^{\frac{1}{2}-\frac{2-r}{4(r-1)}} \left\| |\nabla_3|^{1-\frac{2-r}{4(r-1)}} |\nabla_h|^{-\frac{2-r}{4(r-1)}} |\omega|\right\|_{L_v^2}^{\frac{1}{2}} \right\|_{L_h^2}^{\frac{1}{\frac{2}{r}-\frac{1}{2}}} \\
 &\lesssim \left\| \|\nabla_h|^{-\frac{2-r}{4(r-1)}} |\omega|\|_{L^2}^{\frac{1}{2}-\frac{2-r}{4(r-1)}} \|\nabla|\omega|\|_{L^2}^{\frac{1}{2}} \right\|_{L^2}^{\frac{1}{1-\frac{2}{4(r-1)}}} \\
 &\lesssim \|\omega\|_{L^2}^{\frac{r}{2}} \|\nabla|\omega|\|_{L^2}^{\frac{1}{2}} \|\omega\|_{L^2}^{\frac{2-r}{4(r-1)} \frac{1}{1-\frac{2}{4(r-1)}} + \frac{1}{1-\frac{2}{4(r-1)}}}.
 \end{aligned}$$

Let us turn to the estimate of I_{22} . First, we consider the case $p \in (2, 4)$ which can be easily dealt with by anisotropic Hölder inequality and Sobolev embedding. More precisely,

$$\begin{aligned}
 I_{22} &:= \int \partial_2 v^3 \partial_2 \Delta_h^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx \\
 &\leq \|\partial_2 v^3\|_{L_h^2 L_v^{\frac{p}{p-2}}} \|\nabla_h|^{-1} \partial_3 \omega\|_{L_h^{\frac{3}{\delta}} L_v} \|\omega|\omega|^{r-2}\|_{L_h^{\frac{r}{r-1}} L_v^{\left(\frac{2}{p}-\frac{1}{r}\right)^{-1}}}
 \end{aligned}$$

$$\begin{aligned} &\lesssim \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\partial_3 \omega\|_{L^r} \|\omega\|^{\frac{r}{2}} \left\| \frac{2(r-1)}{L_h^2 L_v^{2r-p}} \right\| \\ &\lesssim \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \|\omega\|^{\frac{r}{2}} \left\| \frac{2}{L^2} \|\nabla|\omega|\|^{\frac{r}{2}} \right\|_{L^2}^{2-\frac{2}{p}}. \end{aligned}$$

Next we consider $p = 4$:

$$\begin{aligned} I_{22} &:= \int \partial_2 v^3 \partial_2 \Delta_h^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx \\ &\leq \| \partial_2 v^3 \|_{L_h^2} \| |\nabla_h|^{-1} \partial_3 \omega \|_{L_h^{\frac{3}{2}}} \| \omega |\omega|^{r-2} \|_{L_h^{\frac{r}{r-1}}} \| L_v^1 \\ &\lesssim \| \partial_2 v^3 \|_{L_h^2} \| \partial_3 \omega \|_{L_h^r} \| \omega \|^{\frac{r}{2}} \| L_h^{\frac{2(r-1)}{r}} \| L_v^1 \\ &\lesssim \| \partial_2 v^3 \|_{L_h^2} \| \partial_3 |\omega|^{\frac{r}{2}} \|_{L_h^2} \| \omega \|^{\frac{r}{2}} \| L_h^2 \| L_v^1 \\ &\lesssim \| v^3 \|_{\dot{H}^1} \| \nabla |\omega|^{\frac{r}{2}} \|_{L^2} \| \omega \|^{\frac{r}{2}} \| L_h^2 L_v^\infty. \end{aligned}$$

Here we remark that in the third inequality above, we have used the fact that for smooth ω , the set $\{x : \omega = 0, \partial_3 \omega \neq 0\}$ has Lebesgue measure zero. Therefore when bounding the term $\partial_3 \omega$ one can up to measure zero regard it as $\partial_3 \omega \cdot 1_{\omega>0} + \partial_3 \omega \cdot 1_{\omega<0}$ and proceed to use interpolation inequalities involving $|\omega|^{\frac{r}{2}}$ which has no differentiability issues.

Now since

$$\| |\omega|^{\frac{r}{2}} \|_{L_h^2 L_v^\infty} \lesssim \| |\omega|^{\frac{r}{2}} \|_{L^2}^{\frac{1}{2}} \| \partial_3 |\omega|^{\frac{r}{2}} \|_{L^2}^{\frac{1}{2}},$$

one has

$$I_{22} \lesssim \| v^3 \|_{\dot{H}^1} \| \nabla |\omega|^{\frac{r}{2}} \|_{L^2}^{\frac{3}{2}} \| \omega \|^{\frac{r}{2}} \| L^2.$$

Finally, we consider $p = 2$. In this case, Sobolev embedding is not enough. We need to apply Littlewood–Paley decomposition in the vertical direction and obtain

$$\begin{aligned} I_{22} &:= \int \partial_2 v^3 \partial_2 \Delta_h^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx \\ &= \int \partial_2 P_{[-J_0, J_0]}^z v^3 |\nabla_h|^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx \end{aligned} \tag{4.3}$$

$$+ \sum_{j>J_0} \int \partial_2 P_j^z v^3 |\nabla_h|^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx \tag{4.4}$$

$$+ \sum_{j<-J_0} \int \partial_2 P_j^z v^3 |\nabla_h|^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx, \tag{4.5}$$

where P_j^z denotes the Littlewood–Paley decomposition on the vertical variable, and J_0 is a positive number which will be determined later.

Estimate of (4.3):

$$\begin{aligned} & \int \partial_2 P_{[-J_0, J_0]}^z v^3 |\nabla_h|^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx \\ & \lesssim \|\partial_2 P_{[-J_0, J_0]}^z v^3\|_{L_h^2 L_v^\infty} \|\nabla_h|^{-1} \partial_3 \omega\|_{L_h^{(\frac{1}{r}-\frac{1}{2})^{-1}} L_v^r} \|\omega |\omega|^{r-2}\|_{L_h^{r-1} L_v^{r-1}} \\ & \lesssim \sqrt{J_0} \|v^3\|_{\dot{H}^{\frac{3}{2}}} \|\partial_3 \omega\|_{L^r} \|\omega\|_{L^r}^{r-1}. \end{aligned}$$

Estimate of (4.4):

For (4.4), we observe that ($\varepsilon_1 > 0$ is a sufficiently small constant)

$$\begin{aligned} \|\partial_2 P_j^z v^3\|_{L_h^2 L_v^\infty} & \lesssim 2^{-j\varepsilon_1} \|\partial_2 |\partial_3|^{\frac{1}{2}+\varepsilon_1} P_j^z v^3\|_{L^2} \lesssim 2^{-j\varepsilon_1} \|\nabla|^{\frac{3}{2}+\varepsilon_1} v^3\|_{L^2} \\ & \lesssim 2^{-j\varepsilon_1} \|\nabla|^{1-\delta} v^3\|_{L^2}^{\frac{1}{2}-\delta-\varepsilon_1} \|\nabla|^{2-\delta} v^3\|_{L^2}^{\frac{1}{2}+\delta+\varepsilon_1} \\ & \lesssim 2^{-j\varepsilon_1} \|\nabla_h|^{-\delta} \partial v^3\|_{L^2}^{\frac{1}{2}-\delta-\varepsilon_1} \|\nabla_h|^{-\delta} \partial^2 v^3\|_{L^2}^{\frac{1}{2}+\delta+\varepsilon_1}. \end{aligned}$$

Then

$$\begin{aligned} & \sum_{j>J_0} \int \partial_2 P_j^z v^3 |\nabla_h|^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx \\ & \lesssim 2^{-J_0\varepsilon_1} \|\nabla_h|^{-\delta} \partial v^3\|_{L^2}^{\frac{1}{2}-\delta-\varepsilon_1} \|\nabla_h|^{-\delta} \partial^2 v^3\|_{L^2}^{\frac{1}{2}+\delta+\varepsilon_1} \|\nabla \omega\|_{L^r} \|\omega\|_{L^r}^{r-1}. \end{aligned}$$

Estimate of (4.5): Note that

$$\|\partial_2 P_j^z v^3\|_{L_h^2 L_v^\infty} \lesssim 2^{j\varepsilon_1} \cdot \|\partial_2 |\partial_3|^{\frac{1}{2}-\varepsilon_1} P_j^z v^3\|_{L^2}, \quad \text{for negative } j.$$

By an argument similar to (4.4), we have

$$\begin{aligned} & \sum_{j<-J_0} \int \partial_2 P_j^z v^3 |\nabla_h|^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx \\ & \lesssim 2^{-J_0\varepsilon_1} \|\nabla_h|^{-\delta} \partial v^3\|_{L^2}^{\frac{1}{2}-\delta+\varepsilon_1} \|\nabla_h|^{-\delta} \partial^2 v^3\|_{L^2}^{\frac{1}{2}+\delta-\varepsilon_1} \|\nabla \omega\|_{L^r} \|\omega\|_{L^r}^{r-1}. \end{aligned}$$

Choosing a suitable J_0 then yields

$$\begin{aligned} & \int \partial_2 v^3 |\nabla_h|^{-1} \partial_3 \omega \omega |\omega|^{r-2} dx \\ & \lesssim \sqrt{\log(10 + \|\nabla_h|^{-\delta} \partial v^3\|_{L^2} + \|\omega\|_{L^r})} (\|v^3\|_{\dot{H}^{\frac{3}{2}}} + 1) \|\nabla \omega\|_{L^r} \|\omega\|_{L^r}^{r-1} \\ & \quad + \left(\|\nabla_h|^{-\delta} \partial^2 v^3\|_{L^2}^{\frac{1}{2}+\delta+\varepsilon_1} + \|\nabla_h|^{-\delta} \partial^2 v^3\|_{L^2}^{\frac{1}{2}+\delta-\varepsilon_1} \right) \cdot \frac{1}{1 + \|\omega\|_{L^r}^{100}} \|\nabla(|\omega|^{\frac{r}{2}})\|_2. \end{aligned}$$

5. Estimate of v^3 : case $2 \leq p \leq 4$

The equation of v^3 is

$$\partial_t v^3 + (v \cdot \nabla)v^3 - \Delta v^3 = -\partial_3 P. \tag{5.1}$$

Applying ∂_k ($k = 1, 2, 3$) to (5.1), one has

$$\partial_t \partial_k v^3 + (v \cdot \nabla)\partial_k v^3 + (\partial_k v \cdot \nabla)v^3 - \Delta \partial_k v^3 = -\partial_3 \partial_k P. \tag{5.2}$$

Taking the L^2 inner product of equation (5.2) with $|\nabla_h|^{-2\delta} \partial_k v^3$, one has

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\sum_{k=1}^3 \| |\nabla_h|^{-\delta} \partial_k v^3 \|_{L^2}^2 \right) + \sum_{k=1}^3 \| |\nabla_h|^{-\delta} \partial_k \nabla v^3 \|_{L^2}^2 \\ &= \sum_{k=1}^3 \left(- \int (\partial_k v \cdot \nabla)v^3 \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx \right. \end{aligned} \tag{5.3}$$

$$- \int (v \cdot \nabla)\partial_k v^3 \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx \tag{5.4}$$

$$\left. - \int \partial_3 \partial_k P \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx \right). \tag{5.5}$$

5.1. Estimate of (5.3)

Case 1: $\int \partial_k v^3 \partial_3 v^3 \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx$. We have for all $2 \leq p < \infty$:

$$\begin{aligned} & \int \partial_k v^3 \partial_3 v^3 \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx \\ & \lesssim \| \partial v^3 \|_{L_v^{\frac{1}{4+\frac{\delta}{2}+\varepsilon}}}^2 \| |\nabla_h|^{-2\delta} \partial v^3 \|_{L_v^{\frac{2}{1-\varepsilon}}} \cdot \| |\nabla_h|^{-2\delta} \partial v^3 \|_{L_v^{\frac{1}{2-\delta-2\varepsilon}}}^{-1} \| |\nabla_h|^{-2\delta} \partial v^3 \|_{L_h^{\frac{1}{\varepsilon}}} \\ & \lesssim \| |\nabla|^{2-\frac{\delta}{2}} v^3 \|_2 \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_2 \\ & \lesssim \| |\nabla|^{\frac{2}{p}+\frac{1}{2}} v^3 \|_2 \cdot \| |\nabla|^{-\delta+2-\frac{2}{p}} v^3 \|_2 \cdot \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_2 \\ & \lesssim \| v^3 \|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_2^{2-\frac{2}{p}} \cdot \| |\nabla_h|^{-\delta} \partial v^3 \|_2^{\frac{2}{p}}. \end{aligned}$$

Case 2: $\int (\partial_k v^h \cdot \nabla_h)v^3 \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx$.

Case 2a: $\int (\partial_k |\nabla_h|^{-1} \omega \cdot \nabla_h)v^3 \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx$.

Applying Littlewood–Paley decomposition on the horizontal direction (see Lemma 2.3, here \tilde{P}_j^h corresponds to projection in x_h -variable only), one has

$$\begin{aligned} & \int (\partial_k |\nabla_h|^{-1} \omega \cdot \nabla_h)v^3 \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx \\ &= \sum_j \left[\int (\partial_k |\nabla_h|^{-1} \tilde{P}_j^h \omega \cdot \nabla_h) \tilde{P}_{\ll j}^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_j^h v^3 dx \right] \end{aligned} \tag{5.6}$$

$$+ \int (\partial_k |\nabla_h|^{-1} \tilde{P}_j^h \omega \cdot \nabla_h) \tilde{P}_j^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_{\ll j}^h v^3 dx \tag{5.7}$$

$$+ \int (\partial_k |\nabla_h|^{-1} \tilde{P}_{\ll j}^h \omega \cdot \nabla_h) \tilde{P}_j^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_j^h v^3 dx \tag{5.8}$$

$$+ \int (\partial_k |\nabla_h|^{-1} \tilde{P}_j^h \omega \cdot \nabla_h) \tilde{P}_j^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_j^h v^3 dx \Big]. \tag{5.9}$$

Estimate of (5.6): For $2 \leq p < \infty$, by taking r sufficiently close to 2, we have

$$\begin{aligned} & \sum_j \int (\partial_k |\nabla_h|^{-1} \tilde{P}_j^h \omega \cdot \nabla_h) \tilde{P}_{\ll j}^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_j^h v^3 dx \\ & \lesssim \|(2^j \partial_k |\nabla_h|^{-1} \tilde{P}_j^h \omega)\|_{l_j^2 L_h^r L_v^r} \|(2^{-j} \nabla_h \tilde{P}_{\ll j}^h v^3)\|_{l_j^\infty L_h^{(\frac{1}{8} + \frac{\delta}{4} - \frac{\varepsilon}{2})^{-1}} L_v^{\frac{1}{\varepsilon}}} \\ & \quad \cdot \|(|\nabla_h|^{-2\delta} \tilde{P}_j^h \partial_k v^3)\|_{l_j^2 L_h^{(\frac{3}{8} - \frac{7\delta}{12} + \frac{\varepsilon}{2})^{-1}} L_v^{(1 - \frac{1}{r} - \varepsilon)^{-1}}} \\ & \lesssim \|\nabla \omega\|_r \|\nabla\|_{\frac{5}{4} - \frac{\delta}{2}} \|v^3\|_2^2 \\ & \lesssim \|\nabla \omega\|_r \cdot \|\nabla\|_{\frac{2}{p} + \frac{1}{2}} \|v^3\|_2 \cdot \|\nabla\|^{-\delta + 2 - \frac{2}{p}} \|v^3\|_2 \\ & \lesssim \|\nabla \omega\|_r \cdot \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \cdot \|\nabla_h\|^{-\delta} \|\partial v^3\|_2^{\frac{2}{p}} \cdot \|\nabla_h\|^{-\delta} \|\partial^2 v^3\|_2^{1 - \frac{2}{p}}. \end{aligned}$$

Estimate of (5.7): the estimate is similar to the above (one only need to swap l_j^∞ and l_j^2 in second and third) and therefore omitted.

Estimate of (5.8): Clearly, for $2 \leq p < \infty$,

$$\begin{aligned} & \sum_j \int (\partial_k |\nabla_h|^{-1} \tilde{P}_{\ll j}^h \omega \cdot \nabla_h) \tilde{P}_j^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_j^h v^3 dx \\ & \lesssim \|(\partial_k |\nabla_h|^{-1} \tilde{P}_{\ll j}^h \omega)\|_{l_j^\infty L_h^{(\frac{1}{r} - \frac{1}{2})^{-1}} L_v^r} \|(2^{-j} (\frac{7}{6} \delta - \varepsilon + \frac{1}{4}) \nabla_h \tilde{P}_j^h v^3)\|_{l_j^2 L_h^{(1 - \frac{1}{r})^{-1}} L_v^{\frac{1}{\varepsilon}}} \\ & \quad \cdot \|(2^j (\frac{7}{6} \delta - \varepsilon + \frac{1}{4}) |\nabla_h|^{-2\delta} \tilde{P}_j^h \partial_k v^3)\|_{l_j^2 L_h^2 L_v^{(1 - \frac{1}{r} - \varepsilon)^{-1}}} \\ & \lesssim \|\nabla \omega\|_r \|\nabla\|_{\frac{1}{4} - \frac{\delta}{2}} \|\partial v^3\|_2^2 \\ & \lesssim \|\nabla \omega\|_r \cdot \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \cdot \|\nabla_h\|^{-\delta} \|\partial v^3\|_2^{\frac{2}{p}} \cdot \|\nabla_h\|^{-\delta} \|\partial^2 v^3\|_2^{1 - \frac{2}{p}}. \end{aligned}$$

Estimate of (5.9): This is similar to the above and it is omitted.

Case 2b: $\int (\partial_k |\nabla_h|^{-1} \partial_3 v^3 \cdot \nabla_h) v^3 \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx$. Thus

$$\begin{aligned} & \int (\partial_k |\nabla_h|^{-1} \partial_3 v^3 \cdot \nabla_h) v^3 \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx \\ & = \sum_j \left[\int (\partial_k |\nabla_h|^{-1} \tilde{P}_j^h \partial_3 v^3 \cdot \nabla_h) \tilde{P}_{\ll j}^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_j^h v^3 dx \tag{5.10} \right. \\ & \quad \left. + \int (\partial_k |\nabla_h|^{-1} \tilde{P}_{\ll j}^h \partial_3 v^3 \cdot \nabla_h) \tilde{P}_j^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_{\ll j}^h v^3 dx \tag{5.11} \right] \end{aligned}$$

$$+ \int (\partial_k |\nabla_h|^{-1} \tilde{P}_{\ll j}^h \partial_3 v^3 \cdot \nabla_h) \tilde{P}_j^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_j^h v^3 dx \tag{5.12}$$

$$+ \int (\partial_k |\nabla_h|^{-1} \tilde{P}_j^h \partial_3 v^3 \cdot \nabla_h) \tilde{P}_j^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_j^h v^3 dx \Big]. \tag{5.13}$$

Estimate of (5.10):

We have, for all $2 \leq p < \infty$,

$$\begin{aligned} & \sum_j \int (\partial_k |\nabla_h|^{-1} \tilde{P}_j^h \partial_3 v^3 \cdot \nabla_h) \tilde{P}_{\ll j}^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_j^h v^3 dx \\ & \lesssim \|(2^{j(1-\delta)} \partial_k |\nabla_h|^{-1} \tilde{P}_j^h \partial_3 v^3)\|_{l_j^2 L_h^2 L_v^2} \|(2^{-j} \nabla_h \tilde{P}_{\ll j}^h v^3)\|_{l_j^\infty L_h^{(\frac{1}{8} + \frac{\delta}{4} - \frac{\varepsilon}{2})^{-1}} L_v^{\frac{1}{\varepsilon}}} \\ & \|(2^{j\delta} |\nabla_h|^{-2\delta} \partial_k \tilde{P}_j^h v^3)\|_{l_j^2 L_h^{(\frac{3}{8} - \frac{\delta}{4} + \frac{\varepsilon}{2})^{-1}} L_v^{1-\frac{2}{2\varepsilon}}} \\ & \lesssim \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2} \cdot \| |\nabla|^{1-\frac{\delta}{2}} \partial v^3 \|_2^2 \\ & \lesssim \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2} \| |\nabla|^{-\delta} |\nabla|^{1-\frac{2}{p}} \partial v^3 \|_{L^2} \| v^3 \|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \\ & \lesssim \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2}^{2-\frac{2}{p}} \| |\nabla_h|^{-\delta} \partial v^3 \|_2^{\frac{2}{p}} \| v^3 \|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}. \end{aligned}$$

Estimate of (5.11): For all $2 \leq p < \infty$,

$$\begin{aligned} & \sum_j \int (\partial_k |\nabla_h|^{-1} \tilde{P}_j^h \partial_3 v^3 \cdot \nabla_h) \tilde{P}_j^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_{\ll j}^h v^3 dx \\ & \lesssim \|(2^{j(1-\delta)} \partial_k |\nabla_h|^{-1} \tilde{P}_j^h \partial_3 v^3)\|_{l_j^2 L_h^2 L_v^2} \|(2^{j(-\frac{1}{4} - \frac{1}{2}\delta + \varepsilon)} \nabla_h \tilde{P}_j^h v^3)\|_{l_j^2 L_h^2 L_v^{\frac{1}{\varepsilon}}} \\ & \cdot \|(2^{j(-\frac{3}{4} + \frac{3}{2}\delta - \varepsilon)} |\nabla_h|^{-2\delta} \tilde{P}_{\ll j}^h \partial_k v^3)\|_{l_j^\infty L_h^\infty L_v^{1-\frac{2}{2\varepsilon}}} \\ & \lesssim \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2} \cdot \| |\nabla|^{1-\frac{\delta}{2}} \partial v^3 \|_2^2 \\ & \lesssim \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2}^{2-\frac{2}{p}} \| |\nabla_h|^{-\delta} \partial v^3 \|_2^{\frac{2}{p}} \| v^3 \|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}. \end{aligned}$$

Estimate of (5.12): For $2 \leq p < \infty$, we have

$$\begin{aligned} & \sum_j \int (\partial_k |\nabla_h|^{-1} \tilde{P}_{\ll j}^h \partial_3 v^3 \cdot \nabla_h) \tilde{P}_j^h v^3 \cdot |\nabla_h|^{-2\delta} \partial_k \tilde{P}_j^h v^3 dx \\ & \lesssim \| (\partial_k |\nabla_h|^{-1} \tilde{P}_{\ll j}^h \partial_3 v^3) \|_{l_j^\infty L_h^{\frac{2}{\delta}} L_v^2} \cdot \|(2^{j(-\frac{1}{4} - \delta + \varepsilon)} \nabla_h \tilde{P}_j^h v^3)\|_{l_j^2 L_h^{\frac{4}{2-\delta}} L_v^{\frac{1}{\varepsilon}}} \\ & \cdot \|(2^{j(\frac{1}{4} + \delta - \varepsilon)} |\nabla_h|^{-2\delta} \tilde{P}_j^h \partial_k v^3)\|_{l_j^2 L_h^{\frac{4}{2-\delta}} L_v^{1-\frac{2}{2\varepsilon}}} \\ & \lesssim \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2} \cdot \| |\nabla|^{1-\frac{\delta}{2}} \partial v^3 \|_2^2 \\ & \lesssim \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2}^{2-\frac{2}{p}} \| |\nabla_h|^{-\delta} \partial v^3 \|_2^{\frac{2}{p}} \| v^3 \|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}. \end{aligned}$$

Estimate of (5.13): The estimate of this term is similar to the above, thus we omit the details.

5.2. Estimate of (5.4)

By using integration by parts, one has

$$\int (v \cdot \nabla) \partial_k v^3 \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx = - \int v \partial_k v^3 \cdot \nabla |\nabla_h|^{-2\delta} \partial_k v^3 dx.$$

Case 1: $\int v^3 \partial_k v^3 \partial_3 |\nabla_h|^{-2\delta} \partial_k v^3 dx$. For all $2 \leq p < \infty$, we have

$$\begin{aligned} & \int v^3 \partial_k v^3 \partial_3 |\nabla_h|^{-2\delta} \partial_k v^3 dx \\ & \lesssim \|v^3\|_{L_h^{\frac{2}{\delta}} L_v^{(\frac{1}{4}-\frac{\delta}{2})^{-1}}} \cdot \|\partial v^3\|_{L_h^2 L_v^{(\frac{1}{4}+\frac{\delta}{2})^{-1}}} \cdot \| |\nabla_h|^{-2\delta} \partial^2 v^3 \|_{L_h^{\frac{2}{1-\delta}} L_v^2} \\ & \lesssim \| |\nabla_h|^{\frac{5}{4}-\frac{\delta}{2}} v^3 \|_2 \cdot \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_2 \\ & \lesssim \| |\nabla_h|^{-\delta} \partial v^3 \|_2^{\frac{2}{p}} \cdot \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_2^{2-\frac{2}{p}}. \end{aligned}$$

Case 2: $\int v^h \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3 dx$.

Case 2a: $\int |\nabla_h|^{-1} \omega \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3 dx$. If $2 < p \leq 4$, then

$$\begin{aligned} & \int |\nabla_h|^{-1} \omega \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3 dx \\ & \lesssim \| |\nabla_h|^{-1} \omega \|_{L_h^{\frac{3}{\delta}} L_v^{(\frac{\delta}{3}+\frac{2}{p}-\frac{1}{2})^{-1}}} \cdot \|\partial_k v^3\|_{L_h^2 L_v^{(1-\frac{2}{p})^{-1}}} \cdot \| |\nabla_h|^{-2\delta} \nabla_h \partial v^3 \|_{(\frac{1}{2}-\frac{\delta}{3})^{-1}} \\ & \lesssim \| |\nabla_h|^{1-\frac{2}{p}} \omega \|_r \cdot \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_2 \\ & \lesssim \|\omega\|_r^{\frac{2}{p}} \|\nabla \omega\|_r^{1-\frac{2}{p}} \cdot \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_2. \end{aligned}$$

On the other hand if $p = 2$, then

$$\begin{aligned} & \int |\nabla_h|^{-1} \omega \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3 dx \\ & \leq \| |\nabla_h|^{-1} \omega \|_{L_h^{\frac{3}{\delta}} L_v^r} \|\partial_k v^3\|_{L_h^{1-\frac{2}{\varepsilon}} L_v^{\frac{1}{\varepsilon}}} \| |\nabla_h|^{-2\delta} \partial_k v^3 \|_{L_h^{(\frac{1+\varepsilon}{2}-\frac{\delta}{3})^{-1}} L_v^{(1-\varepsilon-\frac{1}{r})^{-1}}} \\ & \lesssim \|\omega\|_{L^r} \|v^3\|_{\dot{H}^{\frac{3}{2}}} \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2}. \end{aligned}$$

Case 2b: $\int |\nabla_h|^{-1} \partial_3 v^3 \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3$. If $p = 2$, then

$$\begin{aligned} & \int |\nabla_h|^{-1} \partial_3 v^3 \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3 \, dx \\ & \leq \| |\nabla_h|^{-1} \partial_3 v^3 \|_{L_h^{\frac{2}{\delta}} L_v^2} \| \partial_k v^3 \|_{L_h^{\frac{2}{1-\varepsilon}} L_v^{\frac{1}{\varepsilon}}} \| |\nabla_h|^{-2\delta} \partial_k v^3 \|_{L_h^{\frac{2}{1+\varepsilon-\delta}} L_v^{\frac{2}{1-2\varepsilon}}} \\ & \lesssim \| |\nabla_h|^{-\delta} \partial v^3 \|_{L^2} \| v^3 \|_{\dot{H}^{\frac{3}{2}}} \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2}. \end{aligned}$$

If $2 < p \leq 4$, then

$$\begin{aligned} & \int |\nabla_h|^{-1} \partial_3 v^3 \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3 \, dx \\ & \lesssim \| |\nabla_h|^{-1} \partial_3 v^3 \|_{L_h^{\frac{2}{\delta-\varepsilon}} L_v^{(\varepsilon+\frac{2}{p}-\frac{1}{2})^{-1}}} \cdot \| \partial v^3 \|_{L_h^2 L_v^{(1-\frac{2}{p})^{-1}}} \\ & \quad \cdot \| |\nabla_h|^{-2\delta} \partial v^3 \|_{L_h^{\frac{2}{1+\varepsilon-\delta}} L_v^{\frac{2}{1-2\varepsilon}}} \\ & \lesssim \| |\nabla_h|^{-\delta} |\nabla|^{1-\frac{2}{p}} \partial v^3 \|_2 \cdot \| v^3 \|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_2 \\ & \lesssim \| |\nabla_h|^{-\delta} \partial v^3 \|_2^{\frac{2}{p}} \cdot \| v^3 \|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_2^{2-\frac{2}{p}}. \end{aligned}$$

5.3. Estimate of (5.5)

$$\int \partial_3 \partial_k P \cdot |\nabla_h|^{-2\delta} \partial_k v^3 \, dx = - \int \left(\sum_{l,m=1}^3 \partial_l v^m \partial_m v^l \right) \cdot \partial_3 \partial_k \Delta^{-1} (|\nabla_h|^{-2\delta} \partial_k v^3) \, dx.$$

Case 1: $l, m \in \{1, 2\}$. First observe that if $p = 2$, then (below $\mathcal{R} := \partial_3 \partial_k \Delta^{-1}$)

$$\begin{aligned} & \int \partial_l v^m \partial_m v^l \cdot \partial_3 \partial_k \Delta^{-1} (|\nabla_h|^{-2\delta} \partial_k v^3) \, dx \\ & \lesssim \| \partial_l v^m \|_{L_h^{(\frac{1}{4}+\frac{\varepsilon}{4}+\frac{\delta}{2})^{-1}} L_v^{\frac{2}{1-\varepsilon}}} \| \partial_m v^l \|_{L_h^{(\frac{1}{4}+\frac{\varepsilon}{4}+\frac{\delta}{2})^{-1}} L_v^{\frac{2}{1-\varepsilon}}} \| |\nabla_h|^{-2\delta} \partial_k \mathcal{R} v^3 \|_{L_h^{(\frac{1}{2}-\frac{\varepsilon}{2}-\delta)^{-1}} L_v^{\frac{1}{\varepsilon}}} \\ & \lesssim \left(\| \omega \|^2_{L_h^{(\frac{1}{4}+\frac{\varepsilon}{4}+\frac{\delta}{2})^{-1}} L_v^{\frac{2}{1-\varepsilon}}} + \| \partial_3 v^3 \|^2_{L_h^{(\frac{1}{4}+\frac{\varepsilon}{4}+\frac{\delta}{2})^{-1}} L_v^{\frac{2}{1-\varepsilon}}} \right) \| v^3 \|_{\dot{H}^{\frac{3}{2}}} \\ & \lesssim \left(\| |\nabla|^{\frac{1}{2}} \omega \|_{L^r}^2 + \| |\nabla|^{\frac{1}{2}} |\nabla_h|^{-\delta} \partial_3 v^3 \|_{L^2}^2 \right) \| v^3 \|_{\dot{H}^{\frac{3}{2}}} \\ & \lesssim \| \omega \|_{L^r} \| \nabla \omega \|_{L^r} \| v^3 \|_{\dot{H}^{\frac{3}{2}}} + \| |\nabla_h|^{-\delta} \partial v^3 \|_{L^2} \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2} \| v^3 \|_{\dot{H}^{\frac{3}{2}}}. \end{aligned}$$

On the other hand if $2 < p \leq 4$, then

$$\begin{aligned} & \int \partial_l v^m \partial_m v^l \cdot \partial_3 \partial_k \Delta^{-1} (|\nabla_h|^{-2\delta} \partial_k v^3) dx \\ & \lesssim (\|\omega\|_{L_h^{\frac{4}{1+2\delta}} L_v^p}^2 + \|\partial_3 v^3\|_{L_h^{\frac{4}{1+2\delta}} L_v^p}^2) \|\nabla_h|^{-2\delta} \partial \mathcal{R} v^3\|_{L_h^{(\frac{1}{2}-\delta)^{-1}} L_v^{(1-\frac{2}{p})^{-1}}} \\ & \lesssim \|\omega\|_r^{\frac{2}{p}} \|\nabla \omega\|_r^{2-\frac{2}{p}} \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} + \|\nabla_h|^{-\delta} \partial v^3\|_2^{\frac{2}{p}} \|\nabla_h|^{-\delta} \partial^2 v^3\|_2^{2-\frac{2}{p}} \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}. \end{aligned}$$

Case 2: $l = 3$ or $m = 3$.

The estimate of this term is similar to (5.3) and therefore omitted.

6. The Case $4 < p < \infty$

We shall adopt the same notation as in previous sections. In the estimates to follow, we need to use the homogeneous horizontal Besov norm $\|\cdot\|_{\dot{B}_{p,q}^{h,s}}$ defined for a three-variable function $f = f(x_h, x_3) = f(x_1, x_2, x_3)$ as

$$\|f(\cdot, x_3)\|_{\dot{B}_{p,q}^{h,s}} := \|(2^{js} \|P_j^h f(\cdot, x_3)\|_{L_{x_h}^p})\|_{l_j^q},$$

where $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, and P_j^h is the Littlewood–Paley projection operator in the x_h variable.

6.1. Estimate of ω

Estimate of I_1 : Denote $g = |\omega|^{\frac{r}{2}}$. Then

$$\begin{aligned} |I_1| & \lesssim \|\nabla|^{-\frac{1}{2}+\frac{2}{p}} \partial_3 v^3\|_2 \cdot \|\nabla|^{-\frac{1}{2}-\frac{2}{p}} (g^2)\|_2 \\ & \lesssim \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot \|\nabla|^{-\frac{1}{2}-\frac{2}{p}} g\|_{\left(\frac{1}{2}-\frac{1+\frac{2}{p}}{3}\right)^{-1}} \cdot \|g\|_{\left(\frac{1+\frac{2}{p}}{3}\right)^{-1}} \\ & \lesssim \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot \|\nabla|g\|_2 \cdot \|\nabla|^{-1-\frac{2}{p}} g\|_2 \\ & \lesssim \|v^3\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot \|\omega\|_r^{\frac{r}{2}} \| \nabla (|\omega|^{\frac{r}{2}}) \|_2^{2-\frac{2}{p}}. \end{aligned}$$

Estimate of I_{21} . We have

$$\begin{aligned} |I_{21}| & \lesssim \left\| \|\nabla_h|^{-\frac{\delta}{3}+\frac{2}{p}-\frac{1}{2}} \partial_2 v^3\|_{L_h^{(\frac{1}{r}-\frac{\delta}{2}+\frac{\varepsilon}{2})^{-1}}} \right. \\ & \cdot \left(\|\nabla_h|^{-1} \partial_3^2 v^3\|_{L_h^{\frac{2}{\delta}}} \cdot \|\nabla_h|^{-\frac{\delta}{3}+\frac{1}{2}-\frac{2}{p}} (\omega|\omega|^{r-2})\|_{L_h^{(1-\frac{1}{r}-\frac{\varepsilon}{2})^{-1}}} \right. \\ & \left. \left. + \|\nabla_h|^{-\frac{\delta}{3}-\frac{1}{2}-\frac{2}{p}} \partial_3^2 v^3\|_{L_h^{(\frac{1}{4}+\frac{2}{3}\delta-\frac{1}{p})^{-1}}} \cdot \|\omega\|^{r-1}\|_{L_h^{(\frac{1}{4}-\frac{1}{2}\delta+\frac{1}{p}-\frac{\varepsilon}{2})^{-1}}} \right) \right\|_{L_v^1} \end{aligned}$$

$$\begin{aligned}
 &\lesssim \left\| \left\| |\nabla_h|^{\frac{1}{2} + \frac{2}{p} - \varepsilon} v^3 \right\|_{L_h^2} \cdot \left\| |\nabla_h|^{-\delta} \partial_3^2 v^3 \right\|_{L_h^2} \right. \\
 &\quad \cdot \left(\left\| \omega \right\|_{\dot{B}_{r,2(r-1)}^{h, \frac{\delta}{3} + \frac{1}{2} - \frac{2}{p} + \varepsilon}}^{r-1} + \left\| |\nabla_h|^{\frac{\delta}{3} + \frac{1}{2} - \frac{2}{p} + \varepsilon} \omega \right\|_{L_h^r}^{r-1} \right) \left\| \right\|_{L_v^1} \\
 &\lesssim \left\| |\nabla_h|^{\frac{1}{2} + \frac{2}{p} - \varepsilon} v^3 \right\|_{L_v^{\left(\frac{1}{2} - \varepsilon\right)^{-1}} L_h^2} \cdot \left\| |\nabla_h|^{-\delta} \partial_3^2 v^3 \right\|_{L_v^2 L_h^2} \cdot \left\| \omega \right\|_{L_v^{\frac{r-1}{\varepsilon}} \dot{B}_{r,2(r-1)}^{h, \frac{\delta}{3} + \frac{1}{2} - \frac{2}{p} + \varepsilon}}^{r-1} \\
 &\lesssim \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \cdot \left\| |\nabla_h|^{-\delta} \partial_3^2 v^3 \right\|_2 \cdot \left\| \omega \right\|_{L^2}^{\frac{r}{2}} \cdot \left\| |\nabla_h|^{-\frac{1}{2} + \frac{2}{p}} \omega \right\|_{L^2}^{1 - \frac{2}{p}}.
 \end{aligned}$$

Estimate of I_{22} .

$$\begin{aligned}
 |I_{22}| &\lesssim \left\| \left\| |\nabla_h|^{-1 + \frac{2}{p} + \varepsilon} \partial_2 v^3 \right\|_{L_h^2} \cdot \left(\left\| |\nabla_h|^{-1} \partial_3 \omega \right\|_{L_h^{\frac{3}{2}}} \right. \right. \\
 &\quad \cdot \left\| |\nabla_h|^{1 - \frac{2}{p} - \varepsilon} (\omega|\omega|^{r-2}) \right\|_{L_h^{\left(\frac{1}{2} - \frac{\delta}{3}\right)^{-1}}} \\
 &\quad \left. \left. + \left\| |\nabla_h|^{-\frac{2}{p} - \varepsilon} \partial_3 \omega \right\|_{L_h^{\left(\frac{1}{p} - \frac{1}{p} - \frac{\varepsilon}{2}\right)^{-1}}} \cdot \left\| \omega \right\|_{L_h^{\left(\frac{1}{2} + \frac{1}{p} - \frac{1}{r} + \frac{\varepsilon}{2}\right)^{-1}}} \right) \right\|_{L_v^1} \\
 &\lesssim \left\| \left\| |\nabla_h|^{\frac{2}{p} + \varepsilon} v^3 \right\|_{L_h^2} \cdot \left\| \partial_3 \omega \right\|_{L_h^r} \cdot \left\| \omega \right\|_{L_h^{\frac{1 - \frac{2}{p} - \varepsilon}{r-1}} \dot{B}_{r,2(r-1)}^{h, \frac{1 - \frac{2}{p} - \varepsilon}{r-1}}}^{r-1} \right\|_{L_v^1} \\
 &\lesssim \left\| |\nabla_h|^{\frac{2}{p} + \varepsilon} v^3 \right\|_{L_v^{\frac{1}{\varepsilon}} L_h^2} \cdot \left\| \nabla \omega \right\|_{L_v^r L_h^r} \cdot \left\| \omega \right\|_{L_v^{(1 - \frac{1}{r} - \varepsilon)^{-1}(r-1)} \dot{B}_{r,2(r-1)}^{h, \frac{1 - \frac{2}{p} - \varepsilon}{r-1}}}^{r-1} \\
 &\lesssim \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \cdot \left\| \nabla(|\omega|^{\frac{r}{2}}) \right\|_{L^2}^{\frac{2 - \frac{2}{p}}{r}} \cdot \left\| \omega \right\|_{L^2}^{\frac{2}{p}}.
 \end{aligned}$$

6.2. Estimate of v^3

6.2.1. Estimate of (5.3) This is already done for $2 \leq p < \infty$ in the previous sections.

6.2.2. Estimate of (5.4) Recall by using integration by parts, one has

$$\int (v \cdot \nabla) \partial_k v^3 \cdot |\nabla_h|^{-2\delta} \partial_k v^3 dx = - \int v \partial_k v^3 \cdot \nabla |\nabla_h|^{-2\delta} \partial_k v^3 dx.$$

Case 1: $\int v^3 \partial_k v^3 \partial_3 |\nabla_h|^{-2\delta} \partial_k v^3 dx$. This is already done for $2 \leq p < \infty$.

Case 2: $\int v^h \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3 dx$.

Case 2a: $\int |\nabla_h|^{-1} \omega \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3 dx$.

Denote $T = \nabla_h |\nabla_h|^{-2\delta}$. Then (in the following computation we used a commutator estimate which is proved in [12] for more general operators)

$$\begin{aligned} & 2 \int |\nabla_h|^{-1} \omega \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3 dx \\ &= - \int (T(|\nabla_h|^{-1} \omega \cdot \partial_k v^3) - |\nabla_h|^{-1} \omega T \partial_k v^3) \partial_k v^3 dx \\ &\lesssim \left\| \left(T(|\nabla_h|^{-1} \omega \cdot \partial_k v^3) - |\nabla_h|^{-1} \omega T \partial_k v^3 \right) \right\|_{L_h \left(\frac{5}{8} + \frac{1}{2p} - \frac{\delta}{4} - \frac{\varepsilon}{4} \right)^{-1}} \\ &\quad \cdot \left\| \partial_k v^3 \right\|_{L_h \left(\frac{3}{8} - \frac{1}{2p} + \frac{\delta}{4} + \frac{\varepsilon}{4} \right)^{-1}} \left\| \right\|_{L_v^1} \\ &\lesssim \left\| \left\| |\nabla_h|^{-2\delta} \omega \right\|_{L_h \left(\frac{1}{4} + \frac{1}{p} - \frac{\delta}{2} - \frac{\varepsilon}{2} \right)^{-1}} \cdot \left\| \partial_k v^3 \right\|_{L_h \left(\frac{3}{8} - \frac{1}{2p} + \frac{\delta}{4} + \frac{\varepsilon}{4} \right)^{-1}} \right\|_{L_v^1}^2 \\ &\lesssim \left\| \left\| |\nabla_h|^{-2\delta} \omega \right\|_{L_v^{\frac{1}{\varepsilon}} L_h \left(\frac{1}{4} + \frac{1}{p} - \frac{\delta}{2} - \frac{\varepsilon}{2} \right)^{-1}} \cdot \left\| \partial_k v^3 \right\|_{L_v^{\frac{2}{1-\varepsilon}} L_h \left(\frac{3}{8} - \frac{1}{2p} + \frac{\delta}{4} + \frac{\varepsilon}{4} \right)^{-1}} \right\|_{L_v^1}^2 \\ &\lesssim \left\| |\nabla|^{1-\frac{2}{p}} \omega \right\|_r \cdot \left\| |\nabla|^{\frac{5}{4} + \frac{1}{p} - \frac{\delta}{2}} v^3 \right\|_2^2 \\ &\lesssim \left\| \omega \right\|_r^{\frac{2}{p}} \left\| \nabla \omega \right\|_r^{1-\frac{2}{p}} \cdot \left\| v^3 \right\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \cdot \left\| |\nabla_h|^{-\delta} \partial^2 v^3 \right\|_2. \end{aligned}$$

Case 2b: $\int |\nabla_h|^{-1} \partial_3 v^3 \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3 dx$. We can use a similar commutator estimate as above to derive

$$\begin{aligned} & \int |\nabla_h|^{-1} \partial_3 v^3 \partial_k v^3 \cdot \nabla_h |\nabla_h|^{-2\delta} \partial_k v^3 dx \\ &\lesssim \left\| \left\| |\nabla_h|^{-2\delta} \partial_3 v^3 \right\|_{L_v^2 L_h \left(\frac{3}{8} - \frac{3}{4}\delta \right)^{-1}} \cdot \left\| \partial_k v^3 \right\|_{L_v^4 L_h \left(\frac{5}{16} + \frac{3}{8}\delta \right)^{-1}} \right\|_{L_v^1}^2 \\ &\lesssim \left\| |\nabla|^{\frac{5}{4} - \frac{\delta}{2}} v^3 \right\|_2 \cdot \left\| |\nabla|^{\frac{13}{8} - \frac{3}{4}\delta} v^3 \right\|_2^2 \\ &\lesssim \left\| |\nabla|^{\frac{5}{4} - \frac{\delta}{2}} v^3 \right\|_2 \cdot \left\| |\nabla|^{2-\delta} v^3 \right\|_2 \\ &\lesssim \left\| |\nabla_h|^{-\delta} \partial v^3 \right\|_2^{\frac{2}{p}} \cdot \left\| v^3 \right\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \cdot \left\| |\nabla_h|^{-\delta} \partial^2 v^3 \right\|_2^{2-\frac{2}{p}}. \end{aligned}$$

6.2.3. Estimate of (5.5) We only need to deal with the expression for $l, m \in \{1, 2\}$:

$$\begin{aligned} & \int \partial_l v^m \partial_m v^l \mathcal{R}_3(|\nabla_h|^{-2\delta} \partial_k v^3) dx \\ &= \int \mathcal{R}_2(\partial_3 v^3) \cdot \mathcal{R}_2(\partial_3 v^3) \cdot \mathcal{R}_3(|\nabla_h|^{-2\delta} \partial_k v^3) dx \tag{6.1} \end{aligned}$$

$$+ \int \mathcal{R}_2(\omega) \cdot \mathcal{R}_2(\partial_3 v^3) \cdot \mathcal{R}_3(|\nabla_h|^{-2\delta} \partial_k v^3) dx \tag{6.2}$$

$$\begin{aligned}
 &+ \int \mathcal{R}_2(\omega) \cdot \mathcal{R}_2(\omega) \cdot \mathcal{R}_3(|\nabla_h|^{-2\delta} \partial_k v^3) dx, \\
 &+ \dots
 \end{aligned} \tag{6.3}$$

where in the above $\mathcal{R}_2, \mathcal{R}_3$ denote Riesz type operators in $x_h = (x_1, x_2)$ and the whole space \mathbb{R}^3 respectively. The notation “ \dots ” denotes other omitted terms in the summation which can be represented by either (6.1), (6.2) or (6.3). Clearly

$$\begin{aligned}
 |(6.1)| &\lesssim \| |\nabla_h|^{-2\delta} \partial_k v^3 \|_{L_v^2 L_h^{(\frac{3}{8}-\frac{3}{4}\delta)^{-1}}} \cdot \| \partial_3 v^3 \|^2_{L_v^4 L_h^{(\frac{5}{16}+\frac{3}{8}\delta)^{-1}}} \\
 &\lesssim \| |\nabla_h|^{-\delta} \partial v^3 \|_2^{\frac{2}{p}} \cdot \| v^3 \|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_2^{2-\frac{2}{p}}.
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 |(6.2)| &\lesssim \| \omega \|_{L_v^{\frac{1}{\varepsilon}} L_h^{(\frac{1}{4}+\frac{1}{p}+\frac{\delta}{2}-\frac{\varepsilon}{2})^{-1}}} \cdot \| \partial v^3 \|_{L_v^{\frac{2}{1-\varepsilon}} L_h^{(\frac{3}{8}-\frac{1}{2p}+\frac{\delta}{4}+\frac{\varepsilon}{4})^{-1}}} \\
 &\quad \cdot \| |\nabla_h|^{-2\delta} \partial v^3 \|_{L_v^{\frac{2}{1-\varepsilon}} L_h^{(\frac{3}{8}-\frac{1}{2p}-\frac{3\delta}{4}+\frac{\varepsilon}{4})^{-1}}} \\
 &\lesssim \| |\nabla|^{1-\frac{2}{p}} \omega \|_r \cdot \| |\nabla|^{\frac{5}{4}+\frac{1}{p}-\frac{\delta}{2}} v^3 \|_2^2 \\
 &\lesssim \| \omega \|_r^{\frac{2}{p}} \| \nabla \omega \|_r^{1-\frac{2}{p}} \cdot \| v^3 \|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \cdot \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_2.
 \end{aligned}$$

Finally for (6.3) we can integrate by parts in ∂_k . Then

$$\begin{aligned}
 |(6.3)| &\lesssim \| \nabla \omega \|_r \cdot \| \omega \|_{L_v^{\frac{1}{\varepsilon}} L_h^{(\frac{1}{4}+\frac{1}{p}+\frac{\delta}{2}-\frac{\varepsilon}{2})^{-1}}} \cdot \| |\nabla_h|^{-2\delta} v^3 \|_{L_v^{(1-\frac{1}{p}-\varepsilon)^{-1}} L_h^{(\frac{1}{4}-\frac{\delta}{8}-\frac{1}{p}+\frac{\varepsilon}{2})^{-1}}} \\
 &\lesssim \| \nabla \omega \|_r \cdot \| |\nabla|^{1-\frac{2}{p}} \omega \|_r \cdot \| v^3 \|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}.
 \end{aligned}$$

7. Gronwall and Proof of Main Theorem

7.1. Gronwall for $p = 2$

The estimate of ω is

$$\begin{aligned}
 &\frac{1}{r} \frac{d}{dt} \| \omega \|_{L^r}^r + (r-1) \| \nabla |\omega|^{\frac{r}{2}} \|_{L^2}^2 \\
 &\lesssim \| v^3 \|_{\dot{H}^{\frac{3}{2}}} \| \omega \|_{L^r}^{r-1} \| \nabla \omega \|_{L^r} + \| v^3 \|_{\dot{H}^{\frac{3}{2}}} \| |\nabla_h|^{-\delta} \partial_3^2 v^3 \|_{L^2} \| \omega \|_{L^r}^{r-1} \\
 &\quad + \left(\| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2}^{\frac{1}{2}+\delta+\varepsilon_1} + \| |\nabla_h|^{-\delta} \partial^2 v^3 \|_{L^2}^{\frac{1}{2}+\delta-\varepsilon_1} \right) \cdot \frac{1}{1 + \| \omega \|_{L^r}^{100}} \| \nabla |\omega|^{\frac{r}{2}} \|_{L^2} \\
 &\quad + \sqrt{\log(10 + \| |\nabla_h|^{-\delta} \partial v^3 \|_{L^2} + \| \omega \|_{L^r})} (\| v^3 \|_{\dot{H}^{\frac{3}{2}}} + 1) \| \nabla \omega \|_{L^r} \| \omega \|_{L^r}^{r-1}.
 \end{aligned} \tag{7.1}$$

The estimate of v^3 is

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\sum_{k=1}^3 \|\lvert \nabla_h \rvert^{-\delta} \partial_k v^3\|_{L^2}^2 \right) + \sum_{k=1}^3 \|\lvert \nabla_h \rvert^{-\delta} \partial_k \nabla v^3\|_{L^2}^2 \\ & \lesssim \|v^3\|_{\dot{H}^{\frac{3}{2}}} \|\lvert \nabla_h \rvert^{-\delta} \partial^2 v^3\|_{L^2} \|\lvert \nabla_h \rvert^{-\delta} \partial_k v^3\|_{L^2} + \|\nabla \omega\|_{L^r} \|\lvert \nabla_h \rvert^{-\delta} \partial v^3\|_{L^2} \|v^3\|_{\dot{H}^{\frac{3}{2}}} \\ & \quad + \|\omega\|_{L^r} \|v^3\|_{\dot{H}^{\frac{3}{2}}} \|\lvert \nabla_h \rvert^{-\delta} \partial^2 v^3\|_{L^2} + \|\omega\|_{L^r} \|\nabla \omega\|_{L^r} \|v^3\|_{\dot{H}^{\frac{3}{2}}}. \end{aligned} \tag{7.2}$$

Multiplying inequality (7.1) with $\|\omega\|_{L^r}^{2-r}$ yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^r}^2 + (r-1) \|\omega\|_{L^r}^{2-r} \|\nabla \omega\|_{L^2}^{\frac{r}{2}} \|\omega\|_{L^2}^2 \\ & \lesssim \|v^3\|_{\dot{H}^{\frac{3}{2}}} \|\omega\|_{L^r} \|\nabla \omega\|_{L^r} + \|v^3\|_{\dot{H}^{\frac{3}{2}}} \|\lvert \nabla_h \rvert^{-\delta} \partial_3^2 v^3\|_{L^2} \|\omega\|_{L^r} \\ & \quad + \left(\|\lvert \nabla_h \rvert^{-\delta} \partial^2 v^3\|_{L^2}^{\frac{1}{2} + \delta + \varepsilon_1} + \|\lvert \nabla_h \rvert^{-\delta} \partial^2 v^3\|_{L^2}^{\frac{1}{2} + \delta - \varepsilon_1} \right) \\ & \quad \cdot \frac{1}{1 + \|\omega\|_{L^r}^{100}} \|\nabla \omega\|_{L^2}^{\frac{r}{2}} \|\omega\|_{L^r}^{2-r} \\ & \quad + \sqrt{\log(10 + \|\lvert \nabla_h \rvert^{-\delta} \partial v^3\|_{L^2} + \|\omega\|_{L^r})} (\|v^3\|_{\dot{H}^{\frac{3}{2}}} + 1) \|\nabla \omega\|_{L^r} \|\omega\|_{L^r}. \end{aligned} \tag{7.3}$$

Then, it follows from (7.3) that

$$\begin{aligned} & \frac{d}{dt} \|\omega\|_{L^r}^2 + \|\omega\|_{L^r}^{2-r} \|\nabla \omega\|_{L^2}^{\frac{r}{2}} \|\omega\|_{L^2}^2 \\ & \leq C \|v^3\|_{\dot{H}^{\frac{3}{2}}}^2 \|\omega\|_{L^r}^2 + \frac{1}{100} \|\lvert \nabla_h \rvert^{-\delta} \partial^2 v^3\|_{L^2}^2 \\ & \quad + C \log(10 + \|\lvert \nabla_h \rvert^{-\delta} \partial v^3\|_{L^2} + \|\omega\|_{L^r}) (\|v^3\|_{\dot{H}^{\frac{3}{2}}}^2 + 1) \|\omega\|_{L^r}^2. \end{aligned} \tag{7.4}$$

In addition, we know from (7.2) that

$$\begin{aligned} & \frac{d}{dt} \|\lvert \nabla_h \rvert^{-\delta} \partial v^3\|_{L^2}^2 + \|\lvert \nabla_h \rvert^{-\delta} \partial^2 v^3\|_{L^2}^2 \\ & \leq C \|v^3\|_{\dot{H}^{\frac{3}{2}}}^2 \|\lvert \nabla_h \rvert^{-\delta} \partial v^3\|_{L^2}^2 + C \|v^3\|_{\dot{H}^{\frac{3}{2}}}^2 \|\omega\|_{L^r}^2 + \frac{1}{100} \|\omega\|_{L^r}^{2-r} \|\nabla \omega\|_{L^2}^{\frac{r}{2}} \|\omega\|_{L^2}^2. \end{aligned} \tag{7.5}$$

Adding (7.4) and (7.5) together, one has

$$\begin{aligned} & \frac{d}{dt} (\|\omega\|_{L^r}^2 + \|\lvert \nabla_h \rvert^{-\delta} \partial v^3\|_{L^2}^2) + \frac{1}{2} \|\omega\|_{L^r}^{2-r} \|\nabla \omega\|_{L^2}^{\frac{r}{2}} \|\omega\|_{L^2}^2 + \frac{1}{2} \|\lvert \nabla_h \rvert^{-\delta} \partial^2 v^3\|_{L^2}^2 \\ & \leq C \|v^3\|_{\dot{H}^{\frac{3}{2}}}^2 (\|\omega\|_{L^r}^2 + \|\lvert \nabla_h \rvert^{-\delta} \partial v^3\|_{L^2}^2) \\ & \quad + C \log(10 + \|\lvert \nabla_h \rvert^{-\delta} \partial v^3\|_{L^2} + \|\omega\|_{L^r}) (\|v^3\|_{\dot{H}^{\frac{3}{2}}}^2 + 1) \|\omega\|_{L^r}^2. \end{aligned} \tag{7.6}$$

Using Gronwall inequality, we obtain that

$$\begin{aligned} & \|\omega(t)\|_{L^r}^2 + \||\nabla_h|^{-\delta} \partial v^3(t)\|_{L^2}^2 \\ & \leq (\|\omega_0\|_{L^r}^2 + \||\nabla_h|^{-\delta} \partial v_0^3\|_{L^2}^2 + 10) \exp\{C \int_0^t (\|v^3(s)\|_{\dot{H}^{\frac{3}{2}}}^2 + 1) ds\}. \end{aligned} \tag{7.7}$$

It also follows from (7.6) and (7.7) that

$$\begin{aligned} & \int_0^t \|\omega\|_{L^r}^{2-r} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^2 ds + \int_0^t \||\nabla_h|^{-\delta} \partial^2 v^3\|_{L^2}^2 ds \\ & \leq C(\|\omega\|_{L_t^\infty L_x^r}^2 + \||\nabla_h|^{-\delta} \partial v^3\|_{L_t^\infty L_x^2}^2) \log(10 + \||\nabla_h|^{-\delta} \partial v^3\|_{L_t^\infty L_x^2} + \|\omega\|_{L_t^\infty L_t^r}) \\ & \left[\int_0^t (\|v^3(s)\|_{\dot{H}^{\frac{3}{2}}}^2 + 1) ds \right]. \end{aligned}$$

7.2. Gronwall for $2 < p < \infty$

Now we consider the case when $2 < p < \infty$. The estimate for ω is

$$\begin{aligned} & \frac{1}{r} \frac{d}{dt} \|\omega\|_{L^2}^{\frac{r}{2}} + \frac{4(r-1)}{r^2} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^2 \\ & \leq C \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \||\nabla_h|^{-\delta} \partial_3^2 v^3\|_{L^2} \|\omega\|_{L^2}^{1-\frac{2}{r} + \frac{2}{p}} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}} \\ & + C \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \|\omega\|_{L^2}^{\frac{2}{p}} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^{2-\frac{2}{p}}. \end{aligned} \tag{7.8}$$

Multiplying inequality (7.8) with $\|\omega\|_{L^2}^{\frac{4}{r}-2}$, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\omega\|_{L^2}^{\frac{4}{r}} + \frac{4(r-1)}{r^2} \|\omega\|_{L^2}^{\frac{4}{r}-2} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^2 \\ & \leq C \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \|\omega\|_{L^2}^{\frac{4}{r}} \|\omega\|_{L^2}^{\frac{2}{r}-1} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}} \||\nabla_h|^{-\delta} \partial_3^2 v^3\|_{L^2} \\ & + C \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \|\omega\|_{L^2}^{\frac{4}{r}} \|\omega\|_{L^2}^{\frac{2}{r}-1} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^{2-\frac{2}{p}} \\ & \leq C \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p \|\omega\|_{L^2}^{\frac{4}{r}} + \frac{1}{100} \||\nabla_h|^{-\delta} \partial_3^2 v^3\|_{L^2}^2. \end{aligned} \tag{7.9}$$

The estimate for $\partial_k v^3$ is

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\sum_{k=1}^3 \||\nabla_h|^{-\delta} \partial_k v^3\|_{L^2}^2 \right) + \sum_{k=1}^3 \||\nabla_h|^{-\delta} \partial_k \nabla v^3\|_{L^2}^2 \\ & \leq C \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \||\nabla_h|^{-\delta} \partial_k v^3\|_{L^2}^{\frac{2}{p}} \||\nabla_h|^{-\delta} \partial_k \nabla v^3\|_{L^2}^{2-\frac{2}{p}} \\ & + C \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \|\omega\|_{L^2}^{\frac{4}{r}} \|\omega\|_{L^2}^{\frac{2}{r}-1} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^{2-\frac{2}{p}} \\ & + C \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \|\omega\|_{L^2}^{\frac{4}{r}} \|\omega\|_{L^2}^{\frac{2}{r}-1} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^{1-\frac{2}{p}} \||\nabla_h|^{-\delta} \partial_k \nabla v^3\|_{L^2} \\ & + C \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}} \cdot \|\nabla\omega\|_r \cdot \||\nabla_h|^{-\delta} \partial v^3\|_{L^2}^{\frac{2}{p}} \cdot \||\nabla_h|^{-\delta} \partial^2 v^3\|_{L^2}^{1-\frac{2}{p}}. \end{aligned} \tag{7.10}$$

Adding (7.9) and (7.10) together and using the Young inequality, one has

$$\begin{aligned} & \frac{d}{dt} \left(\|\omega\|_{L^2}^{\frac{r}{2}} \|\omega\|_{L^2}^{\frac{4}{r}} + \sum_{k=1}^3 \|\nabla_h |^{-\delta} \partial_k v^3\|_{L^2}^2 \right) \\ & + \left(\|\omega\|_{L^2}^{\frac{r}{2}} \|\omega\|_{L^2}^{\frac{4}{r}-2} \|\nabla|\omega|^{\frac{r}{2}}\|_{L^2}^2 + \sum_{k=1}^3 \|\nabla_h |^{-\delta} \nabla \partial_k v^3\|_{L^2}^2 \right) \\ & \leq C \|v^3\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p \left(\|\omega\|_{L^2}^{\frac{r}{2}} \|\omega\|_{L^2}^{\frac{4}{r}} + \sum_{k=1}^3 \|\nabla_h |^{-\delta} \partial_k v^3\|_{L^2}^2 \right). \end{aligned}$$

Therefore, standard Gronwall inequality shows that

$$\begin{aligned} & \|\omega\|_{L^2}^{\frac{r}{2}}(t) \|\omega\|_{L^2}^{\frac{4}{r}} + \sum_{k=1}^3 \|\nabla_h |^{-\delta} \partial_k v^3(t)\|_{L^2}^2 \\ & + \int_0^t \|\omega\|_{L^2}^{\frac{r}{2}}(s) \|\omega\|_{L^2}^{\frac{4}{r}-2} \|\nabla|\omega|^{\frac{r}{2}}(s)\|_{L^2}^2 ds + \sum_{k=1}^3 \int_0^t \|\nabla_h |^{-\delta} \nabla \partial_k v^3(s)\|_{L^2}^2 ds \\ & \leq \left(\|\omega_0\|_{L^2}^{\frac{r}{2}} \|\omega_0\|_{L^2}^{\frac{4}{r}} + \sum_{k=1}^3 \|\nabla_h |^{-\delta} \partial_k v_0^3\|_{L^2}^2 \right) \exp \left\{ C \int_0^t \|v^3(s)\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}}^p ds \right\}. \end{aligned} \tag{7.11}$$

Now we are ready to prove the main theorem.

Proof of Theorem 1.1 (for $2 \leq p < \infty$). By smoothing estimates we may assume without loss of generality that $v_0 \in \dot{H}^{\frac{1}{2}} \cap \dot{H}^1$, and $\Omega_0, \nabla^4 \Omega_0 \in L^{r_0}$. With these assumptions (and propagation of regularity) we note that the auxiliary norms $\|\omega\|_r, \|\nabla_h |^{-\delta} \partial v^3\|_2$ are well defined for any $r \in (2 - \varepsilon_0, 2]$ with $\varepsilon_0 > 0$ sufficiently small, during the life span of the local solution.

Now to control the local solution, by using Proposition 3.1 and Remark 3.3, it suffices for us to control $\|\omega\|_{L_t^p \dot{H}^{-\frac{1}{2} + \frac{2}{p}}(0, T^*)}$ if $2 \leq p \leq 4$ and $\|\omega\|_{L_t^\infty L_x^{\tilde{r}}}$, for some \tilde{r} satisfying $\frac{1}{2} < \frac{1}{\tilde{r}} < \frac{2}{3}(1 - \frac{1}{p})$, if $4 < p < \infty$. Consider first the case $4 < p < \infty$. We shall take r sufficiently close to 2. By the Gronwall estimates derived in previous sections, we have uniform estimates on $\|\omega\|_r$. It follows easily that the solution remains regular.

Next for $2 \leq p \leq 4$ we can take r sufficiently close to 2 satisfying also $\frac{2}{p} + \frac{3}{r} - 2 > 0$. Then

$$\begin{aligned} \|\omega\|_{\dot{H}^{-\frac{1}{2} + \frac{2}{p}}} & \lesssim \|\nabla|\omega|^{\frac{2}{p} + \frac{3}{r} - 2}\|_r \\ & \lesssim \|\omega\|_r^{3 - \frac{3}{r} - \frac{2}{p}} \|\nabla\omega\|_r^{\frac{2}{p} + \frac{3}{r} - 2} \\ & \lesssim \|\nabla(|\omega|^{\frac{r}{2}})\|_2^{\frac{2}{p} + \frac{3}{r} - 2} \cdot \|\omega\|_r^{r - \frac{r}{p} - \frac{1}{2}}. \end{aligned}$$

Noting that $0 < \frac{2}{p} + \frac{3}{r} - 2 < \frac{2}{p}$ and $\|\nabla(|\omega|^{\frac{r}{2}})\|_{L^2_x L^2_x} \lesssim 1$, it follows easily that

$$\|\omega\|_{L^p_t \dot{H}^{-\frac{1}{2} + \frac{2}{p}}(0, T^*)} < \infty.$$

Thus the solution remains regular. \square

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Compliance with Ethical standards

Conflict of interest

The authors declare that they have no conflict of interest.

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