TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 376, Number 2, February 2023, Pages 945–967
https://doi.org/10.1090/tran/8708
Article electronically published on November 4, 2022

REMARKS ON THE BERNSTEIN INEQUALITY FOR HIGHER ORDER OPERATORS AND RELATED RESULTS

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Abstract. This note is devoted to several results about frequency localized functions and associated Bernstein inequalities for higher order operators. In particular, we construct some counterexamples for the frequency-localized Bernstein inequalities for higher order Laplacians. We show also that the heat semi-group associated to powers larger than one of the laplacian does not satisfy the strict maximum principle in general. Finally, in a suitable range we provide several positive results.

Contents

1. Introduction

This note is devoted to several results about frequency localized functions and associated Bernstein inequalities for higher order operators. We consider a class of fractional Laplacian operators acting on frequency localized functions on the whole space $\mathbb{R}^d$ or the periodic torus. To fix the notation, we use the following convention for Fourier transform on $\mathbb{R}^d$, $d \ge 1$:

$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{f}(\xi) e^{i\xi\cdot x} d\xi; \tag{1.1}$$

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x} dx. \tag{1.2}$$

For $s>0$, we define the fractional Laplacian operator $\Lambda^s = (-\Delta)^{\frac{s}{2}}$ via the Fourier transform:

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \widehat{f}(\xi), \qquad \xi \in \mathbb{R}^d. \tag{1.3}$$

In yet other words Λ^s corresponds to the Fourier multiplier $|\xi|^s$. Note that for $s=2$ we have $-\Lambda^s = \Delta$, i.e. the usual Laplacian operator. For $0 < s \le 2$, it is known (cf. [2], [7], [13] and the references therein) that the following frequency-localized

Received by the editors September 17, 2021, and, in revised form, March 4, 2022.
2020 *Mathematics Subject Classification.* Primary 35Q35.

Bernstein-type inequality hold: for $1 < p < \infty$ and any band-limited $f \in L^p(\mathbb{R}^d, \mathbb{R})$ with

$$\operatorname{supp}(\widehat{f}) \subset \{\xi : \gamma_1 \le |\xi| \le \gamma_2\}, \tag{1.4}$$

there are constants $A_1 > 0$, $A_2 > 0$ depending only on $(d, p, s, \gamma_1, \gamma_2)$ such that

$$A_2 \|f\|_p^p \le \int_{\mathbb{R}^d} (\Lambda^s f)|f|^{p-2} f dx \le A_1 \|f\|_p^p = A_1 \int_{\mathbb{R}^d} |f|^p dx. \tag{1.5}$$

Note that for $p = 2$, the above inequality is trivial thanks to the usual Plancherel theorem. The main point of (1.5) is that it continues to hold for $p \neq 2$ where the Fourier support of the associated functions have nontrivial overlapping interactions.

By a scaling argument, if $h \in \mathcal{S}(\mathbb{R}^d)$ has frequency localized into $\{|\xi| \sim N\}$ where $N \gg 1$, then it follows from (1.5) that (below $0 < s \le 2$ and $1 < p < \infty$)

$$\int_{\mathbb{R}^d} \Lambda^s h |h|^{p-2} h dx \ge \text{ const} \cdot N^s \|h\|_p^p. \tag{1.6}$$

Such powerful estimates have important applications in the regularity theory of fluid dynamics equations (cf. [13]). For example, consider the dissipative two-dimensional surface quasi-geostrophic equation

$$\partial_t \theta = -\Lambda^s \theta + \Lambda^{-1} \nabla^\perp \theta \cdot \nabla \theta, \tag{1.7}$$

where $0 < s \le 2$. Applying the Littlewood-Paley projection P_j which is localized to $\{|\xi| \sim 2^j\}$ and calculating the L^p norm of $P_j\theta$, we obtain

$$\begin{align} \frac{1}{p}\partial_t(\|P_j\theta\|_p^p) &= -\int_{\mathbb{R}^d} (\Lambda^s P_j\theta)|P_j\theta|^{p-2} P_j\theta dx + \text{Nonlinear terms} \tag{1.8} \\ &\le -\text{const} \cdot 2^{js} \|P_j\theta\|_p^p + \text{Nonlinear terms}, \qquad \text{(by Bernstein)}. \tag{1.9} \end{align}$$

From this and using additional (nontrivial) commutator estimates, one can deduce fine regularity results in various critical and subcritical Besov spaces (see recent [8] for an optimal Gevrey regularity result and the references therein for earlier results). On the other hand, it has been long speculated[1] whether the above Bernstein inequalities also hold for higher order Laplacian operators Λ^s for $s > 2$. The purpose of this note is to demonstrate some counterexamples around these higher operators Λ^s. Our main results are the following.

- Biharmonic operator. See Theorem 2.1. We show via an explicit construction the failure of Bernstein inequalities for the biharmonic operator Δ^2 with $p = 4$.
- Lack of positivity for higher order $e^{-\Lambda^s}\delta_0$, $s > 2$. See Theorem 3.1. We give two proofs to show the general lack of positivity for the higher order heat operators. Some sharp asymptotic decay at spatial infinity is also shown.
- Counterexamples for Bernstein for $s > 2$, $p \in (1, p_0)$ or $p \in (p_1, \infty)$ for some $p_0 < 2 < p_1$. See Theorem 3.2 and Theorem 3.3. For general operators Λ^s with $s > 2$, we show generic failure of Bernstein inequalities for $p = 1+$ or $p = \infty-$.
- Some periodic Bernstein inequalities for Λ^s, $0 < s \le 2$. See Theorem 4.1 and Theorem 4.2. By using a nontrivial complex interpolation argument together with some concentration inequality, we prove a family of Bernstein

[1]We would like to thank Professor Jiahong Wu for raising this intriguing question.

inequalities for mean-zero periodic functions for all $p \in (1, \infty)$. We also show frequency-localized versions in Theorem 4.2.

- A Liouville theorem for Λ^s, $s > 0$. See Theorem 5.1. We prove a rigidity type for the ancient solutions to a fractional heat equation.

The rest of this note is organized according to the above summary.

Notation. For any two positive quantities X and Y, we write $X \lesssim Y$ or $X = O(Y)$ if $X \leq CY$ for some unimportant constant $C > 0$. We write $X \ll Y$ if $X \leq cY$ for some sufficiently small constant $c > 0$. The needed smallness is clear from the context. We write $f \in L^p(\Omega, Y)$ if $f : \Omega \to Y$ and is in L^p. For example $f \in L^2(\mathbb{R}^3, \mathbb{R}^2)$ means f is $\mathbb{R}^2$-valued and

$$\|f\|_2^2 = \int_{\mathbb{R}^3} |f|^2 dx = \int_{\mathbb{R}^3} (f_1^2 + f_2^2) dx < \infty, \qquad \text{here } f = (f_1, f_2)^{\mathrm{T}}. \tag{1.10}$$

For a complex number $z = a + bi$ with $a, b \in \mathbb{R}$, we denote $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$.

We denote the usual sign function $\mathrm{sgn}(x) = 1$ for $x > 0$, -1 for $x < 0$ and 0 if $x = 0$.

We use the Japanese bracket notation $\langle x \rangle = \sqrt{1 + |x|^2}$ for any $x \in \mathbb{R}^d$, $d \geq 1$.

2. Failure of Bernstein inequalities

We start by proving the following result on the failure of Bernstein inequalities for the bi-Laplacian.

Theorem 2.1. *Let the dimension $d \geq 1$. There exists a sequence of Schwartz functions $f_j : \mathbb{R}^d \to \mathbb{R}$ with frequency localized around $N_j \to \infty$, such that*

$$-C_2 < \frac{\int_{\mathbb{R}^d} \Delta^2 f_j f_j^3 dx}{N_j^4 \int_{\mathbb{R}^d} f_j^4 dx} < -C_1 < 0.$$

In the above $C_1 > 0$, $C_2 > 0$ are constants depending only on d. More precisely, the frequency support of f_j satisfies

$$\mathrm{supp}(\widehat{f_j}) \subset \{\xi : \alpha_1 N_j < |\xi| < \alpha_2 N_j\},$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ are constants depending only on d.

Lemma 2.1. *Consider $f_1(x) = \log(1 + x^2)$, $x \in \mathbb{R}$. Denote $f_1^{(4)}(x) = \frac{d^4}{dx^4}(f_1(x))$. Then*

$$I_1 = \int_{\mathbb{R}} f_1^{(4)}(x) f_1(x)^3 dx < 0.$$

Remark 2.1. One can take for example $f_2(x) = x + e^{-x^2}$ to obtain

$$\int f_2^{(4)}(x)(f_2(x))^3 dx \approx -2.47784 < 0. \tag{2.1}$$

However the issue with f_2 is that it is not amenable to localization. Namely if we consider

$$\int (f_2(x)\phi(\frac{x}{R}))^{(4)} (f_2(x)\phi(\frac{x}{R}))^3 \tag{2.2}$$

for a bump function ϕ and R large, then the main order term is

$$\int (x\phi(\frac{x}{R}))^{(4)}(x\phi(\frac{x}{R}))^3 dx = R\int (x\phi(x))^{(4)}(x\phi(x))^3 dx \tag{2.3}$$

which may not take a favorable sign. This subtle issue disappears for the function $f_1(x) = \log(1+x^2)$ due to its mild growth at spatial infinity.

Remark 2.2. Interestingly, if we work with $\frac{d^8}{dx^8}$ instead of $\frac{d^4}{dx^4}$, then we have for $f_2(x) = x + e^{-x^2}$,

$$\int f_2^{(8)}(x)(f_2(x))^3 dx \approx -219.804 < 0. \tag{2.4}$$

One may then take $f_{2,R}(x) = f_2(x)\phi(x/R)$ for R sufficiently large to show

$$\int f_{2,R}^{(8)}(x)(f_{2,R}(x))^3 dx < 0. \tag{2.5}$$

This can be used to construct frequency localized counterexamples for $\Lambda^8 = (-\partial_{xx})^4$.

Remark 2.3. To obtain $I_1 < 0$, we can also adopt a more *numerical* approach in lieu of exact contour integral computation. To this end, denote

$$g(x) = f_1^{(4)}(x)f_1(x)^3 = -12\frac{(1-6x^2+x^4)(\log(1+x^2))^3}{(1+x^2)^4}. \tag{2.6}$$

A schematic drawing of $g(x)$ for $x \in [0, 0.5]$ and $x \in [0, 10]$ can be found in Figure 1. By examining the polynomial $1 - 6x^2 + x^4$ in the definition of $g(x)$, it is easy

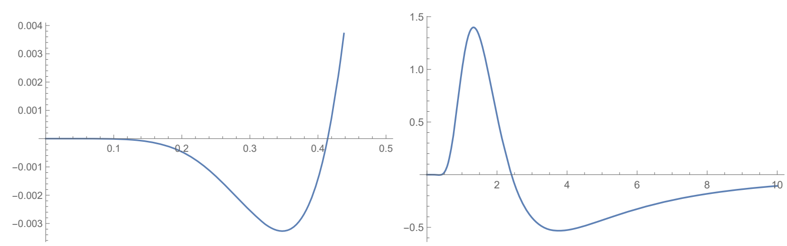

FIGURE 1. The plot of $g(x)$ for $0 \le x \le 0.4$ and $0 \le x \le 10$

to check that $g(x) > 0$ for $x \in (\sqrt{2}-1, \sqrt{2}+1)$ and $g(x) < 0$ for $x < \sqrt{2}-1$ or $x > \sqrt{2}+1$. In particular

$$I_1 < 2\int_{0\le x\le 10} g(x)dx \approx -1.65835. \tag{2.7}$$

Proof. To ease the notation we write f_1 as f and $\int_{\mathbb{R}} dx$ as $\int$. By successive integration by parts, we have

$$\begin{aligned} I_1 &= \int f''(f^3)'' = \int f''(3f^2f')' \\ &= 3\int f^2(f'')^2 + \int 6f''(f')^2 f \\ &= 3\int f^2(f'')^2 - 2\int (f')^4. \end{aligned}$$

Note that $f'(x) = 2x/(1+x^2)$. By a contour integral computation, it is not difficult to check that

$$2\int (f')^4 = 2\pi.$$

On the other hand (see Appendix A), we have

$$3\int f^2(f'')^2 = -\frac{29}{6}\pi + \pi^3 + (\log 4)(-7+\log 64)\pi. \tag{2.8}$$

Thus

$$I_1 = -\frac{41}{6}\pi + \pi^3 + (\log 4)(-7+\log 64)\pi \approx -2.83.$$

□

We now complete the proof of Theorem 2.1.

Proof of Theorem 2.1. We proceed in several steps.

Step 1. We first construct f_j in the one dimensional case. To ease the notation we shall denote

$$I(f) = \int_{\mathbb{R}} \partial_x^4 f f^3 dx.$$

We choose f_1 as in Lemma 2.1. Clearly

$$I(f_1) = \int_{\mathbb{R}} \partial_x^4 f_1 f_1^3 dx \approx -2.83 < 0.$$

Define for $R \geq 2$

$$f_R(x) = f_1(x)\phi(x/R) = \log(1+x^2)\phi(x/R),$$

where $\phi \in C_c^\infty(\mathbb{R})$ is such that $\phi(z) = 1$ for $|z| \leq 1$ and $\phi(z) = 0$ for $|z| \geq 2$. By taking R sufficiently large, it is not difficult to check that $I(f_R) < 0$. As a matter of fact $I(f_R) \to I(f_1)$ as $R \to \infty$. We now fix $R = R_0$ such that $I_{R_0} < 0$. Clearly $f_{R_0} \in C_c^\infty(\mathbb{R})$.

Next we take $\epsilon > 0$ and define $h_\epsilon \in \mathcal{S}(\mathbb{R})$ such that

$$\widehat{h_\epsilon}(\xi) = \phi(\epsilon\xi)\left(1 - \phi\Big(\frac{\xi}{\epsilon}\Big)\right)\widehat{f_{R_0}}(\xi), \qquad \xi \in \mathbb{R}.$$

Clearly $I(h_\epsilon) \to I(f_{R_0})$ as $\epsilon \to 0$. Thus we can fix $\epsilon_0 > 0$ sufficiently small such that $I(h_{\epsilon_0}) < 0$.

Finally we define $f_j \in \mathcal{S}(\mathbb{R})$ such that

$$\widehat{f_j}(\xi) = N_j^{-\frac{3}{4}}\widehat{h_{\epsilon_0}}\Big(\frac{\xi}{N_j}\Big).$$

On the real side, we have

$$f_j(x) = N_j^{\frac{1}{4}} h_{\epsilon_0}(N_j x).$$

Apparently $\|f_j\|_4 = \|h_{\epsilon_0}\|_4$ for all j. Clearly f_j satisfies the desired constraints in dimension $d = 1$.

Step 2. Higher dimensions. With no loss we consider dimension $d = 2$. The case for $d \geq 3$ is similar and omitted. Define

$$f_j(x_1, x_2) = N_j^{\frac{1}{4}} h_{\epsilon_0}(N_j x_1)\psi(x_2),$$

where h_{ϵ_0} was specified in Step 1, and $\psi \in \mathcal{S}(\mathbb{R})$ is chosen to have frequency localized to $\frac{1}{2} \leq |\xi| \leq 1$. Clearly

$$\int_{\mathbb{R}^2} \Delta^2 f_j f_j^3 dx = \int_{\mathbb{R}^2} \partial_{x_1}^4 f_j f_j^3 dx + \text{l.o.t.}$$

The desired conclusion follows easily.

$\square$

Remark. For the end-point case $p = \infty$, a somewhat related issue is the maximum principle for general fractional operators and their spectral discretizations. In this connection a new effective maximum principle is worked out in recent [9]. See also [10] for some related stability estimates on singular phase field models.

3. Lack of positivity for the kernel of $e^{-\Lambda^s}$, $s > 2$

Consider $s > 2$, and fix any $p \neq 2$. A general question is whether one can find smooth frequency localized f such that

$$\int_{\mathbb{R}^d} (-\Delta)^{\frac{s}{2}} f |f|^{p-2} f dx < 0.$$

In this section we investigate somewhat more general situations concerning Λ^s, $s > 2$ and the connections with the lack of positivity of the fundamental solution for higher order heat propagators. We first start with Lemma 3.1.

Lemma 3.1 ([16]). *Let $\alpha > 0$. Define*

$$F_\alpha(x) = \int_0^\infty e^{-t^\alpha} \cos xt dt, \quad x > 0.$$

Then

$$\lim_{x \to \infty} x^{\alpha+1} F_\alpha(x) = \Gamma(\alpha + 1) \sin \frac{\pi\alpha}{2}.$$

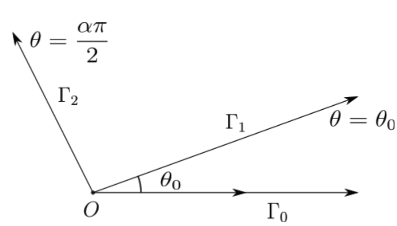

FIGURE 2. Contours Γ_1 and Γ_2

Proof. We briefly recall the argument of Polya as follows. First by using partial integration one has

$$\begin{aligned} x^{\alpha+1}F_\alpha(x) &= x^\alpha \int_0^\infty (\sin xt) e^{-t^\alpha} d(t^\alpha) \\ &= \int_0^\infty \sin u^{\frac{1}{\alpha}} e^{-x^{-\alpha}u} du \qquad (\ u = t^\alpha x^\alpha\ ) \\ &= \mathrm{Im}\Big(\int_0^\infty e^{iu^{\frac{1}{\alpha}} - x^{-\alpha}u} du\Big). \end{aligned}$$

Now first deform the contour $\Gamma_0 = [0, \infty)$ (see Figure 2) to $\Gamma_1 = \{u = re^{i\theta_0} : 0 \le r < \infty\}$ for $0 < \theta_0 \ll 1$, one has[2]

$$\lim_{x\to\infty} x^{\alpha+1} F_\alpha(x) = \mathrm{Im}\Big(\int_{\Gamma_1} e^{iu^{\frac{1}{\alpha}}} du\Big).$$

One can then deform the latter integral from Γ_1 to $\Gamma_2 = \{u = re^{i\frac{\alpha\pi}{2}} : 0 \le r < \infty\}$ to obtain

$$\mathrm{Im}\Big(\int_{\Gamma_1} e^{iu^{\frac{1}{\alpha}}} du\Big) = \sin(\frac{\pi\alpha}{2}) \int_0^\infty e^{-r^{\frac{1}{\alpha}}} dr = \Gamma(\alpha+1) \sin\frac{\pi\alpha}{2}.$$

□

Remark 3.1. We shall need to use the standard Bessel functions: for $\nu > 0$ and $z = \rho e^{i\theta}$ with $-\pi < \theta \le \pi$,

$$J_\nu(z) = C_\nu z^\nu \int_{-1}^1 (1-s^2)^{\nu-\frac{1}{2}} e^{isz} ds,$$

where $C_\nu > 0$ depends only on ν. In particular we recall the usual formula for Bessel functions (cf. pp. 11 of [3]):

$$\frac{d}{dz}(J_{\nu+1}(z) z^{\nu+1}) = J_\nu(z) z^{\nu+1}. \tag{3.1}$$

We also recall the definition of the Hankel functions (cf. pp. 168 of [14] and pp. 149 of [12]): for $\mathrm{Re}(\nu+\frac{1}{2}) > 0$, $-\frac{1}{2}\pi < \arg(z) < \frac{3}{2}\pi$,

$$H_\nu^{(1)}(z) := \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \frac{1}{\Gamma(\nu+\frac{1}{2})} e^{i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)} \int_0^\infty e^{-u} u^{\nu-\frac{1}{2}} (1+\frac{iu}{2z})^{\nu-\frac{1}{2}} du. \tag{3.2}$$

Lemma 3.2 ([1]). *Consider $d \ge 2$. Let $\alpha > 0$ and define*

$$F_\alpha(x) = \int_{\mathbb{R}^d} e^{-|\xi|^\alpha} e^{ix\cdot\xi} d\xi, \qquad x \in \mathbb{R}^d. \tag{3.3}$$

Then

$$\lim_{|x|\to\infty} |x|^{d+\alpha} F_\alpha(x) = C_{d,\alpha} \sin\frac{\alpha\pi}{2}, \tag{3.4}$$

where $C_{d,\alpha} > 0$ depends only on $(d,\ \alpha)$.

[2]This step is necessary since the integrand contains $e^{-x^{-\alpha}u} = e^{-x^{-\alpha}(r\cos\theta + ir\sin\theta)}$ for $u = re^{i\theta}$, and $r\cos\theta$ may become negative if θ goes from 0 to $\frac{\alpha\pi}{2}$ especially when $\alpha > 1$.

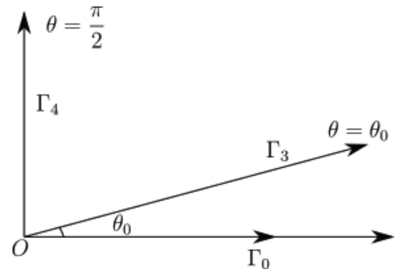

FIGURE 3. Contours Γ_3 and Γ_4

Proof. To simplify the notation we shall denote by C a positive constant depending only on $(d,\ \alpha)$ which may vary from line to line. Denote $r = |x|$ and $t = |\xi|$. By passing to hyper-spherical coordinates, we have

$$\begin{aligned} r^{d+\alpha} F_\alpha(x) &= C r^{d+\alpha} \int_0^\infty e^{-t^\alpha} t^{d-1} \int_0^1 (1-s^2)^{\frac{d-3}{2}} \cos(rts)\, dsdt \\ &= C r^\alpha \int_0^\infty e^{-r^{-\alpha} t^\alpha} t^{d-1} \int_0^1 (1-s^2)^{\frac{d-3}{2}} \cos(ts) dsdt \qquad (rt \to t). \end{aligned}$$

It is not difficult to check that (see (3.1), or one can verify directly the computation)

$$\frac{d}{dt}\Big(t^d \int_0^1 (1-s^2)^{\frac{d-1}{2}} \cos(ts) ds\Big) = C t^{d-1} \int_0^1 (1-s^2)^{\frac{d-3}{2}} \cos(ts) ds.$$

Thus

$$\begin{aligned} r^{d+\alpha} F_\alpha(x) &= C \int_0^\infty e^{-r^{-\alpha} t^\alpha} t^{d+\alpha-1} \int_0^1 (1-s^2)^{\frac{d-1}{2}} \cos(ts) dsdt \\ &= C \int_0^\infty e^{-r^{-\alpha} t^\alpha} t^{\frac{d}{2}+\alpha-1} J_{\frac{d}{2}}(t) dt = C \mathrm{Re} \left(\int_0^\infty e^{-r^{-\alpha} t^\alpha} t^{\frac{d}{2}+\alpha-1} H^{(1)}_{\frac{d}{2}}(t) dt \right). \end{aligned}$$

By (3.2), it suffices for us to examine (below $0 < \theta_0 \ll 1$ is a fixed angle)

$$\begin{aligned} &\lim_{\epsilon \to 0} \mathrm{Re} \left(\int_0^\infty e^{-\epsilon t^\alpha} t^{\frac{d}{2}+\alpha-1} \Big(t^{-\frac{1}{2}} e^{it - i\frac{d+1}{4}\pi} \int_0^\infty e^{-s} s^{\frac{n-1}{2}} (1 + \frac{is}{2t})^{\frac{d-1}{2}} ds \Big) dt \right) \\ &= \mathrm{Re} \left(\int_{\Gamma_3} z^{\frac{d}{2}+\alpha-\frac{3}{2}} e^{iz - i\frac{d+1}{4}\pi} \Big(\int_0^\infty e^{-s} s^{\frac{d-1}{2}} (1 + \frac{is}{2z})^{\frac{d-1}{2}} ds \Big) dz \right) \\ &\qquad\qquad (\Gamma_3 : \{z = re^{i\theta_0} :\ 0 \le r < \infty\}),\ \text{see Fig. 3} \\ &= \mathrm{Re} \left(\int_{\Gamma_4} z^{\frac{d}{2}+\alpha-\frac{3}{2}} e^{iz - i\frac{d+1}{4}\pi} \Big(\int_0^\infty e^{-s} s^{\frac{d-1}{2}} (1 + \frac{is}{2z})^{\frac{d-1}{2}} ds \Big) dz \right) \\ &\qquad\qquad (\Gamma_4 : \{z = \rho i :\ 0 \le \rho < \infty\}),\ \text{see Fig. 3} \\ &= (\sin \frac{\alpha\pi}{2}) \int_0^\infty \rho^{\frac{d}{2}+\alpha-\frac{3}{2}} e^{-\rho} \Big(\int_0^\infty e^{-s} s^{\frac{d-1}{2}} (1 + \frac{s}{2\rho})^{\frac{d-1}{2}} ds \Big) d\rho. \end{aligned}$$

The desired result clearly follows. □

Theorem 3.1 (Lack of positivity for the propagator $e^{-\Lambda^s}$ when $s > 2$)**.** *Define for $s > 0$,*

$$K_s(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-|\xi|^s} e^{i\xi \cdot x} d\xi.$$

If $s > 2$, then

$$\min_{x\in\mathbb{R}} K_s(x) < 0.$$

More generally define

$$K_{s,d}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-|\xi|^s} e^{i\xi\cdot x} d\xi.$$

If $s > 2$, then

$$\min_{x\in\mathbb{R}^d} K_{s,d}(x) < 0.$$

Proof. We first consider the 1D case. By Lemma 3.1, it is not difficult to check that $\inf K_s(x) < 0$ for $2 < s < 4$. We claim that for any $s \geq 4$, we must have $\inf K_s(x) < 0$. Assume this is not true and for some $s_0 \geq 4$, it holds that $K_{s_0}(\cdot)$ is always nonnegative. By using the usual subordination principle, for any $\beta \in (0,1)$, $t > 0$, it holds that

$$e^{-t^\beta} = \int_0^\infty e^{-\lambda t} d\mu_\beta(\lambda),$$

where $d\mu_\beta$ is a positive measure. Taking $\beta_0 \in (0,1)$ sufficiently small, we have

$$e^{-|\xi|^{s_0\beta_0}} = \int_0^\infty e^{-\lambda|\xi|^{s_0}} d\mu_{\beta_0}(\lambda),$$

where $s_0\beta_0 \in (2,4)$. But then it follows that $K_{s_0\beta_0}$ must be nonnegative. This is clearly a contradiction. This finishes the proof for the 1D case. The higher dimensional case is similar by using Lemma 3.2. $\square$

We now give yet another proof of Theorem 3.1 based on a contradiction argument. We first recall the usual Bochner theorem: namely if $F(\xi) = \mathbb{E}e^{-i\xi\cdot x}$ ($\mathbb{E}(\cdot)$ denotes taking expectation with respect to some probability measure on $\mathbb{R}^n$), then $F(\cdot)$ must be a positive definite function, i.e. for all $n \in \mathbb{N}$ and all $\xi_i, \xi_j \in \mathbb{R}^d$, the $n \times n$-matrix whose entries are $F(\xi_i - \xi_j)$ is positive semi-definite.

In particular we must have

$$|F(\xi)| \leq |F(0)|, \qquad \forall\, \xi \in \mathbb{R}^d. \tag{3.5}$$

With this we now give an alternative proof of Theorem 3.1.

2[nd] *proof of Theorem* 3.1. We argue by contradiction. Assume that

$$e^{-|\xi|^s} = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx, \tag{3.6}$$

where f is nonnegative for all $x \in \mathbb{R}^d$. We shall deduce a contradiction.

By using Fourier transform it is not difficult to check that $|\cdot|^2 f(\cdot) \in L^1_x(\mathbb{R}^d)$. In particular we have

$$\tilde{F}(\xi) = -\Delta_\xi(e^{-|\xi|^s}) = \int_{\mathbb{R}^d} f(x)|x|^2 e^{-ix\cdot\xi} dx \tag{3.7}$$

and $\tilde{F}(\cdot)$ is continuous and positive definite (i.e. for all $n \in \mathbb{N}$ and all $\xi_i, \xi_j \in \mathbb{R}^d$, the $n \times n$-matrix whose entries are $\tilde{F}(\xi_i - \xi_j)$ is positive semi-definite). Thus we must have

$$|\tilde{F}(\xi)| \leq |\tilde{F}(0)|, \qquad \forall\, \xi \in \mathbb{R}^d. \tag{3.8}$$

However since $s > 2$, it is easy to check that $\tilde{F}(0) = 0$ which clearly gives a contradiction! $\square$

We now draw some consequences of the previous theorem.

Theorem 3.2. *Let the dimension $d \geq 1$. Let $s > 2$. There exists $p_0 > 2$ depending only on (s, d) such that for any $p \in [p_0, \infty)$, we can find $f \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ such that*

$$\int_{\mathbb{R}^d} (\Lambda^s f)|f|^{p-2} f dx \leq -C_{p,d,s} < 0, \tag{3.9}$$

where $C_{p,d,s} > 0$ depends only on (p, d, s).

Furthermore for the same $p \in [p_0, \infty)$, there exists a sequence of Schwartz functions $f_j : \mathbb{R}^d \to \mathbb{R}$ with frequency localized around $N_j \to \infty$, such that

$$-D_2 < \frac{\int_{\mathbb{R}^d} \Lambda^s f_j |f_j|^{p-2} f_j dx}{N_j^s \int_{\mathbb{R}^d} |f_j|^p dx} < -D_1 < 0.$$

In the above $D_1 > 0$, $D_2 > 0$ are constants depending on (p, d, s). More precisely, the frequency support of f_j satisfies

$$\operatorname{supp}(\widehat{f_j}) \subset \{\xi : \alpha_1 N_j < |\xi| < \alpha_2 N_j\},$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ are constants depending on (p, d, s).

Proof. It suffices for us to prove (3.9). The frequency localized version follows from similar arguments as in Theorem 2.1. Fix $s > 2$ and denote $K = e^{-\Lambda^s}\delta_0$ as the kernel function corresponding to $e^{-\Lambda^s}$. By Theorem 3.1, we clearly have $\|K\|_{L_x^1(\mathbb{R}^d)} > 1$. By using the spatial decay of K, we have for some L_0 sufficiently large

$$\int_{\mathbb{R}^d} K(y) \operatorname{sgn}(K(y)) \chi_{|y| \leq L_0} dy > 1. \tag{3.10}$$

By suitably mollifying the function $\operatorname{sgn}(K(y))\chi_{|y|\leq L_0}$, we obtain for some $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\|\psi\|_\infty \leq 1$ that

$$\int_{\mathbb{R}^d} K(y)\psi(y) dy > 1. \tag{3.11}$$

Thus for some $\beta_1 > 0$,

$$\|e^{-\Lambda^s}\psi\|_\infty \geq (1 + 2\beta_1)\|\psi\|_\infty. \tag{3.12}$$

Since $\lim_{p\to\infty} \|\psi\|_p = \|\psi\|_\infty$ and $\lim_{p\to\infty} \|e^{-\Lambda^s}\psi\|_p = \|e^{-\Lambda^s}\psi\|_\infty$, we can find p_0 sufficiently large such that for all $p \in [p_0, \infty)$,

$$\|e^{-\Lambda^s}\psi\|_p \geq (1 + \beta_1)\|\psi\|_p. \tag{3.13}$$

Define $\psi_1 = \psi/\|\psi\|_p$. Clearly $\|\psi_1\|_p = 1$ and

$$\int_0^1 \frac{d}{dt}\Big(\|e^{-t\Lambda^s}\psi_1\|_p^p\Big) dt = \|e^{-\Lambda^s}\psi_1\|_p^p - 1 \geq (1 + \beta_1)^p - 1 =: \tilde{c}_1 > 0. \tag{3.14}$$

Thus for some $t_0 \in (0, 1)$, the mean value theorem ensures

$$\frac{d}{dt}\Big(\|e^{-t\Lambda^s}\psi_1\|_p^p\Big)\Big|_{t=t_0} \geq \frac{1}{2}\tilde{c}_1 > 0. \tag{3.15}$$

Denoting $\psi_2(t) = e^{-t\Lambda^s}\psi_1$, we now compute

$$\frac{d}{dt}\Big(\|e^{-t\Lambda^s}\psi_1\|_p^p\Big) = -p\int_{\mathbb{R}^d} \Lambda^s\psi_2(t)|\psi_2(t)|^{p-2}\psi_2(t)\,dx.$$

Denoting now $\psi_2^0 = \psi_2(t_0) = e^{-t_0\Lambda^s}\psi_1$, one gets (for some $\tilde{c}_2 > 0$)

$$\int_{\mathbb{R}^d} \Lambda^s\psi_2^0|\psi_2^0|^{p-2}\psi_2^0 dx \le -\tilde{c}_2 < 0. \tag{3.16}$$

It follows from a density argument that for some $\psi_3 \in C_c^\infty(\mathbb{R}^d)$ and some $\tilde{c}_3 > 0$,

$$\int_{\mathbb{R}^d} \Lambda^s\psi_3|\psi_3|^{p-2}\psi_3 dx \le -\tilde{c}_3 < 0. \tag{3.17}$$

Thus (3.9) is proved. □

Theorem 3.3. *Let the dimension $d \ge 1$. Let $s > 2$. There exists $1 < p_1 < 2$ depending only on (s, d) such that for any $p \in (1, p_1]$, we can find $f \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ such that*

$$\int_{\mathbb{R}^d} (\Lambda^s f)|f|^{p-2} f dx \le -C_{p,d,s} < 0, \tag{3.18}$$

where $C_{p,d,s} > 0$ depends only on (p, d, s).

Furthermore for the same $p \in (1, p_1]$, there exists a sequence of Schwartz functions $f_j : \mathbb{R}^d \to \mathbb{R}$ with frequency localized around $N_j \to \infty$, such that

$$-D_2 < \frac{\int_{\mathbb{R}^d} \Lambda^s f_j|f_j|^{p-2} f_j dx}{N_j^s \int_{\mathbb{R}^d} |f_j|^p dx} < -D_1 < 0. \tag{3.19}$$

In the above $D_1 > 0$, $D_2 > 0$ are constants depending on (p, d, s). More precisely, the frequency support of f_j satisfies

$$\operatorname{supp}(\widehat{f_j}) \subset \{\xi : \alpha_1 N_j < |\xi| < \alpha_2 N_j\}, \tag{3.20}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$ are constants depending on (p, d, s).

Proof. We only need to show (3.18). The idea is to use the construction in Theorem 3.2 and duality. Denote $T = e^{-\Lambda^s}$. Let $p_0 > 2$ be the same as in Theorem 3.2. Denote $p_1 = p_0/(p_0 - 1)$. For $p \in (1, p_1]$, denote $p' = p/(p-1) \in [p_0, \infty)$. By using the proof of Theorem 3.2, we can find $f \in C_c^\infty(\mathbb{R}^d)$ with $\|f\|_{p'} = 1$ such that for some constant $\gamma_1 > 0$,

$$\|Tf\|_{p'} \ge 1 + 2\gamma_1. \tag{3.21}$$

Since

$$\|Tf\|_{p'} = \sup_{\|\psi\|_p = 1} \langle \psi, Tf\rangle, \tag{3.22}$$

we can find $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\|\psi\|_p = 1$ such that

$$1 + \gamma_1 \le \langle \psi, Tf\rangle = \langle T\psi, f\rangle \le \|T\psi\|_p \|f\|_{p'} = \|T\psi\|_p. \tag{3.23}$$

We can then use the inequality

$$\int_0^1 \frac{d}{dt}\Big(\|e^{-t\Lambda^s}\psi\|_p^p\Big) dt = \|e^{-\Lambda^s}\psi\|_p^p - 1 \ge (1+\gamma_1)^p - 1 \tag{3.24}$$

to obtain (3.18). □

Remark 3.2. The previous theorems show the failure also for $s>2$ of the Stroock-Varopoulos inequality. More precisely, the Stroock-Varopoulos inequality states that for $s\in(0,2)$ and $p>1$ there holds

$$\int_{\mathbb{R}^d} |v|^{p-2}v(-\Delta)^{s/2}v \geq \frac{4(p-1)}{p^2}\int_{\mathbb{R}^d} |(-\Delta)^{s/4}|v|^{p/2}|^2 \tag{3.25}$$

for all $v\in L^p(\mathbb{R}^d)$ such that $(-\Delta)^{s/2}v\in L^p(\mathbb{R}^d)$. Therefore, the results in Theorem 3.3 show that only the trivial function satisfies (3.25). This result is expected in view of our results since the Stroock-Varopoulos inequality holds for *positive* sub-markovian processes (recall that the fractional laplacian is the infinitesimal generator of a Levy process with pure jumps) (see [19]).

Remark 3.3. In [6], Lieb considered maximizers for the problem:

$$\sup_f \frac{\|\mathcal{G}f\|_q}{\|f\|_p}, \tag{3.26}$$

where $\mathcal{G}$ is an integral operator with Gaussian kernel G, and $1<p,q<\infty$. For degenerate and centered Gaussian kernel G (see equation (1.3) in [6]) the supremum can be shown to be taken over centered Gaussian functions. In particular if we consider the problem[3]

$$\sup_f \frac{\|e^{\Delta}f\|_p}{\|f\|_p}, \tag{3.27}$$

for $p\in(1,\infty)$, then it is clear that one may take $f_n=e^{t_n\Delta}\delta_0$, with $t_n\to\infty$ as $n\to\infty$ in order to saturate the optimal operator norm bound 1. On the other hand, for general signed kernel G an intriguing problem is to classify the maximizers or the maximizing sequence. These type of results will improve our understanding of the Bernstein-type inequalities.

4. Bernstein inequality for the periodic case

We now turn to positive results, i.e. a situation where positivity occurs and one can deduce several estimates. This section is of independent interest and will be useful to deal with sharp wellposedness for surface quasi-geostrophic equation (SQG) in periodic domains. We shall show some results for the fractional Laplacian operator Λ^s, $0<s\leq 2$ on the periodic torus. Let $\mathbb{T}^d=\mathbb{R}^d/\mathbb{Z}^d=[-\frac{1}{2},\frac{1}{2}]^d$.

For any integrable $f:\mathbb{T}^d\to\mathbb{C}$, denote

$$\boxed{\langle f\rangle = \int_{\mathbb{T}^d} f(x)dx.}$$

We use the following convention for Fourier transform on $\mathbb{T}^d$:

$$\widehat{f}(k)=\int_{\mathbb{T}^d} f(x)e^{-2\pi i k\cdot x}dx, \qquad k\in\mathbb{Z}^d; \tag{4.1}$$

$$f(x)=\sum_{k\in\mathbb{Z}^d}\widehat{f}(k)e^{2\pi i k\cdot x}, \qquad x\in\mathbb{T}^d. \tag{4.2}$$

[3]Note that the kernel corresponding to e^{Δ} is $K(x,y)=(4\pi)^{-d}e^{-\frac{|x-y|^2}{4}}$ which is degenerate in the language of [6].

The fractional laplacian operator $\Lambda^s = (-\Delta)^{s/2}$, $s > 0$ on $\mathbb{T}^d$ is defined as

$$\widehat{\Lambda^s f}(k) = |k|^s \widehat{f}(k), \qquad k \in \mathbb{Z}^d. \tag{4.3}$$

In yet other words it corresponds to the Fourier multiplier $|k|^s$. Note that $\widehat{f}(0) = \langle f \rangle$.

Theorem 4.1 (Bernstein inequality on the torus). *Let $0 < s \le 2$ and consider Λ^s on $\mathbb{T}^d = [-\frac{1}{2}, \frac{1}{2}]^d$, $d \ge 1$. Let $1 < p < \infty$. For any smooth $f : \mathbb{T}^d \to \mathbb{R}$ with $\langle f \rangle = 0$, we have*

$$\|e^{-t\Lambda^s} f\|_p \le e^{-c_{p,s,d}t} \|f\|_p, \qquad \forall\, t > 0. \tag{4.4}$$

Here $c_{p,s,d} > 0$ depends only on (p, s, d). We also have for any smooth f with $\langle f \rangle = 0$, it holds that

$$\int_{\mathbb{T}^d} (\Lambda^s f)|f|^{p-2} f dx \ge \tilde{c}_{p,s,d} \|f\|_p^p, \tag{4.5}$$

where $\tilde{c}_{p,s,d} > 0$ depends only on (p, s, d).

Remark 4.1. The inequality (4.5) follows from (4.4) by examining the time derivative at $t = 0+$. Conversely one may also deduce (4.4) from (4.5). However we shall not dwell on these technical issues here.

Remark 4.2. Similar results hold if f is complex-valued or vector-valued. For example if $f : \mathbb{T}^d \to \mathbb{R}^{d_1}$ and $\int_{\mathbb{T}^d} f dx = 0$, then we have

$$\| |e^{-t\Lambda^s f}| \|_p \le e^{-ct} \| |f| \|_p, \tag{4.6}$$

where $|f| = \sqrt{f_1^2 + \cdots + f_{d_1}^2}$. For complex-valued f, (4.5) should be replaced by

$$\int_{\mathbb{T}^d} (\Lambda^s f)|f|^{p-2} f^* dx \ge \tilde{c}_{p,s,d} \|f\|_p^p, \tag{4.7}$$

where f^* denotes the complex conjugate of f.

Remark 4.3. We briefly explain the heuristics as follows. Consider the case $s = 2$, i.e. the usual Laplacian $\Delta = -\Lambda^2$. Clearly we have

$$\begin{aligned} &\|e^{t\Delta} f\|_2 \le e^{-ct} \|f\|_2, \qquad \forall\, f \text{ with } \langle f \rangle = 0; \\ &\|e^{t\Delta} f\|_\infty \le \|f\|_\infty; \\ &\|e^{t\Delta} f\|_1 \le \|f\|_1. \end{aligned}$$

By *formally* interpolating the above two inequalities, it is natural to expect that for any $p \in (1, \infty)$,

$$\|e^{t\Delta} f\|_p \le e^{-c_p t} \|f\|_p, \qquad \forall\, f \text{ with } \langle f \rangle = 0,$$

where $c_p > 0$. However due to the presence of the constraint $\langle f \rangle = 0$, this requires some nontrivial interpolation of Riesz-Thorin type. The technical difficulty is that the usual Riesz-Thorin interpolation employs a nonlinear functor which in general does not preserve the condition $\langle f \rangle = 0$. This interpolation argument might be known but we could not find it in the literature. So we provide full details. Nevertheless in Theorem 4.1 we overcome this difficulty by proving some nontrivial concentration-type inequalities.

Lemma 4.1 (Strong Phragmen-Lindelof estimate). *Suppose h is an analytic function on the strip $0 < \mathrm{Re}(z) < 1$ and is continuous up to the boundary. Assume for some constant $\alpha < \pi$ and constant A,*

$$|h(z)| \le e^{Ae^{a|\mathrm{Re}(z)|}}, \qquad \forall\, z \text{ in the closed strip.} \tag{4.8}$$

Then for any $0 < \theta < 1$, we have

$$|h(\theta)| \le \exp\Big(\frac{\sin \pi\theta}{2}\int_{-\infty}^{\infty}\Big(\frac{\log|h(iy)|}{\cosh \pi y - \cos \pi\theta} + \frac{\log|h(1+iy)|}{\cosh \pi y + \cos\pi\theta}\Big)dy\Big). \tag{4.9}$$

Proof. See for example Chapter 5.4 of [18]. $\square$

Lemma 4.2 (Small mean implies short-time decay). *Let $s > 0$ and consider the torus $\mathbb{T}^d$, $d \ge 1$. Suppose $f:\ \mathbb{T}^d \to \mathbb{C}$ and $f \in L^2$. If $\frac{1}{\|f\|_2}|\int_{\mathbb{T}^d} f dx| \le \lambda < 1$, then*

$$\|e^{-t\Lambda^s} f\|_2 \le e^{-\alpha_1 t}\|f\|_2, \quad \forall\, 0 < t < t_0. \tag{4.10}$$

Here $\alpha_1 > 0$, $t_0 > 0$ are constants depending only on $(s,\, d,\, \lambda)$.

Proof. With no loss we assume $\|f\|_2 = 1$. Denote $\langle f\rangle = \int_{\mathbb{T}^d} f dx$. Clearly

$$\|e^{-t\Lambda^s} f\|_2^2 \le e^{-ct}\|f - \langle f\rangle\|_2^2 + |\langle f\rangle|^2 \tag{4.11}$$

$$\le e^{-ct}(\|f\|_2^2 - |\langle f\rangle|^2) + |\langle f\rangle|^2 \tag{4.12}$$

$$\le e^{-ct} + (1-e^{-ct})\lambda^2 \le e^{-c_1 t}, \qquad \text{for } 0 < t \ll 1. \tag{4.13}$$

$\square$

Proof of Theorem 4.1. We shall present the proof for the simplest case $s = 2$, $d = 1$ and $2 < p < \infty$. It is not difficult to adapt the proof to the most general situations.

It suffices for us to prove (4.4) for $0 < t \le t_0$ where $t_0 > 0$ can be taken as a small constant depending on $(s,\, d,\, p)$. This is because for $t > t_0$,

$$\|e^{-t\Lambda^s} f\|_p = \|e^{-t_0\Lambda^s} e^{-(t-t_0)\Lambda^s} f\|_p \tag{4.14}$$

$$\le e^{-ct_0}\|e^{-(t-t_0)\Lambda^s} f\|_p, \qquad (\text{since } \langle e^{-(t-t_0)\Lambda^s} f\rangle = 0). \tag{4.15}$$

One can then iterate the estimates to get the decay for all $t > 0$.

Let $\frac{1}{p} = \frac{1-\theta}{2}$, $0 < \theta < 1$. Take simple real-valued functions f, g with $\langle f\rangle = 0$, $\|f\|_p = 1$, $\|g\|_{p'} = 1$ (here $p' = p/(p-1)$). Consider

$$h(s) = \langle e^{t\Delta}(|f|^{p(\frac{1-s}{2}+\frac{s}{\infty})}\,\mathrm{sgn}(f)), |g|^{p'(\frac{1-s}{2}+\frac{s}{1})}\,\mathrm{sgn}(g)\rangle,$$

where $\langle f_1, f_2\rangle := \int_{\mathbb{T}} \bar f_1 f_2 dx$. Here $\bar f_1$ denotes the complex conjugate of f_1.

(Here we recall the usual Riesz-Thorin setup: namely in going from $L^{p_0} \to L^{q_0}$, $L^{p_1} \to L^{q_1}$ to $L^p \to L^q$, one needs to employ the general interpolation formula for simple functions f and g:

$$f_z = |f|^{p(\frac{1-z}{p_0}+\frac{z}{p_1})}\mathrm{sgn}(f),$$

$$g_z = |g|^{q(\frac{1-z}{q_0'}+\frac{z}{q_1'})}\mathrm{sgn}(g),$$

where q_j' are conjugates of q_j. Our case corresponds to $p_0 = q_0 = 2$, $p_1 = q_1 = \infty$.)

We verify the interpolation as follows.

- The case $\mathrm{Re}(s) = 1$. Clearly

$$|h(s)| \le \|e^{t\Delta}(|f|^{p\frac{-i\mathrm{Im}(s)}{2}}\mathrm{sgn}(f))\|_\infty \||g|^{p'}\|_1 \le \|g\|_{p'}^{p'} = 1. \tag{4.16}$$

- The case $\mathrm{Re}(s) = 0$, i.e. $s = iy$, $y \in \mathbb{R}$. First for all $y \in \mathbb{R}$, we clearly have

$$|h(iy)| \le \|e^{t\Delta}(|f|^{p\frac{1-iy}{2}}\mathrm{sgn}(f))\|_2 \||g|^{p'(\frac{1-iy}{2}+iy)}\mathrm{sgn}(y)\|_2 \tag{4.17}$$

$$\le \||f|^{\frac{p}{2}}\|_2 \||g|^{\frac{1}{2}p'}\|_2 \le 1. \tag{4.18}$$

It remains for us to show that for $|y| \le 1$ and $0 < t \le t_0$ (for some small $t_0 > 0$),

$$|h(iy)| \le e^{-ct}, \tag{4.19}$$

where $c > 0$ is some constant. If this holds, we can just use Strong Phragman-Lindelof Theorem to conclude the interpolation argument. Indeed by using (4.9), we have

$$|h(\theta)| \le \exp\left(\frac{\sin \pi\theta}{2}\int_{|y|\le 1}\frac{-ct}{\cosh \pi y - \cos\theta}dy\right) \le e^{-\tilde{c}t}, \tag{4.20}$$

where $\tilde{c} > 0$ is a constant.

- It remains for us to verify (4.19). By Lemma 4.2, it suffices for us to establish for $|y| \le 1$,

$$|\mathbb{E}((|f|^p)^{\frac{1-iy}{2}}\mathrm{sgn}(f))| \le \lambda < 1, \tag{4.21}$$

where $\lambda > 0$ is some constant. Here and below we denote

$$\mathbb{E}h = \int_{\mathbb{T}} h dx. \tag{4.22}$$

We recall that $\|f\|_p = 1$ and $\langle f\rangle = 0 = \mathbb{E}f$.

The proof of (4.21) follows from the following steps.

- If $\mathbb{E}|f| \le \lambda_1 < 1$, then by using the interpolation $\|f\|_{\frac{p}{2}} \le \|f\|_p^{\frac{p-2}{p-1}}\|f\|_1^{\frac{1}{p-1}}$, we have

$$|\mathbb{E}(|f|^{\frac{p}{2}})| \le \lambda_1^{\frac{1}{p-1}\cdot\frac{p}{2}} < 1.$$

This clearly implies (4.21). Therefore we can assume $\lambda_1 < \mathbb{E}|f| \le 1$ and $\lambda_1 \to 1-$. Since $\|f\|_2 \le \|f\|_p \le 1$, we have

$$\mathbb{E}||f|-1|^2 \le 2 - 2\mathbb{E}|f| \le 2(1-\lambda_1) =: \delta_1 \ll 1.$$

We shall view δ_1 as a tunable parameter which can be taken sufficiently small.

- By using the inequality $|x^{\frac{p}{2}} - 1| \lesssim |x-1|\langle x\rangle^{\frac{p-2}{2}}$, we have

$$\mathbb{E}||f|^{\frac{p}{2}} - 1| \le \delta_2 = O(\delta_1^{\frac{1}{2}}) \ll 1.$$

- We now take $\eta > 0$ whose smallness will be specified momentarily. Clearly

$$|\mathbb{E}((|f|^p)^{\frac{1-iy}{2}}\mathrm{sgn}(f))| \le |\mathbb{E}((|f|^p)^{\frac{1-iy}{2}}\mathrm{sgn}(f))\chi_{|f|\ge\eta}| + \eta^{\frac{p}{2}} \tag{4.23}$$

$$\le \mathbb{E}||f|^{\frac{p}{2}} - 1| + \mathbb{E}|e^{-i\frac{p}{2}y\log|f|} - 1|\chi_{|f|\ge\eta} + |\mathbb{E}\mathrm{sgn}(f)| + \eta^{\frac{p}{2}} \tag{4.24}$$

$$\le \mathbb{E}||f|^{\frac{p}{2}} - 1| + \frac{p}{2}\mathbb{E}|\log|f||\chi_{|f|\ge\eta} + |\mathbb{E}\mathrm{sgn}(f)| + \eta^{\frac{p}{2}}. \tag{4.25}$$

- Observe that for $x \geq \eta$ ($\eta < 1$ will be taken sufficiently small), we have

$$|\log x| = |\log x - \log 1| \leq \frac{1}{\eta}|x-1|. \tag{4.26}$$

Thus

$$\mathbb{E}|\log|f||\chi_{|f|\geq\eta} \leq \frac{1}{\eta}\mathbb{E}||f|-1|. \tag{4.27}$$

On the other hand, observe (below we use the crucial property that $\mathbb{E}f = \mathbb{E}|f|\mathrm{sgn}(f) = 0$)

$$|\mathbb{E}\mathrm{sgn}(f)| = |\mathbb{E}\,\mathrm{sgn}\, f(|f|-1)| \leq \mathbb{E}||f|-1|.$$

- Thus we obtain

$$|\mathbb{E}((|f|^p)^{\frac{1-iy}{2}}\,\mathrm{sgn}(f))| \leq \mathbb{E}||f|^{\frac{p}{2}}-1| + (\frac{p}{2\eta}+1)\mathbb{E}||f|-1| + \eta^{\frac{p}{2}} \tag{4.28}$$

$$\leq O(\delta_1^{\frac{1}{2}})\cdot(1+\frac{p}{2\eta}) + \eta^{\frac{p}{2}}. \tag{4.29}$$

Taking $\eta = \delta_1^{\frac{1}{4}}$ with δ_1 sufficiently small clearly yields the result.

□

In what follows, we shall explain a somewhat more simplified approach to the proof of Theorem 4.1.

We begin with a simple yet powerful lemma.

Lemma 4.3. *Let $\Omega = \mathbb{R}^d$ or the periodic torus $\mathbb{T}^d$. Suppose $K \in L^1(\Omega)$ is nonnegative with unit L^1 mass. For any $p \in [2,\infty)$, we have*

$$\|K * f\|_{L^p(\Omega)} \leq \|K * (|f|^{\frac{p}{2}})\|_{L^2(\Omega)}^{\frac{2}{p}}. \tag{4.30}$$

Here $*$ denotes the usual convolution, i.e.

$$(K * f)(x) = \int K(x-y)f(y)dy. \tag{4.31}$$

For $p \in (1,2]$, we have

$$\|K * f\|_p \leq \|K * (|f|^{\frac{p}{2}})\|_2^2 \cdot \|f\|_p^{2-p}. \tag{4.32}$$

Proof. Observe that for each fixed x, $K(x-y)dy$ can be viewed as a probability measure. Thus if $p \in [2,\infty)$, then

$$\int |f(y)|K(x-y)dy \leq \left(\int |f(y)|^{\frac{p}{2}}K(x-y)dy\right)^{\frac{2}{p}}. \tag{4.33}$$

This yields the first inequality. Now for $p \in (1,2)$, by using the inequality $\|g\|_{\frac{2}{p}} \leq \|g\|_1^{p-1}\|g\|_2^{2-p}$ with $g = |f|^{\frac{2}{p}}$ and $d\mu = K(x-y)dy$, we have

$$\int |f(y)|K(x-y)dy \leq \left(\int |f(y)|^{\frac{p}{2}}K(x-y)dy\right)^{\frac{2(p-1)}{p}}\left(\int |f(y)|^p K(x-y)dy\right)^{\frac{2-p}{p}}. \tag{4.34}$$

Thus

$$\|K * f\|_p \leq \|K * (|f|^{\frac{p}{2}})\|_2^2 \cdot \|f\|_p^{2-p}. \tag{4.35}$$

□

We now sketch a different proof of Theorem 4.1 for the Laplacian case (i.e. $-\Lambda^2 = \Delta$) as follows. With no loss we consider the case $\mathbb{T}^d = \mathbb{T}$ and $p \in (2, \infty)$. Take f with mean zero and $\|f\|_p = 1$. Discuss two cases.

- Case 1 $\|f\|_{\frac{p}{2}+1} \ll 1$. Clearly then $\|f\|_{\frac{p}{2}} \ll 1$. By Lemma 4.3, we obtain

$$\|e^{t\Delta} f\|_p \le \|e^{t\Delta}(|f|^{\frac{p}{2}})\|_2^{\frac{2}{p}}. \tag{4.36}$$

By Lemma 4.2, since $\mathbb{E}|f|^{\frac{p}{2}} \ll 1$ and $\||f|^{\frac{p}{2}}\|_2 = 1$, we obtain

$$\|e^{t\Delta}(|f|^{\frac{p}{2}})\|_2 \le e^{-ct}\||f|^{\frac{p}{2}}\|_2 = e^{-ct}, \qquad 0 < t \le t_0. \tag{4.37}$$

This clearly implies the desired estimate $\|e^{t\Delta} f\|_p \le e^{-\tilde{c}t}\|f\|_p$ for $0 < t \le t_0$. Note that this part of the argument can be adapted to $e^{-t\Lambda^s}$ for $0 < s < 2$.

- Case 2 $\|f\|_{\frac{p}{2}+1} \gtrsim 1$. Note that

$$\langle |f|^{\frac{p}{2}} \operatorname{sgn}(f), f\rangle = \int |f|^{\frac{p}{2}+1} dx \gtrsim 1.$$

Since f is spectrally localized to $|k| \ge 1$, it follows that (below P_k is the Fourier projection to all modes $|k| \ge 1$)

$$\|P_{|k|\ge 1}(|f|^{\frac{p}{2}} \operatorname{sgn}(f))\|_2 \|f\|_2 \gtrsim 1.$$

On the torus, we obviously have $\|f\|_2 \le \|f\|_p = 1$. Thus

$$\|P_{|k|\ge 1}(|f|^{\frac{p}{2}} \operatorname{sgn}(f))\|_2 \gtrsim 1. \tag{4.38}$$

In yet other words, the L^2-mass of $|f|^{\frac{p}{2}} \operatorname{sgn}(f)$ must have a nontrivial portion in $|k| \ge 1$. Now observe that

$$\int (-\Delta f)|f|^{p-2} f dx = \text{const} \int |\nabla f|^2 |f|^{p-2} dx \tag{4.39}$$

$$= \text{const} \int |\nabla(|f|^{\frac{p}{2}} \operatorname{sgn}(f))|^2 dx \tag{4.40}$$

$$\gtrsim \|P_{|k|\ge 1}(|f|^{\frac{p}{2}} \operatorname{sgn}(f))\|_2^2 \gtrsim 1 = \|f\|_p^p. \tag{4.41}$$

Thus the desired inequality follows.

Next we shall state and prove a frequency localized Bernstein inequality on the torus. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be such that $\psi(\xi) = 1$ for $|\xi| \le 1$ and $\psi(\xi) = 0$ for $|\xi| \ge 1.01$. For integer $N \ge 2$ and $f : \mathbb{T}^d \to \mathbb{C}$, define

$$\widehat{P_N f}(k) = \widehat{f}(k)(\psi(\frac{k}{2N}) - \psi(\frac{k}{N})). \tag{4.42}$$

We need the following lemma from Kato [11]. The inequality stated therein[4] is for the whole space. We adapt it here for the torus with essentially the same proof.

Lemma 4.4 (Kato [11]). *Let $1 < p < \infty$. Assume $\phi \in W^{2,p}(\mathbb{T}^d \to \mathbb{R}^{d_1})$, where $d \ge 1$, $d_1 \ge 1$. Then*

$$-\langle |\phi|^{p-2}\phi, \Delta\phi\rangle \ge \min\{1, p-1\} \int_{\phi \ne 0} |\nabla\phi|^2 |\phi|^{p-2} dx. \tag{4.43}$$

[4]Note that there is a minor typo in the definition of Q_p in formula (2.2), pp 55 of [11]: the lower limit for the integration therein should be $\phi(x) \ne 0$ instead of $\partial\phi(x) \ne 0$.

Here for $f,\, g : \mathbb{T}^d \to \mathbb{R}^{d_1}$,

$$\langle f,\, g\rangle = \sum_{j=1}^{d_1} \int_{\mathbb{T}^d} f_j g_j dx. \tag{4.44}$$

Proof. We briefly recall Kato's proof. For $p > 2$, we use the identity

$$-\langle |\phi|^{p-2}\phi, \Delta\phi\rangle = \int |\phi|^{p-2}|\partial_k\phi_j|^2 + (p-2)|\phi|^{p-2}\frac{\phi_l\phi_j}{|\phi|^2}\partial_k\phi_l\partial_k\phi_j \tag{4.45}$$

$$\geq \int |\phi|^{p-2}|\partial_k\phi_j|^2. \tag{4.46}$$

For $1 < p < 2$, we use

$$-\langle |\phi|^{p-2}\phi, \Delta\phi\rangle = -\lim_{\epsilon\to 0}\langle (|\phi|^2+\epsilon)^{\frac{p-2}{2}}\phi,\, \Delta\phi\rangle. \tag{4.47}$$

Denote $\phi_\epsilon = \sqrt{|\phi|^2+\epsilon}$. Then (note below $p-2<0$ and $\frac{\phi_l\phi_l}{|\phi_\epsilon|^2}$ is bounded by 1 in matrix norm)

$$-\langle |\phi_\epsilon|^{p-2}\phi, \Delta\phi\rangle = \int |\phi_\epsilon|^{p-2}|\partial_k\phi_j|^2 + (p-2)|\phi_\epsilon|^{p-2}\frac{\phi_l\phi_j}{|\phi_\epsilon|^2}\partial_k\phi_l\partial_k\phi_j \tag{4.48}$$

$$\geq (p-1)\int |\phi_\epsilon|^{p-2}|\partial_k\phi_j|^2 \geq (p-1)\int_{\phi\neq 0} |\phi_\epsilon|^{p-2}|\nabla\phi|^2. \tag{4.49}$$

The result follows from dominated convergence (for the LHS) and monotone convergence (for the RHS). □

Theorem 4.2 (Bernstein inequality on the torus, frequency localized version)**.** *Let $0 < s \leq 2$ and consider Λ^s on $\mathbb{T}^d = [-\frac{1}{2}, \frac{1}{2}]^d$, $d \geq 1$. Let $1 < p < \infty$. For any smooth $f : \mathbb{T}^d \to \mathbb{R}$ and any integer $N \geq 2$, we have*

$$\|e^{-t\Lambda^s}P_N f\|_p \leq e^{-c_{p,s,d}N^s t}\|P_N f\|_p, \qquad \forall\, t > 0. \tag{4.50}$$

Here $c_{p,s,d} > 0$ depends only on $(p,\, s,\, d,\, \psi)$ (Recall ψ is the same cut-off function used in the definition of the operator P_N). We also have

$$\int_{\mathbb{T}^d} (\Lambda^s P_N f)|P_N f|^{p-2}P_N f dx \geq \tilde{c}_{p,s,d}N^s\|P_N f\|_p^p, \tag{4.51}$$

where $\tilde{c}_{p,s,d} > 0$ depends only on $(p,\, s,\, d,\, \psi)$.

Remark 4.4. See [7] for a proof using a nontrivial perturbation of the Lévy semigroup near low frequencies.

Proof. We follow [2]. For the Laplacian case, the idea is based on an ingenious partial integration trick dating back to Danchin [4] (p being even integers), Planchon [15] ($p > 2$) and Danchin [5] ($1 < p < 2$).

To simplify the notation we shall write $f = P_N f$ keeping in mind that f is frequency-localized. We shall write $\int_{\mathbb{T}^d} dx$ simply as $\int$.

Step 1. Laplacian case. We first show

$$-\int \Delta f|f|^{p-2}f \gtrsim N^2\|f\|_p^p. \tag{4.52}$$

We first deal with the case $p > 2$. We have

$$\|f\|_p^p = \int f^2|f|^{p-2} = \int (\nabla \cdot \nabla\Delta^{-1} f) f|f|^{p-2} \tag{4.53}$$

$$\lesssim \int |\nabla\Delta^{-1} f||\nabla f||f|^{p-2} \tag{4.54}$$

$$\le C_\epsilon N^{-2} \int |\nabla f|^2|f|^{p-2} + \epsilon N^2 \int |\nabla\Delta^{-1} f|^2|f|^{p-2} \tag{4.55}$$

$$\le C_\epsilon N^{-2} \int |\nabla f|^2|f|^{p-2} + \epsilon \cdot \mathrm{Const}\|f\|_p^p. \tag{4.56}$$

Choosing ϵ to be sufficiently small then yields the result for $p > 2$. For $1 < p < 2$, we use

$$N^p\|f\|_p^p \lesssim \|\nabla f\|_p^p = \int_{f\neq 0} (|\nabla f|^2|f|^{p-2})^{\frac{p}{2}} (|f|^p)^{\frac{2-p}{2}}\, dx \tag{4.57}$$

$$\lesssim (\int_{f\neq 0} |\nabla f|^2|f|^{p-2})^{\frac{p}{2}} (\int |f|^p)^{\frac{2-p}{2}}. \tag{4.58}$$

The desired result then follows from Lemma 4.4.

Step 2. The estimate (4.50) in the case $s = 2$ follows from Step 1 by examining $\frac{d}{dt}\Big(\|e^{-t\Lambda^s} f\|_p^p\Big)$ and an energy estimate. The general case $0 < s < 2$ follows from subordination. The estimate (4.51) follows from differentiating at $t = 0$.

□

5. Liouville theorem for general fractional Laplacian operators

This last section provides a useful application of decay estimates of the heat propagator despite its lack of positivity.

We now consider the fractional heat equation of the form

$$\partial_t u = -\Lambda^s u, \quad (t, x) \in (-\infty, 0] \times \mathbb{R}^d. \tag{5.1}$$

Here $\Lambda^s = (-\Delta)^{s/2}$ is the fractional Laplacian of order s, and we assume $s > 0$. Note that for $0 < s \le 2$ the corresponding semigroup has positivity but this is no longer the case for $s > 2$, i.e. the higher order Laplacians.

Theorem 5.1. *Suppose u is a solution to* (5.1) *defined on* $(-\infty, 0] \times \mathbb{R}^d$ *(i.e. an ancient solution) satisfying*

$$|u(t,x)| \le \frac{C}{|x|^a}, \quad \forall\, (t,x) \in (-\infty, 0] \times \mathbb{R}^d,$$

where $0 < a < d$ and $C > 0$ are constants. Then u must be identically zero.

Proof. Take any $\phi \in C_c^\infty(\mathbb{R}^d)$ and consider $u_\phi = \phi * u$. By splitting into $|y| \le 1$ and $|y| > 1$ respectively, it is not difficult to check that u_ϕ is smooth and $u_\phi \in L^p$ for any $\frac{d}{a} < p < \infty$. Fix any $t_0 \in (-\infty, 0]$. We then have $u_\phi(t_0) = e^{-(t_0-t)\Lambda^s} u_\phi(t)$ for any $t < t_0$. By sending t to $-\infty$ and invoking the usual decay estimates (for the kernel $e^{-\tau\Lambda^s}$), i.e.

$$\|u_\phi(t_0)\|_\infty \lesssim (t_0 - t)^{-\frac{d}{s}\cdot\frac{1}{p-1}} \|u_\phi(t)\|_p \lesssim (t_0 - t)^{-\frac{d}{s}\cdot\frac{1}{p-1}},$$

we obtain $\|u_\phi(t_0)\|_\infty = 0$. Thus $u_\phi(t_0) = 0$ for any $\phi \in C_c^\infty$. This implies that u must be identically zero. □

Remark 5.1. The hypothesis that $|u| \lesssim |x|^{-a}$ can be replaced by the more general condition that

$$\sup_t \|u(t)\|_{L^{p_1}+L^{p_2}} < \infty$$

for some $1 \leq p_1, p_2 < \infty$.

Remark 5.2. The rigidity result Theorem 5.1 plays an important role in e.g. the infinite-time blow up problem for the half-harmonic map flow from $\mathbb{R}$ to $\mathbb{S}^1$ [17].

Appendix A. Computation of the contour integral

In this appendix we show (2.8). Recall that $f(x) = \log(1+x^2)$, $f'(x) = \frac{2x}{1+x^2}$, $f''(x) = 2\frac{1-x^2}{(1+x^2)^2}$ and we need to show

$$\begin{aligned} 3\int f^2(f'')^2 &= 12\int_{-\infty}^{\infty} (\log(1+x^2))^2 \frac{(1-x^2)^2}{(1+x^2)^4} dx \\ &= -\frac{29}{6}\pi + \pi^3 + (\log 4)(-7+\log 64)\pi. \end{aligned} \tag{A.1}$$

For this we need to compute

$$I_j = \int_{-\infty}^{\infty} (\log(1+x^2))^2 \frac{1}{(1+x^2)^j} dx, \quad j = 1, \cdots, 4. \tag{A.2}$$

We shall proceed in several steps.

Step 1. Preliminary reduction. Observe that

$$I_{n+1} = \int_{-\infty}^{\infty} (\log(1+x^2))^2 \frac{1+x^2-x^2}{(1+x^2)^{n+1}} dx \tag{A.3}$$

$$= I_n + \int_{-\infty}^{\infty} (\log(1+x^2))^2 \cdot \frac{x}{2n}\frac{d}{dx}\Big((1+x^2)^{-n}\Big) dx \tag{A.4}$$

$$= (1-\frac{1}{2n})I_n - \frac{2}{n}\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)^{n+1}} \log(1+x^2) dx \tag{A.5}$$

$$\begin{aligned} &= (1-\frac{1}{2n})I_n - \frac{2}{n}\Big(\int_{-\infty}^{\infty} (1+x^2)^{-n}\log(1+x^2)dx \\ &\qquad - \int_{-\infty}^{\infty} (1+x^2)^{-n-1}\log(1+x^2)dx\Big). \end{aligned} \tag{A.6}$$

By using the above iterative relation, to compute I_n for all n, it suffices for us to compute

$$I_1 = \int_{-\infty}^{\infty} (\log(1+x^2))^2 \frac{1}{1+x^2} dx \tag{A.7}$$

and

$$F_n = \int_{-\infty}^{\infty} (1+x^2)^{-n} \log(1+x^2) dx, \qquad n = 1, \cdots, 4. \tag{A.8}$$

Step 2. Computation of I_1. We shall perform a contour integral computation.

For $z = \rho e^{i\theta}$ with $-\pi \leq \theta < \pi$, we denote

$$\mathrm{Log} z = \log\rho + i\theta. \tag{A.9}$$

In yet other words we use the standard principal branch of the multi-valued function $\log z$ with argument in $[-\pi, \pi)$. By a slight abuse of notation, we shall write $\mathrm{Log} z$ simply as $\log z$.

Denote $g(z) = (1+z^2)^{-1}$. Note that g has poles at $z = \pm i$. First we observe that (see Fig. 4)

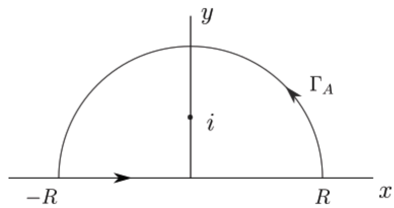

FIGURE 4. Contour Γ_A

$$\tag{A.10} \begin{aligned} \int_{-\infty}^{\infty} (\log(x+i))^2 g(x)dx &= \lim_{R\to\infty} \int_{\Gamma_A} (\log(z+i))^2 g(z)dz \\ &= 2\pi i \mathrm{Res}((\log(z+i))^2 g(z); i). \end{aligned}$$

By using our choice of the branch cut for the logarithm function, we have for $x > 0$

$$\tag{A.11} \log(x+i) = \frac{1}{2}\log(x^2+1) + i\theta_x, \qquad \theta_x = \frac{\pi}{2} - \arctan x;$$

$$\tag{A.12} \log(-x+i) = \frac{1}{2}\log(x^2+1) + i(\pi - \theta_x).$$

Thus (A.10) becomes

$$\tag{A.13} \begin{aligned} \int_0^{\infty} \frac{\frac{1}{2}(\log(1+x^2))^2}{1+x^2} dx - \int_0^{\infty} \frac{\theta_x^2 + (\pi-\theta_x)^2}{1+x^2} dx \\ = \mathrm{Re}\Big(2\pi i \mathrm{Res}((\log(z+i))^2 g(z); i)\Big). \end{aligned}$$

This implies

$$\tag{A.14} I_1 = 2\int_0^{\infty} \frac{(\log(1+x^2))^2}{1+x^2} dx = 4\frac{\pi^3}{3} + 4\mathrm{Re}\Big(2\pi i \mathrm{Res}((\log(z+i))^2 g(z); i)\Big).$$

Since $\mathrm{Res}((\log(z+i))^2 g(z); i) = \frac{1}{8} i(\pi - 2i\log 2)^2$, we obtain

$$\tag{A.15} \boxed{I_1 = \frac{1}{3}\pi^3 + 4\pi(\log 2)^2.}$$

Step 3. Computation of F_n. This is analogous to the previous step. Note that

$$\tag{A.16} \int_{-\infty}^{\infty} \frac{\log(x+i)}{(1+x^2)^n} dx = 2\pi i \mathrm{Res}(\log(z+i)(1+z^2)^{-n}; i).$$

This yields

$$\tag{A.17} \begin{aligned} \int_0^{\infty} \frac{\frac{1}{2}\log(x^2+1) + i\theta_x}{(1+x^2)^n} dx + \int_0^{\infty} \frac{\frac{1}{2}\log(x^2+1) + i(\pi-\theta_x)}{(1+x^2)^n} dx \\ = 2\pi i \mathrm{Res}(\log(z+i)(1+z^2)^{-n}; i). \end{aligned}$$

Thus

$$F_n = 2\int_0^\infty \frac{\log(x^2+1)}{(1+x^2)^n} dx = 4\pi \mathrm{Re}\Big(i\mathrm{Res}(\log(z+i)(1+z^2)^{-n}; i)\Big). \tag{A.18}$$

We obtain for $n = 1, \cdots, 4$,

$$F_1 = \pi \log 4, \qquad F_2 = \pi(-\frac{1}{2} + \log 2); \tag{A.19}$$

$$F_3 = \pi(-\frac{7}{16} + \frac{3}{4}\log 2); \qquad F_4 = \pi(-\frac{37}{96} + \frac{5}{8}\log 2). \tag{A.20}$$

Step 4. Verification of (A.1). Clearly

$$\text{LHS of (A.1)} = 12I_2 - 48I_3 + 48I_4. \tag{A.21}$$

By using Step 1, we have

$$I_{n+1} = (1 - \frac{1}{2n})I_n - \frac{2}{n}(F_n - F_{n+1}). \tag{A.22}$$

Clearly

$$I_2 = \frac{1}{2}I_1 - 2(F_1 - F_2) = -\pi + \frac{1}{6}\pi^3 + \frac{1}{2}\pi(-2 + \log 4)\log 4. \tag{A.23}$$

Similarly

$$I_3 = \frac{1}{16}\pi(-11 + 2\pi^2 + (\log 16)(-7 + \log 64)); \tag{A.24}$$

$$I_4 = \frac{1}{288}\pi\Big(-155 + 30\pi^2 + 6(\log 4)(-37 + 15\log 4)\Big). \tag{A.25}$$

The identity (A.1) then follows easily.

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