On the refined analyticity radius of 3-D generalized Navier-Stokes equations

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A R T I C L E   I N F O

A B S T R A C T

We analyze the instantaneous growth of analyticity radius for three dimensional generalized Navier-Stokes equations. For the subcritical $H^\gamma(\mathbb{R}^3)$ case with $\gamma > \frac{1}{2}$, we prove that there exists a positive time t_0 so that for any $t \in]0, t_0]$, the radius of analyticity of the solution u satisfies the pointwise-in-time lower bound

$$\text{rad}(u)(t) \geq \sqrt{(2\gamma - 1)t(|\ln t| + \ln|\ln t| + K_t)},$$

where $K_t \to \infty$ as $t \to 0^+$. This in particular gives a nontrivial improvement of the previous result by Herbst and Skibsted in [17] for the case $\gamma \in]1/2, 3/2[$ and also settles the decade-long open question in [17], namely, whether or not

$$\liminf_{t \to 0^+} \frac{\text{rad}(u)(t)}{\sqrt{t|\ln t|}} \geq \sqrt{2\gamma - 1}$$

for all $\gamma \geq \frac{3}{2}$. In the critical case $H^{\frac{1}{2}}(\mathbb{R}^3)$ we prove that there exists $t_1 > 0$ so that for any $t \in]0, t_1]$, $\text{rad}(u)(t) \geq \lambda(t)\sqrt{t}$ with $\lambda(t)$ satisfying $\lim_{t \to 0^+} \lambda(t) = \infty$.

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1. Introduction

In this paper, we consider the instantaneous growth of analyticity radius for the solutions to the following 3-D generalized Navier-Stokes equations in $\mathbb{R}^+ \times \mathbb{R}^3$:

$$(GNS) \qquad \begin{cases} \partial_t u - \Delta u = Q(u, u), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ u|_{t=0} = u_0. \end{cases}$$

Here $u = (u^1, u^2, u^3) : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ denotes the velocity of the fluid under study. The viscosity preceding the Laplacian term is set to be one. Throughout this paper we shall denote by $Q = (Q^1, Q^2, Q^3)$ any bilinear map of the form:

$$Q^j(u, v) \overset{\text{def}}{=} \sum_{k, \ell, m=1}^{3} q_{k, \ell}^{j, m}(D) \partial_m(u^k v^\ell), \tag{1.1}$$

where $q_{k,\ell}^{j,m}(D)$ is a Fourier multiplier with symbol

$$q_{k,\ell}^{j,m}(\xi) \overset{\text{def}}{=} \sum_{n, p=1}^{3} \alpha_{k,\ell}^{j,m,n,p} \frac{\xi_n \xi_p}{|\xi|^2},$$

and $\alpha_{k,\ell}^{j,m,n,p}$ are real numbers. The precise numerical values of $\alpha_{k,\ell}^{j,m,n,p}$ will not play any role in our analysis. Henceforth from a practical point of view it is often useful to regard $Q(u, v)$ as

$$Q(u, v) = \mathcal{R}\partial(uv), \tag{1.2}$$

where $\mathcal{R}$ denotes a general Riesz transform. Using this abstraction it is easy to deduce scaling transformations associated with (GNS).

Namely if $u = u(t, x)$ is a smooth solution to (GNS), then for $\lambda > 0$,

$$u_\lambda(t, x) \overset{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x) \tag{1.3}$$

forms a one-parameter family of smooth solutions to (GNS). The homogeneous space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is critical in the sense that

$$\left\| u_\lambda(t, \cdot) \right\|_{\dot{H}^{\frac{1}{2}}} = \left\| u(t, \cdot) \right\|_{\dot{H}^{\frac{1}{2}}}$$

for any $\lambda > 0$. By a slight generalization we designate the inhomogeneous spaces $H^s(\mathbb{R}^3)$, $s = \frac{1}{2}$, $s > \frac{1}{2}$ as critical and subcritical spaces respectively.

Our motivation for studying the system (GNS) comes from the following classical 3-D incompressible Navier-Stokes equations:

$$(NS) \qquad \begin{cases} \partial_t u + u \cdot \nabla u - \Delta u = -\nabla P, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases}$$

where u stands for the fluid velocity and P for the scalar pressure function, which guarantees the divergence free condition of the velocity field. In fact, by applying Leray projection operator, $\mathbb{P} = \mathrm{I} + \nabla(-\Delta)^{-1} \operatorname{div}$, to (NS), we obtain equations of the type (GNS). One may check pages 206-207 of [2] for a motivating discussion of the system (GNS). See also Chapter 5 therein for an extensive review of classical wellposedness results for (GNS).

In the seminal paper [26], Leray proved the global existence of weak solution and local existence of strong solution to (NS). It is well-known that strong solutions of (NS) are in fact analytic in both space and time variables (see [25] for instance). In fluid mechanics, the space analyticity radius of solutions to Navier-Stokes equations yields a Kolmogrov type length scale encountered in turbulence theory, one may check [3,4,8,9,11,16,22] and the references therein for more details.

Mathematically, the study of analyticity of solutions to the Navier-Stokes equations goes back to Masuda in [28], where the authors used complex-analytic techniques to investigate the analyticity in both space and time for the solutions of 2-D Navier-Stokes equations in a bounded domain with Dirichlet boundary conditions. Foias and Temam [13] introduced the notion of Gevrey norm, which allows one to study analyticity properties of solutions via energy method. In particular, they [13] proved the analyticity of periodic solutions of (NS) in space and time with initial data $u_0 \in H^1(\mathbb{T}^3)$ (see also [12]). Grujić and Kukavica [15] investigated the analyticity radius of the solution to (NS) with initial data in L^p for p greater than the space dimensions. The related result was later extended by the authors in [5,21,23,24] to show that: there exists a positive time T so that

$$\int\limits_{\mathbb{R}^3} |\xi| \big(\sup_{t \leq T} e^{\sqrt{t}|\xi|} |\widehat{u}(t,\xi)| \big)^2 \, d\xi + \int\limits_0^T \int\limits_{\mathbb{R}^3} |\xi|^3 \big(e^{\sqrt{t}|\xi|} |\widehat{u}(t,\xi)| \big)^2 \, d\xi dt < \infty \,.$$

This in particular implies the Fujita-Kato solution of (NS), which was constructed by Fujita and Kato in [14], with initial data $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is analytic for any positive time t. One may check related results in the survey book [25].

We remark that in the previous works [6,13,17], the authors used Gevrey norm of the form $\|e^{r(t)|D|} u(t)\|_X$ with a L^2 based Sobolev space X. In [23] Lemarié-Rieusset studied

Gevrey regularities of the solution u to (NS) in the L^p framework. One may check [1,9,10,19,29] for more recent development in this direction.

Before proceeding, we recall the definition of Sobolev spaces from [2]:

Definition 1.1.

(1) For $s \in \mathbb{R}$, we define the inhomogeneous Sobolev space $H^s(\mathbb{R}^3)$ to be the space of those tempered distributions f which satisfy

$$\|f\|_{H^s} \stackrel{\text{def}}{=} \|\langle \xi \rangle^s \widehat{f}(\xi)\|_{L^2} < \infty,$$

where $\widehat{f}$ denotes the Fourier transform of f. Here and in all that follows, we always denote the quantity $\langle \xi \rangle \stackrel{\text{def}}{=} (1 + |\xi|^2)^{\frac{1}{2}}$.

(2) For $s \in \mathbb{R}$, we define the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ to be the space of those homogeneous distributions f which satisfy

$$\|f\|_{\dot{H}^s} \stackrel{\text{def}}{=} \||\xi|^s \widehat{f}(\xi)\|_{L^2} < \infty.$$

From a scaling perspective, the $\mathcal{O}(\sqrt{t})$-radius of analyticity of the solution to (NS) seems to be optimal since it almost fully utilizes the heat kernel. Thus it is somewhat surprising that Herbst and Skibsted [17] proved the following sharpened result:

Theorem 1.1 (*Theorem 1.3 of [17]*). *Suppose $u_0 \in H^\gamma$ for some $\gamma \in]1/2, 3/2[$. Then the system (NS) with initial data u_0 has a unique local solution u on $[0, T]$. Let $\varepsilon \in]0, 2\gamma - 1[$. Then there exist constant $t_0 = t_0(\varepsilon, \gamma, \|u_0\|_{H^\gamma}) \in]0, T]$ and $C = C(\varepsilon, \gamma, \|u_0\|_{H^\gamma}) > 0$ such that*

$$\left\| e^{\sqrt{2\gamma - 1 - \varepsilon}\sqrt{t|\ln t|}|D|} u(t) \right\|_{H^\gamma} \leq C t^{\frac{1}{4} + \varepsilon - \frac{\gamma}{2}} \quad \text{for all } t \in]0, t_0]. \tag{1.4}$$

In particular,

$$\liminf_{t \to 0^+} \frac{\text{rad}(u(t))}{\sqrt{t|\ln t|}} \geq \sqrt{2\gamma - 1}. \tag{1.5}$$

Henceforth, we always denote $\text{rad}(u(t))$ to be the space analyticity radius of $u(t)$.

Remark 1.1. Herbst and Skibsted asked the questions below (see page 194 of [17]):

(i) Are the bounds (1.4) and (1.5) optimal for $\gamma \in]1/2, 3/2[$?

(ii) Are there better bounds than those deducible from Theorem 1.1 if $\gamma > \frac{3}{2}$?

(iii) Can the asymptotic

$$\lim_{t \to 0^+} \frac{\mathrm{rad}(u(t))}{\sqrt{t}} = \infty, \tag{1.6}$$

be improved for the critical case $\gamma = \frac{1}{2}$?

The purpose of this paper is to settle the questions in Remark 1.1 proposed by Herbst and Skibsted. Our first main result addresses the subcritical case H^γ with $\gamma > \frac{1}{2}$.

Theorem 1.2. *Let $u_0 \in H^\gamma$ with $\gamma > \frac{1}{2}$ and be divergence-free. There exists $T > 0$ so that the system (GNS) has a unique solution $u \in C([0,T]; H^\gamma) \cap L^2(]0,T[; \dot{H}^{\gamma+1})$. Furthermore, there exists $t_0 \leq T$ so that for any sufficiently small $\delta > 0$ with $\delta < \frac{1}{2}(\gamma - \frac{1}{2})$ there holds*

$$\left\| e^{\lambda(t)\sqrt{t}|D|} u(t) \right\|_{\dot{H}^{\frac{1}{2}+\delta}} \leq C t^{-\frac{1}{2}(\gamma+\delta-\frac{1}{2})} |\ln t|^{\frac{1}{2}(\gamma-\frac{1}{2})} e^{\frac{3}{4}\beta(t)} \quad \textit{for all } t \in]0, t_0], \tag{1.7}$$

where

$$
\begin{aligned}
\lambda(t) &\overset{\text{def}}{=} \sqrt{(2\gamma-1)\big(|\ln t| + \ln|\ln t|\big) + 3\beta(t)} \quad \textit{with} \\
\eta_J^\gamma(t) &\overset{\text{def}}{=} \max_{0 \leq \tau \leq t} \|1_{|\xi| \geq 0.01 J} |\xi|^\gamma \widehat{u}(\tau, \xi)\|_{L_\xi^2} \quad \textit{and} \\
\beta(t) &\overset{\text{def}}{=} \begin{cases} \min\{\, |\ln \eta_{t^{-\frac{1}{2}}}^\gamma(t)|,\ \frac{1}{2}(\gamma - \frac{1}{2})|\ln t| \,\}, & \textit{if } \eta_{t^{-\frac{1}{2}}}^\gamma(t) > 0; \\ \frac{1}{2}(\gamma - \frac{1}{2})|\ln t|, & \textit{if } \eta_{t^{-\frac{1}{2}}}^\gamma(t) = 0. \end{cases}
\end{aligned} \tag{1.8}
$$

In particular, we have

$$\frac{\mathrm{rad}(u(t))}{\lambda(t)\sqrt{t}} \geq 1 \quad \textit{for all } t \in]0, t_0]. \tag{1.9}$$

Remark 1.2. We deduce from (1.8) and (1.9) that

$$\frac{\mathrm{rad}(u(t))}{\sqrt{t\big(|\ln t| + \ln|\ln t|\big)}} \geq \sqrt{2\gamma - 1} \qquad \text{for all } t \in]0, t_0],$$

which not only improves the analyticity radius derived for the solution of (NS) in (1.5), but also generalizes Theorem 1.1 to the conjectured range $\gamma \geq \frac{3}{2}$. Consequently, we fully resolve questions (i) and (ii) listed in Remark 1.1, which were originally raised by Herbst and Skibsted in [17].

Remark 1.3.

(1) For any given solution $u \in C([0,T]; H^\gamma)$ of the system (GNS) with initial data $u_0 \in H^\gamma$, we shall prove in Lemma 2.2 that $\eta_J^\gamma(t) \to 0$ as $J \to \infty$. We should point it out that the case $\eta_{t-\frac{1}{2}}^\gamma(t) = 0$ in the definition of $\beta(t)$ is trivial in the following sense: according to the definition of $\eta_J^\gamma(t)$, when $\eta_{t-\frac{1}{2}}^\gamma(t) = 0$, it holds that

$$\mathrm{supp}(\widehat{u}(\tau, \cdot)) \subset \{\xi : |\xi| \leq 0.01 t^{-\frac{1}{2}}\}, \qquad \forall\, \tau \in]0, t].$$

In particular the space analyticity radius of $u(\tau)$ is arbitrarily large for $\tau \in]0, t]$. This scenario does not seem to be easily ruled out for the nonlinear problem. For example if we consider two-dimensional Navier-Stokes in vorticity form, the solution $\omega(t) = e^{t\Delta}\omega_0$ with ω_0 being radial is an explicit solution to the nonlinear equation. If one takes $\widehat{\omega_0}$ to be compactly supported, then clearly $\omega(t)$ also has the same compact support in the frequency space.

(2) Roughly speaking, the definition of $\beta(t)$ given by (1.8) is to accommodate the situation when the initial data u_0 has higher smoothness (say in H^m with $m > \gamma > \frac{1}{2}$) whereas the working space is H^γ. Apparently in the case $u_0 \in H^m$ with $m > \gamma > \frac{1}{2}$, we have $\eta_J^\gamma(t) \lesssim J^{-(m-\gamma)}$ and $|\log \eta_{t-\frac{1}{2}}^\gamma(t)| \geq \frac{1}{2}(m-\gamma)|\ln t| - C$ ($C > 0$ is a constant). If m is large, we clearly see an "upgrade" of analyticity radius of the amount $\frac{1}{2}(\gamma - \frac{1}{2})|\ln t|$ thanks to our definition of $\beta(t)$.

(3) The cut-off $\frac{1}{2}(\gamma - \frac{1}{2})|\ln t|$ is for the convenience of analysis only. In principle it can be replaced by other suitable $\mathcal{O}(|\ln t|)$ term but the corresponding running parameters (in our nonlinear analysis, see for example the estimate of the low frequency piece (2.4)) will have to be adjusted accordingly. In practice we tacitly assume that the working space H^γ "saturates" the smoothness of u_0 so that $\eta_J^\gamma(t)$ decays suitably slowly as $J \to \infty$. For this reason we chose the working cut-off $\frac{1}{2}(\gamma - \frac{1}{2})|\ln t|$ in order to ease the presentation. We shall not dwell on this subtle technical issue here.

A fundamental insight leading to the proof of Theorem 1.2 is that the high frequency part of the solution to (GNS) controls its space analyticity radius. In fact, we shall recast (GNS) into the following form:

$$u = e^{t\Delta}u_0 + \mathcal{B}(u, u). \tag{1.10}$$

We employ the classical iteration scheme to construct the approximate solution sequence $\{u_n\}_{n \in \mathbb{N}}$ of (1.10). We first prove that there exists a positive time T so that $\{u_n\}$ converges to the solution u of (GNS) in $L^\infty([0,T]; H^\gamma) \cap L^2(]0, T[; \dot{H}^{\gamma+1})$. Then we prove that there exists a positive time $t_0 \leq T$ so that $\|u_n\|_{X_{t_0}}$ is uniformly bounded, where the working norm $\|\cdot\|_{X_T}$ is judiciously chosen as

$$\|u\|_{X_T} \overset{\text{def}}{=} \left\| t^{\frac{\delta}{2}} |\xi|^{\delta + \frac{1}{2}} \mathbf{1}_{|\xi| \geq 0.01 \lambda T^{-\frac{1}{2}}} e^{-\frac{\lambda^2 t}{4T} + \lambda \frac{t}{\sqrt{T}}|\xi|} \widehat{u}(t, \xi) \right\|_{L_T^\infty(L_\xi^2)}. \tag{1.11}$$

Henceforth, δ is a positive constant satisfying $\gamma > \frac{1}{2} + 2\delta$. Finally we prove the convergence of the approximate solution sequence $\{u_n\}_{n \in \mathbb{N}}$ in the norm:

$$\left\| t^{\frac{\delta}{2}} |\xi|^{\delta + \frac{1}{2}} e^{-\frac{\lambda^2 t}{4T} + \lambda \frac{t}{\sqrt{T}} |\xi|} \widehat{u}(t, \xi) \right\|_{L_T^\infty (L_\xi^2)}$$

for $\lambda = \lambda(T)$ with $\lambda(T)$ being given by (1.8).

To answer the question (iii) of Remark 1.1 for the critical case $\gamma = \frac{1}{2}$, we have the following result:

Theorem 1.3. *Let $u_0 \in \dot{H}^{\frac{1}{2}}$ and be divergence free. The system (GNS) with initial data u_0 has a unique solution $u \in C([0,T]; \dot{H}^{\frac{1}{2}}) \cap L^2(]0,T[; \dot{H}^{\frac{3}{2}})$ for some positive time T. We denote*

$$\zeta_J^\gamma(t) \stackrel{\text{def}}{=} \max_{0 \le \tau \le t} \|1_{|\xi| \ge J} |\xi|^\gamma \widehat{u}(\tau, \xi)\|_{L_\xi^2} \quad and \quad \lambda(t) \stackrel{\text{def}}{=} \sqrt{3 \min\{|\ln \zeta_{t^{-\frac{1}{4}}}^{\frac{1}{2}}(t)|, |\ln t|\}}, \quad (1.12)$$

where $|\log \zeta_{t^{-\frac{1}{4}}}^{\frac{1}{2}}(t)|$ is tacitly defined as ∞ if $\zeta_{t^{-\frac{1}{4}}}^{\frac{1}{2}}(t) = 0$. Then there exists a positive time $t_1 \le T$ so that for all $t \in]0, t_1]$, there holds

$$\left\| e^{\lambda(t)\sqrt{t}|D|} u(t) \right\|_{\dot{H}^{\frac{1}{2}+\delta}} \le C(\|u_0\|_{\dot{H}^{\frac{1}{2}}}) \left(t^{-\frac{\delta}{4}} + t^{-\frac{\delta}{2}} e^{(\frac{1}{4} + 10^{-4})\lambda^2(t)} \zeta_{t^{-\frac{1}{4}}}^{\frac{1}{2}}(t) \right). \quad (1.13)$$

Remark 1.4.

(1) It is easy to observe from (1.13) that

$$\text{rad}(u(t)) \ge \lambda(t)\sqrt{t} \quad \text{for all } t \in]0, t_1]. \quad (1.14)$$

In Proposition 3.1 we show that $\zeta_J^{\frac{1}{2}}(t) \to 0$ as $J \to \infty$. This implies $\lambda(t) \to \infty$ as $t \to 0^+$. By (1.12) and (1.13), we deduce (1.6), i.e.

$$\lim_{t \to 0^+} \frac{\text{rad}(u(t))}{\sqrt{t}} = \infty. \quad (1.15)$$

In this sense the point-wise-in-time bound (1.14) offers a minuscule yet nontrivial improvement. Note that (1.14) also gives an "ϵ"-improvement of [6].

(2) In [6], the authors proved that for any global solution $u \in C([0, \infty[; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$ of (NS), there holds

$$\lim_{t \to \infty} \frac{\text{rad}(u(t))}{\sqrt{t}} = \infty.$$

We expect that similar result as (1.14) should be true for any global solution u of (NS) with time t being large enough. However we shall not pursue this interesting direction here.

We conclude this section by introducing notation used throughout this paper.

Notations:

- Throughout, $C > 0$ denotes an absolute constant whose value may vary at each occurrence. For positive quantities X and Y, we use $X \lesssim Y$ to indicate $X \leq CY$, and $X \lesssim_{Z_1,\ldots,Z_k} Y$ to indicate that the constant C depends on the parameters $Z_1, \ldots, Z_k$. Occasionally we write $X \ll Y$ (equivalently $Y \gg X$) if $X \leq cY$ for some sufficiently small constant $c > 0$.

- We adopt the following convention for Fourier transform. For Schwartz function $a = a(x) : \mathbb{R}^3 \to \mathbb{C}$, we denote the Fourier transform

$$(\mathcal{F}a)(\xi) = (\mathcal{F}_{x\to\xi}a)(\xi) = \widehat{a}(\xi) \overset{\text{def}}{=} \int\limits_{\mathbb{R}^3} a(x)e^{-ix\cdot\xi}dx.$$

For Schwartz function $b = b(\xi) : \mathbb{R}^3 \to \mathbb{C}$, we denote the inverse Fourier transform

$$(\mathcal{F}^{-1}b)(x) \overset{\text{def}}{=} (2\pi)^{-3} \int\limits_{\mathbb{R}^3} b(\xi)e^{ix\cdot\xi}d\xi.$$

The action of Fourier transform and inverse Fourier transform on tempered distributions can be defined accordingly.

- We use the notation $t \to 0^+$ to denote $t \to 0$ with $t > 0$.

- We shall use the Japanese bracket notation $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ for $x \in \mathbb{R}^3$. For $s \in \mathbb{R}$, we denote the smoothed fractional Laplacian $\langle D \rangle^s = (\mathrm{I} - \Delta)^{s/2}$ which corresponds to the Fourier multiplier $(1 + |\xi|^2)^{s/2}$. We also use $|D|^s = (-\Delta)^{s/2}$ to denote the fractional Laplacian which corresponds to the symbol $|\xi|^s$. We denote by $(f,g)_{\dot{H}^s}$ the usual $\dot{H}^s$ inner product, namely

$$(f,g)_{\dot{H}^s} = \int\limits_{\mathbb{R}^3} |D|^s f |D|^s \bar{g}\, dx. \tag{1.16}$$

- For vector-valued Schwartz function $u = (u_1, u_2, u_3) : \mathbb{R}^3 \to \mathbb{C}^3$, we denote

$$\|u\|_{L^p} = \|(|u_1|^2 + |u_2|^2 + |u_3|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^3)}, \tag{1.17}$$

where L^p is the usual Lebesgue L^p-norm. The vector-valued Sobolev norm H^s is similarly defined. In yet other words we shall suppress the notational dependence of the vector-valued spaces. For example we write $L^p(\mathbb{R}^3)^3$ simply as $L^p(\mathbb{R}^3)$.

- We use $*$ to denote the convolution of two functions, namely for Schwartz functions $f_1 : \mathbb{R}^3 \to \mathbb{C}$ and $f_2 : \mathbb{R}^3 \to \mathbb{C}$,

$$(f_1 * f_2)(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} f_1(x-y)f_2(y)dy.$$

- For a nonempty set A, we use 1_A to denote the usual indicator function, i.e.

$$1_A \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases} \tag{1.18}$$

For example in Section 2, we have

$$1_{|\xi| \geq 0.01N_1} = \begin{cases} 1, & \text{if } |\xi| \geq 0.01N_1; \\ 0, & \text{otherwise.} \end{cases} \tag{1.19}$$

- For two vectors $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $v = (v_1, v_2, v_3) \in \mathbb{R}^3$, we employ the usual tensor notation

$$(u \otimes v)_{ij} \stackrel{\text{def}}{=} u_i v_j. \tag{1.20}$$

- For a Banach space B, we denote by $\| \cdot \|_{L_T^p(B)}$ or $\| \cdot \|_{L_T^p B}$ the norm $\big\| \| \cdot \|_B \big\|_{L^p(0,T)}$. For example,

$$\|f\|_{L_T^2 L_x^3} := \left(\int_0^T \left(\int_{\mathbb{R}^3} |f(t,x)|^3 dx \right)^{\frac{2}{3}} dt \right)^{\frac{1}{2}}.$$

The space of continuous functions from $[0,T]$ to B is denoted by $C([0,T]; B)$, equipped with the norm

$$\|u\|_{C([0,T];B)} \stackrel{\text{def}}{=} \sup_{0 \leq t \leq T} \|u(t)\|_B.$$

2. The subcritical case: $\gamma > \frac{1}{2}$

In this section we give the proof of Theorem 1.2. For $\lambda, T > 0$ and $a \in \mathcal{S}'(\mathbb{R}^3)$, we denote

$$N_1 \stackrel{\text{def}}{=} \lambda T^{-\frac{1}{2}}, \qquad \widehat{a}(t, \xi) \stackrel{\text{def}}{=} \mathcal{F}_{x \to \xi}(a)(t, \xi), \tag{2.1}$$

where $\mathcal{F}_{x \to \xi}$ denotes the Fourier transform. We decompose a into low frequency and high frequency parts as:

$$a = a_{\text{l}} + a_{\text{h}} \quad \text{with} \quad \widehat{a_{\text{l}}}(t,\xi) \stackrel{\text{def}}{=} \widehat{a}(t,\xi) \cdot 1_{|\xi| < 0.01N_1} \quad \text{and} \quad \widehat{a_{\text{h}}} \stackrel{\text{def}}{=} \widehat{a}(t,\xi) \cdot 1_{|\xi| \geq 0.01N_1}. \tag{2.2}$$

To track analyticity radius, it is convenient to adopt the following notation:

$$\widehat{\mathfrak{T}(a)}(t,\xi) \overset{\text{def}}{=} e^{-\frac{\lambda^2 t}{4T} + \lambda \frac{t}{\sqrt{T}}|\xi|} \widehat{a}(t,\xi). \tag{2.3}$$

Here we suppress the notational dependence on λ and T.

We first deal with the low frequency part of a. Throughout this paper we shall tacitly assume $T \ll 1$ since T will be eventually taken sufficiently small. Since λ will also eventually be taken sufficiently large ($\lambda = \mathcal{O}(\sqrt{|\ln T|})$ in the main order), we shall also tacitly assume $\lambda \gg 1$ to avoid any pathologies in the computation. For example in Lemma 2.1 below, we have $(\lambda \sqrt{T})^{-1} < 0.01 N_1 = 0.01 \lambda T^{-\frac{1}{2}}$.

Lemma 2.1. *Let* $\gamma > \frac{1}{2}$. *Let*[1] $0 < \delta < \gamma - \frac{1}{2}$ *and* $a \in L^\infty([0,T]; H^\gamma)$. *Then for any* $t \le T$, *one has*

$$\|\mathfrak{T}(a_{\mathrm{l}})(t)\|_{\dot{H}^{\frac{1}{2}+\delta}} \lesssim \left(1 + \lambda^{\frac{1}{2}+\delta+\gamma} T^{\frac{1}{2}(\gamma-\frac{1}{2}-\delta)} e^{0.01\lambda^2}\right) e^{-\frac{\lambda^2 t}{4T}} \|a\|_{L^\infty_T(H^\gamma)}. \tag{2.4}$$

Proof. We observe that for $\gamma > \frac{1}{2} + \delta$,

$$|\xi|^{\frac{1}{2}+\delta} \frac{e^{\lambda \frac{t}{\sqrt{T}}|\xi|}}{\langle \xi \rangle^\gamma} 1_{|\xi|<0.01 N_1} \le |\xi|^{\frac{1}{2}+\delta} \frac{e^{\lambda \frac{t}{\sqrt{T}}|\xi|}}{\langle \xi \rangle^\gamma} 1_{|\xi| \le (\lambda\sqrt{T})^{-1}}$$

$$+ |\xi|^{\frac{1}{2}+\delta} \frac{e^{\lambda \frac{t}{\sqrt{T}}|\xi|}}{\langle \xi \rangle^\gamma} \cdot 1_{(\lambda\sqrt{T})^{-1} < |\xi| < 0.01 N_1}$$

$$\lesssim 1 + \lambda^{\frac{1}{2}+\delta+\gamma} T^{\frac{1}{2}(\gamma-\frac{1}{2}-\delta)} e^{0.01\lambda^2}.$$

By (2.3), we obtain

$$\||D|^{\delta+\frac{1}{2}} \mathfrak{T}(a_{\mathrm{l}})(t)\|_{L^2} \lesssim \left\| |\xi|^{\delta+\frac{1}{2}} e^{-\frac{\lambda^2 t}{4T} + \lambda \frac{t}{\sqrt{T}}|\xi|} 1_{|\xi|<0.01 N_1} \widehat{a}(t,\xi) \right\|_{L^2_\xi}$$

$$\lesssim \left(1 + \lambda^{\frac{1}{2}+\delta+\gamma} T^{\frac{1}{2}(\gamma-\frac{1}{2}-\delta)} e^{0.01\lambda^2}\right) e^{-\frac{\lambda^2 t}{4T}} \|a\|_{L^\infty_T(H^\gamma)}. \quad \square$$

We remark that there is a saving of $e^{-\frac{\lambda^2 t}{4T}}$ in the estimate (2.4) which will be used in forthcoming nonlinear estimates.

Notice from (1.11) and (2.3) that

$$\|a\|_{X_T} = \|t^{\frac{\delta}{2}} |D|^{\delta+\frac{1}{2}} \mathfrak{T}(a_{\mathrm{h}})(t)\|_{L^\infty_T(L^2)}.$$

To estimate $\|a\|_{X_T}$, we first observe that for any $t \le T \le 1$,

$$\left\| 1_{0.01 N_1 \le |\xi| \le 0.1 N_1} t^{\frac{\delta}{2}} |\xi|^{\delta+\frac{1}{2}} e^{-\frac{\lambda^2 t}{4T} + \lambda \frac{t}{\sqrt{T}}|\xi|} \widehat{a}(t,\cdot) \right\|_{L^2}$$

$$\lesssim T^{\frac{\delta}{2}} (N_1)^{-(\gamma-\frac{1}{2}-\delta)} e^{-(\frac{1}{4}-0.1)\frac{\lambda^2 t}{T}} \|a_{\mathrm{h}}\|_{L^\infty_T(H^\gamma)}$$

$$\lesssim T^{\frac{1}{2}(\gamma-\frac{1}{2})} \lambda^{-(\gamma-\frac{1}{2}-\delta)} \|a_{\mathrm{h}}\|_{L^\infty_T(H^\gamma)}. \tag{2.5}$$

---

[1] Here in the linear estimate we only need $\gamma > \frac{1}{2} + \delta$.

Therefore to complete the estimate of $\|u\|_{X_T}$ for the solution u of (GNS) in $L^\infty([0,T];H^\gamma)$, it remains for us to treat the main piece

$$\left\|1_{|\xi|\geq 0.1N_1}t^{\frac{\delta}{2}}|\xi|^{\delta+\frac{1}{2}}e^{-\frac{\lambda^2 t}{4T}+\lambda\frac{t}{\sqrt{T}}|\xi|}\widehat{u}(t,\xi)\right\|_{L_T^\infty(L_\xi^2)}. \tag{2.6}$$

For this, we appeal to the following integral reformulation of (GNS):

$$u = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}Q(u,u)(s)\,ds,$$

where the bilinear form $Q(f,g)$ is given by (1.1). The avid reader should think of

$$Q(f,g) \approx \mathcal{R}\partial(fg),$$

where $\mathcal{R}$ is Riesz-type transform. On the Fourier side, we need to estimate the piece

$$1_{|\xi|\geq 0.1N_1}\int_0^t e^{-(t-s)|\xi|^2}\widehat{Q(f,g)}(s,\xi)\,ds.$$

Thanks to the high frequency cut-off $1_{|\xi|\geq 0.1N_1}$, there will be no low-low interactions of f and g entering the nonlinear estimate. Our main technical result is stated in the next proposition. This is the most crucial ingredient used in the proof of Theorem 1.2.

Proposition 2.1. *Let $Q(f,g)$ be the bilinear form given by (1.1). Let $\gamma > \frac{1}{2}$. Let[2] $0 < \delta < \frac{1}{2}(\gamma - \frac{1}{2})$ be sufficiently small. Then for η_0 being a small enough positive constant, one has*

$$\left\|t^{\frac{\delta}{2}}1_{|\xi|\geq 0.1N_1}|\xi|^{\delta+\frac{1}{2}}e^{\lambda\frac{t}{\sqrt{T}}|\xi|-\frac{\lambda^2 t}{4T}}\int_0^t e^{-(t-s)|\xi|^2}\widehat{Q(f,g)}(s,\xi)\,ds\right\|_{L_T^\infty(L_\xi^2)}$$

$$\lesssim \left(e^{4\eta_0\lambda^2} + \lambda^{-\delta}\right)\lambda^{-(\gamma-\frac{1}{2}+\delta)}T^{\frac{1}{2}(\gamma-\frac{1}{2}+2\delta)}$$

$$\times \left(\|f\|_{L_T^\infty(H^\gamma)}\|g_{\mathrm{h}}\|_{L_T^\infty(H^\gamma)} + \|f_{\mathrm{h}}\|_{L_T^\infty(H^\gamma)}\|g\|_{L_T^\infty(H^\gamma)}\right) + \lambda^{-\delta}e^{\frac{\lambda^2}{4}}\|f\|_{X_T}\|g\|_{X_T}$$

$$+ \lambda^{2-\delta}T^{\frac{\delta}{2}}\left(1 + \lambda^{\frac{1}{2}+\delta+\gamma}T^{\frac{1}{2}(\gamma-\frac{1}{2}-\delta)}e^{0.01\lambda^2}\right)\left(\|g\|_{L_T^\infty(H^\gamma)}\|f\|_{X_T} + \|f\|_{L_T^\infty(H^\gamma)}\|g\|_{X_T}\right), \tag{2.7}$$

where f_{h} and g_{h} are given by (2.2).

Proof. For any $\eta_0 > 0$, which will be taken sufficiently small later, we split the integral $\int_0^t = \int_0^{\eta_0 t} + \int_{\eta_0 t}^t$ and shall estimate each piece separately.

---

[2] Note that here in the nonlinear estimate, we need $\gamma > \frac{1}{2} + 2\delta$.

In view of (1.1), we decompose the short-time piece $\int_0^{\eta_0 t}$ into the following two parts:

$$
\begin{aligned}
t^{\frac{\delta}{2}} \big\| |\xi|^{\frac{1}{2}+\delta} 1_{|\xi| \geq 0.1 N_1} e^{\lambda \frac{t}{\sqrt{T}} |\xi| - \frac{\lambda^2 t}{4T} |\xi|^2} \int_0^{\eta_0 t} e^{-(t-s)|\xi|^2} \widehat{Q(f,g)}(s,\xi)\, ds \big\|_{L_\xi^2} \\
= t^{\frac{\delta}{2}} \big\| |\xi|^{\frac{1}{2}+\delta} 1_{|\xi| \geq 0.1 N_1} \int_0^{\eta_0 t} e^{-t\left(|\xi| - \frac{\lambda}{2\sqrt{T}}\right)^2} e^{s|\xi|^2} \widehat{Q(f,g)}(s,\xi)\, ds \big\|_{L_\xi^2} \\
\lesssim \big\| t^{\frac{\delta}{2}} 1_{|\xi| \leq 2N_1} \int_0^{\eta_0 t} |\xi|^{\frac{3}{2}+\delta} e^{s|\xi|^2} 1_{|\xi| \geq 0.1 N_1} |\widehat{f} * \widehat{g}(s,\xi)|\, ds \big\|_{L_\xi^2} \\
+ \big\| t^{\frac{\delta}{2}} 1_{|\xi| \geq 2N_1} \int_0^{\eta_0 t} |\xi|^{\frac{3}{2}+\delta} e^{-\frac{t}{10}|\xi|^2} |\widehat{f} * \widehat{g}(s,\xi)|\, ds \big\|_{L_\xi^2}.
\end{aligned}
\tag{2.8}
$$

Here the smallness of η_0 is needed for the second part so that $e^{s|\xi|^2} e^{-t \cdot \frac{9}{16} |\xi|^2} \leq e^{-\frac{t}{10}|\xi|^2}$. In particular $\eta_0 < 0.1$ suffices.

By frequency localization and (2.2), we have

$$
1_{|\xi| \geq 0.1 N_1} \widehat{f} * \widehat{g} = 1_{|\xi| \geq 0.1 N_1} \big(\widehat{f_l} * \widehat{g_h} + \widehat{f_h} * \widehat{g_l} + \widehat{f_h} * \widehat{g_h} \big).
\tag{2.9}
$$

Then we get, by applying Hausdorff-Young's inequality, $\|\widehat{f}\|_{L^{p'}} \leq C \|f\|_{L^p}$ for $p \in [1, 2]$ and $p' \overset{\text{def}}{=} \frac{p}{p-1}$, that

$$
\begin{aligned}
&\int_0^{\eta_0 t} \big\| 1_{|\xi| \leq 2N_1} |\xi|^{\frac{3}{2}+\delta} e^{s|\xi|^2} 1_{|\xi| \geq 0.1 N_1} |\widehat{f} * \widehat{g}(s,\xi)| \big\|_{L_\xi^2} ds \\
&\lesssim \int_0^{\eta_0 t} e^{4s N_1^2} N_1^{\frac{3}{2}+\delta} \big\| 1_{|\xi| \leq 2N_1} \big\|_{L_\xi^{\frac{6}{1-4\delta}}} \big\| 1_{|\xi| \geq 0.1 N_1} \widehat{f} * \widehat{g}(s,\cdot) \big\|_{L_\xi^{\frac{3}{1+2\delta}}} ds \\
&\lesssim \int_0^{\eta_0 t} e^{4s N_1^2} N_1^{\frac{3}{2}+\delta} \cdot N_1^{\frac{1}{2}-2\delta} \big(\|f_l g_h(s)\|_{L^{\frac{3}{2(1-\delta)}}} + \|f_h g_l(s)\|_{L^{\frac{3}{2(1-\delta)}}} + \|f_h g_h(s)\|_{L^{\frac{3}{2(1-\delta)}}} \big) ds.
\end{aligned}
$$

By Sobolev embedding and (2.2), we obtain

$$
\begin{aligned}
&\int_0^{\eta_0 t} \big\| 1_{|\xi| \leq 2N_1} |\xi|^{\frac{3}{2}+\delta} e^{s|\xi|^2} 1_{|\xi| \geq 0.1 N_1} |\widehat{f} * \widehat{g}(s,\xi)| \big\|_{L_\xi^2} ds \\
&\lesssim \int_0^{\eta_0 t} e^{4s N_1^2} N_1^{2-\delta} \big(\big\| |D|^{\frac{1}{2}+2\delta} f_l \big\|_{L^2} \big\| |D|^{\frac{1}{2}} g_h \big\|_{L^2} + \big\| |D|^{\frac{1}{2}} f_h \big\|_{L^2} \big\| |D|^{\frac{1}{2}+2\delta} g_l \big\|_{L^2}
\end{aligned}
$$

$$
+ \||D|^{\frac{1}{2}+\delta} f_{\mathrm{h}}\|_{L^2} \||D|^{\frac{1}{2}+\delta} g_{\mathrm{h}}\|_{L^2}\Big) \, ds
$$

$$
\lesssim \int_0^{\eta_0 t} e^{4sN_1^2} N_1^{2-\delta} \, ds \Big(N_1^{-\gamma+\frac{1}{2}} \big(\|f\|_{L_t^\infty(H^\gamma)} \|g_{\mathrm{h}}\|_{L_t^\infty(H^\gamma)} + \|f_{\mathrm{h}}\|_{L_t^\infty(H^\gamma)} \|g\|_{L_t^\infty(H^\gamma)} \big)
$$

$$
+ N_1^{-2(\gamma-\frac{1}{2}-\delta)} \|f_{\mathrm{h}}\|_{L_t^\infty(H^\gamma)} \|g_{\mathrm{h}}\|_{L_t^\infty(H^\gamma)} \Big).
$$

Since $\gamma > \frac{1}{2} + 2\delta$, we get

$$
\int_0^{\eta_0 t} \|1_{|\xi|\leq 2N_1} |\xi|^{\frac{3}{2}+\delta} e^{s|\xi|^2} 1_{|\xi|\geq 0.1 N_1} |\widehat{f} * \widehat{g}(s,\xi)|\|_{L_\xi^2} ds
$$

$$
\lesssim N_1^{-\delta} e^{4\eta_0 T N_1^2} N_1^{-\gamma+\frac{1}{2}} \big(\|f\|_{L_t^\infty(H^\gamma)} \|g_{\mathrm{h}}\|_{L_t^\infty(H^\gamma)} + \|f_{\mathrm{h}}\|_{L_t^\infty(H^\gamma)} \|g\|_{L_t^\infty(H^\gamma)} \big)
$$

$$
\lesssim \big(\lambda^{-1} T^{\frac{1}{2}} \big)^{\gamma-\frac{1}{2}+\delta} e^{4\eta_0 \lambda^2} \big(\|f\|_{L_t^\infty(H^\gamma)} \|g_{\mathrm{h}}\|_{L_t^\infty(H^\gamma)} + \|f_{\mathrm{h}}\|_{L_t^\infty(H^\gamma)} \|g\|_{L_t^\infty(H^\gamma)} \big).
$$

Similarly for the piece containing $1_{|\xi|\geq 2N_1}$, we have

$$
t^{\frac{\delta}{2}} \Big\| 1_{|\xi|\geq 2N_1} \int_0^{\eta_0 t} |\xi|^{\frac{3}{2}+\delta} e^{-\frac{t}{10}|\xi|^2} \widehat{f} * \widehat{g}(s,\xi) \, ds \Big\|_{L^2}
$$

$$
\lesssim t^{-1} \big\| |\xi|^{-\frac{1}{2}} 1_{|\xi|\geq 2N_1} \big\|_{L^{\frac{6}{1-4\delta}}} \int_0^{\eta_0 t} \|1_{|\xi|\geq 0.1 N_1} \widehat{f} * \widehat{g}(s,\cdot)\|_{L^{\frac{3}{1+2\delta}}} \, ds
$$

$$
\lesssim t^{-1} N_1^{-2\delta} \int_0^{\eta_0 t} \big(\|f_{\mathrm{l}} g_{\mathrm{h}}(s)\|_{L^{\frac{3}{2(1-\delta)}}} + \|f_{\mathrm{h}} g_{\mathrm{l}}(s)\|_{L^{\frac{3}{2(1-\delta)}}} + \|f_{\mathrm{h}} g_{\mathrm{h}}(s)\|_{L^{\frac{3}{2(1-\delta)}}} \big) \, ds
$$

$$
\lesssim \big(\lambda^{-1} T^{\frac{1}{2}} \big)^{\gamma-\frac{1}{2}+2\delta} \big(\|f\|_{L_t^\infty(H^\gamma)} \|g_{\mathrm{h}}\|_{L_t^\infty(H^\gamma)} + \|f_{\mathrm{h}}\|_{L_t^\infty(H^\gamma)} \|g\|_{L_t^\infty(H^\gamma)} \big).
$$

Plugging the above estimates into (2.8), we obtain

$$
\Big\| t^{\frac{\delta}{2}} |\xi|^{\frac{1}{2}+\delta} 1_{|\xi|\geq 0.1 N_1} e^{\lambda \frac{t}{\sqrt{T}}|\xi| - \frac{\lambda^2 t}{4T}} \int_0^{\eta_0 t} e^{-(t-s)|\xi|^2} \widehat{Q(f,g)}(s,\xi) \, ds \Big\|_{L_T^\infty(L_\xi^2)}
$$

$$
\lesssim \big(e^{4\eta_0 \lambda^2} + \lambda^{-\delta} \big) \lambda^{-(\gamma-\frac{1}{2}+\delta)} T^{\frac{1}{2}(\gamma-\frac{1}{2}+2\delta)}
$$
$$
\times \big(\|f\|_{L_T^\infty(H^\gamma)} \|g_{\mathrm{h}}\|_{L_T^\infty(H^\gamma)} + \|f_{\mathrm{h}}\|_{L_T^\infty(H^\gamma)} \|g\|_{L_T^\infty(H^\gamma)} \big).
$$

(2.10)

For the remaining piece $\int_{\eta_0 t}^t$, we require the following lemma. Its proof is deferred until after that of Proposition 2.1.

Lemma 2.2. *Let $N_0 \overset{\text{def}}{=} \frac{N_1}{2} \gg 1$. Assume $\delta > 0$ is sufficiently small. Then for all $0 < t \le T$, one has*

$$\Big\| t^{\frac{\delta}{2}} \int_{\eta_0 t}^{t} 1_{|\xi| \le 2N_0} e^{N_0^2 s} |\xi|^{\frac{3}{2}+\delta} \widehat{F}(s,\xi) \, ds \Big\|_{L_T^\infty(L_\xi^2)} \lesssim \lambda^{-\delta} e^{N_0^2 T} \cdot \sup_{0 < s \le T} \big(s^\delta \|F(s)\|_{L^{\frac{3}{2(1-\delta)}}} \big),$$

(2.11)

and

$$\Big\| t^{\frac{\delta}{2}} \int_{\eta_0 t}^{t} 1_{|\xi| \ge 2N_0} e^{N_0^2 s} |\xi|^{\frac{3}{2}+\delta} e^{-\frac{1}{10}(t-s)|\xi|^2} \widehat{F}(s,\xi) \, ds \Big\|_{L_T^\infty(L_\xi^2)}$$

$$\lesssim \lambda^{-\delta} e^{N_0^2 T} \cdot \sup_{0 < s \le T} \big(s^\delta \|F(s)\|_{L^{\frac{3}{2(1-\delta)}}} \big).$$

(2.12)

We now continue our estimate of the piece $\int_{\eta_0 t}^{t}$. In view of (2.3), we write

$$t^{\frac{\delta}{2}} \Big\| |\xi|^{\frac{1}{2}+\delta} 1_{|\xi| \ge 0.1 N_1} e^{\lambda \frac{t}{\sqrt{T}} |\xi| - \frac{\lambda^2 t}{4T}} \int_{\eta_0 t}^{t} e^{-(t-s)|\xi|^2} \widehat{Q(f,g)}(s,\xi) \, ds \Big\|_{L_\xi^2}$$

$$\lesssim t^{\frac{\delta}{2}} \Big\| |\xi|^{\frac{1}{2}+\delta} 1_{|\xi| \ge 0.1 N_1} \int_{\eta_0 t}^{t} |\xi| e^{-(t-s)|\xi|^2} e^{\lambda \frac{t}{\sqrt{T}} |\xi| - \frac{\lambda^2 t}{4T}} |\widehat{f} * \widehat{g}(s,\xi)| \, ds \Big\|_{L_\xi^2}$$

(2.13)

$$\lesssim t^{\frac{\delta}{2}} \Big\| |\xi|^{\frac{3}{2}+\delta} \int_{\eta_0 t}^{t} e^{-(t-s)|\xi|^2} e^{\lambda \frac{t-s}{\sqrt{T}} |\xi| + \frac{\lambda^2 s}{2T} - \frac{\lambda^2 t}{4T}} 1_{|\xi| \ge 0.1 N_1} |\widehat{\mathfrak{T}(f)} * \widehat{\mathfrak{T}(g)}(s,\xi)| \, ds \Big\|_{L_\xi^2}.$$

By frequency localization (2.9), it suffices for us to handle the estimates related to the terms:

$$\widehat{\mathfrak{T}(f_{\mathrm{l}})} * \widehat{\mathfrak{T}(g_{\mathrm{h}})}, \quad \widehat{\mathfrak{T}(f_{\mathrm{h}})} * \widehat{\mathfrak{T}(g_{\mathrm{l}})}, \text{ and } \widehat{\mathfrak{T}(f_{\mathrm{h}})} * \widehat{\mathfrak{T}(g_{\mathrm{h}})}.$$

These correspond to low-high, high-low, and high-high interactions. Once again, the saving grace is that there is no low-low interaction piece.

We first estimate the contribution due to $\widehat{\mathfrak{T}(f_{\mathrm{h}})} * \widehat{\mathfrak{T}(g_{\mathrm{h}})}$. For this we employ further frequency cut-offs $1_{|\xi| \le N_1}$, $1_{|\xi| \ge N_1}$, and decompose the integrand accordingly as:

$$t^{\frac{\delta}{2}} \Big\| |\xi|^{\frac{3}{2}+\delta} \int_{\eta_0 t}^{t} e^{-(t-s)|\xi|^2} e^{\lambda \frac{t-s}{\sqrt{T}} |\xi| + \frac{\lambda^2 s}{2T} - \frac{\lambda^2 t}{4T}} |\widehat{\mathfrak{T}(f_{\mathrm{h}})} * \widehat{\mathfrak{T}(g_{\mathrm{h}})}(s,\xi)| \, ds \Big\|_{L_\xi^2}$$

$$\lesssim t^{\frac{\delta}{2}} \Big\| \int_{\eta_0 t}^{t} |\xi|^{\frac{3}{2}+\delta} e^{-(t-s)\left(|\xi| - \frac{\lambda}{2\sqrt{T}}\right)^2 + \frac{\lambda^2 s}{4T}} |\widehat{\mathfrak{T}(f_{\mathrm{h}})} * \widehat{\mathfrak{T}(g_{\mathrm{h}})}(s,\xi)| \, ds \Big\|_{L_\xi^2}$$

$$\lesssim t^{\frac{\delta}{2}} \Big\| \int_{\eta_0 t}^{t} |\xi|^{\frac{3}{2}+\delta} 1_{|\xi| \leq N_1} e^{\frac{\lambda^2 s}{4T}} |\widehat{\mathfrak{T}(f_{\mathrm h})} * \widehat{\mathfrak{T}(g_{\mathrm h})}(s,\xi)| \, ds \Big\|_{L^2_\xi} \tag{2.14}$$

$$+ t^{\frac{\delta}{2}} \Big\| \int_{\eta_0 t}^{t} |\xi|^{\frac{3}{2}+\delta} 1_{|\xi| \geq N_1} e^{-\frac{1}{10}(t-s)|\xi|^2 + \frac{\lambda^2 s}{4T}} |\widehat{\mathfrak{T}(f_{\mathrm h})} * \widehat{\mathfrak{T}(g_{\mathrm h})}(s,\xi)| \, ds \Big\|_{L^2_\xi}.$$

Using Lemma 2.2, we get for $\lambda \geq 1$ and $0 < t \leq T \leq 1$:

$$t^{\frac{\delta}{2}} \Big\| |\xi|^{\frac{3}{2}+\delta} \int_{\eta_0 t}^{t} e^{-(t-s)|\xi|^2} e^{\lambda \frac{t-s}{\sqrt{T}}|\xi| + \frac{\lambda^2 s}{2T} - \frac{\lambda^2 t}{4T}} |\widehat{\mathfrak{T}(f_{\mathrm h})} * \widehat{\mathfrak{T}(g_{\mathrm h})}(s,\xi)| \, ds \Big\|_{L^2_\xi}$$

$$\lesssim \lambda^{-\delta} e^{\frac{\lambda^2}{4}} \cdot \sup_{0 < s \leq T} \big(s^{\delta} \|\mathfrak{T}(f_{\mathrm h}) \mathfrak{T}(g_{\mathrm h})(s,\cdot)\|_{L^{\frac{3}{2(1-\delta)}}} \big) \tag{2.15}$$

$$\lesssim \lambda^{-\delta} e^{\frac{\lambda^2}{4}} \cdot \sup_{0 < s \leq T} \big(s^{\delta} \||D|^{\frac{1}{2}+\delta} \mathfrak{T}(f_{\mathrm h})(s,\cdot)\|_{L^2} \||D|^{\frac{1}{2}+\delta} \mathfrak{T}(g_{\mathrm h})(s,\cdot)\|_{L^2} \big)$$

$$\lesssim \lambda^{-\delta} e^{\frac{\lambda^2}{4}} \|f\|_{X_T} \|g\|_{X_T}.$$

Next we estimate the contribution due to $\widehat{\mathfrak{T}(f_{\mathrm h})} * \widehat{\mathfrak{T}(g_{\mathrm l})}$. Along the same line as the estimate of (2.14), we write

$$t^{\frac{\delta}{2}} \Big\| |\xi|^{\frac{3}{2}+\delta} \int_{\eta_0 t}^{t} e^{-(t-s)|\xi|^2} e^{\lambda \frac{t-s}{\sqrt{T}}|\xi| + \frac{\lambda^2 s}{2T} - \frac{\lambda^2 t}{4T}} |\widehat{\mathfrak{T}(f_{\mathrm h})} * \widehat{\mathfrak{T}(g_{\mathrm l})}(s,\xi)| \, ds \Big\|_{L^2_\xi}$$

$$\lesssim t^{\frac{\delta}{2}} \Big\| \int_{\eta_0 t}^{t} |\xi|^{\frac{3}{2}+\delta} 1_{|\xi| \leq N_1} e^{\frac{\lambda^2 s}{4T}} |\widehat{\mathfrak{T}(f_{\mathrm h})} * \widehat{\mathfrak{T}(g_{\mathrm l})}(s,\xi)| \, ds \Big\|_{L^2_\xi}$$

$$+ t^{\frac{\delta}{2}} \Big\| \int_{\eta_0 t}^{t} |\xi|^{\frac{3}{2}+\delta} 1_{|\xi| \geq N_1} e^{-\frac{1}{10}(t-s)|\xi|^2 + \frac{\lambda^2 s}{4T}} |\widehat{\mathfrak{T}(f_{\mathrm h})} * \widehat{\mathfrak{T}(g_{\mathrm l})}(s,\xi)| \, ds \Big\|_{L^2_\xi}.$$

Using Hausdorff-Young's inequality, we obtain for $0 < t \leq T$:

$$t^{\frac{\delta}{2}} \Big\| \int_{\eta_0 t}^{t} |\xi|^{\frac{3}{2}+\delta} 1_{|\xi| \leq N_1} e^{\frac{\lambda^2 s}{4T}} |\widehat{\mathfrak{T}(f_{\mathrm h})} * \widehat{\mathfrak{T}(g_{\mathrm l})}(s,\xi)| \, ds \Big\|_{L^2_\xi}$$

$$\lesssim t^{\frac{\delta}{2}} N_1^{\frac{3}{2}+\delta} \int_{\eta_0 t}^{t} \big\| 1_{|\xi| \leq N_1} \big\|_{L^{\frac{6}{1-4\delta}}} e^{\frac{\lambda^2 s}{4T}} \|\widehat{\mathfrak{T}(f_{\mathrm h})} * \widehat{\mathfrak{T}(g_{\mathrm l})}(s,\cdot)\|_{L^{\frac{3}{1+2\delta}}} \, ds$$

$$\lesssim N_1^{2-\delta} T^{1-\frac{\delta}{2}} \sup_{0 < s \leq T} \big(s^{\delta} e^{\frac{\lambda^2 s}{4T}} \|\mathfrak{T}(f_{\mathrm h}) \mathfrak{T}(g_{\mathrm l})(s)\|_{L^{\frac{3}{2(1-\delta)}}} \big),$$

and also

$$t^{\frac{\delta}{2}}\Big\|\int_{\eta_0 t}^{t}|\xi|^{\frac{3}{2}+\delta}1_{|\xi|\geq N_1}e^{-\frac{1}{10}(t-s)|\xi|^2+\frac{\lambda^2 s}{4T}}|\widehat{\mathfrak{T}(f_{\mathrm{h}})}*\widehat{\mathfrak{T}(g_{\mathrm{l}})}(s,\xi)|\,ds\Big\|_{L_\xi^2}$$

$$\lesssim t^{\frac{\delta}{2}}\int_{\eta_0 t}^{t}(t-s)^{-1+\frac{\delta}{2}}s^{-\delta}\,ds\sup_{0<s\leq T}\Big(s^\delta e^{\frac{\lambda^2 s}{4T}}\|\mathfrak{T}(f_{\mathrm{h}})\mathfrak{T}(g_{\mathrm{l}})(s)\|_{L^{\frac{3}{2(1-\delta)}}}\Big)$$

$$\lesssim \sup_{0<s\leq T}\Big(s^\delta e^{\frac{\lambda^2 s}{4T}}\|\mathfrak{T}(f_{\mathrm{h}})\mathfrak{T}(g_{\mathrm{l}})(s)\|_{L^{\frac{3}{2(1-\delta)}}}\Big).$$

Consequently, we deduce for $0 < t \leq T$

$$t^{\frac{\delta}{2}}\Big\||\xi|^{\frac{3}{2}+\delta}\int_{\eta_0 t}^{t}e^{-(t-s)|\xi|^2}e^{\lambda\frac{t-s}{\sqrt{T}}|\xi|+\frac{\lambda^2 s}{2T}-\frac{\lambda^2 t}{4T}}|\widehat{\mathfrak{T}(f_{\mathrm{h}})}*\widehat{\mathfrak{T}(g_{\mathrm{l}})}(s,\xi)|\,ds\Big\|_{L_\xi^2}$$

$$\lesssim(\lambda^{2-\delta}+1)\cdot\sup_{0<s\leq T}\Big(s^\delta e^{\frac{\lambda^2 s}{4T}}\|\mathfrak{T}(f_{\mathrm{h}})\mathfrak{T}(g_{\mathrm{l}})(s)\|_{L^{\frac{3}{2(1-\delta)}}}\Big)$$

$$\lesssim\lambda^{2-\delta}\cdot\sup_{0<s\leq T}\Big(s^\delta e^{\frac{\lambda^2 s}{4T}}\||D|^{\frac{1}{2}+\delta}\mathfrak{T}(f_{\mathrm{h}})(s)\|_{L^2}\cdot\||D|^{\frac{1}{2}+\delta}\mathfrak{T}(g_{\mathrm{l}})(s)\|_{L^2}\Big) \qquad (2.16)$$

$$\lesssim\lambda^{2-\delta}T^{\frac{\delta}{2}}\|f\|_{X_T}\sup_{0<s\leq T}\Big(e^{\frac{\lambda^2 s}{4T}}\|\mathfrak{T}(g_{\mathrm{l}})(s)\|_{\dot{H}^{\frac{1}{2}+\delta}}\Big).$$

Together with (2.4), we infer

$$\Big\|t^{\frac{\delta}{2}}|\xi|^{\frac{3}{2}+\delta}\int_{\eta_0 t}^{t}e^{-(t-s)|\xi|^2}e^{\lambda\frac{t-s}{\sqrt{T}}|\xi|+\frac{\lambda^2 s}{2T}-\frac{\lambda^2 t}{4T}}|\widehat{\mathfrak{T}(f_{\mathrm{h}})}*\widehat{\mathfrak{T}(g_{\mathrm{l}})}(s,\xi)|\,ds\Big\|_{L_T^\infty(L_\xi^2)}$$

$$\lesssim\lambda^{2-\delta}T^{\frac{\delta}{2}}\big(1+\lambda^{\frac{1}{2}+\delta+\gamma}T^{\frac{1}{2}(\gamma-\frac{1}{2}-\delta)}e^{0.01\lambda^2}\big)\|f\|_{X_T}\|g\|_{L_T^\infty(H^\gamma)}.$$

Plugging the above estimate together with (2.15) into (2.13), we obtain

$$\Big\|t^{\frac{\delta}{2}}|\xi|^{\frac{1}{2}+\delta}1_{|\xi|\geq 0.1N_1}e^{\lambda\frac{t}{\sqrt{T}}|\xi|-\frac{\lambda^2 t}{4T}}\int_{\eta_0 t}^{t}e^{-(t-s)|\xi|^2}\widehat{Q(f,g)}(s,\xi)\,ds\Big\|_{L_T^\infty(L_\xi^2)}$$

$$\lesssim\lambda^{-\delta}e^{\frac{\lambda^2}{4}}\|f\|_{X_T}\|g\|_{X_T}+\lambda^{2-\delta}T^{\frac{\delta}{2}}\big(1+\lambda^{\frac{1}{2}+\delta+\gamma}T^{\frac{1}{2}(\gamma-\frac{1}{2}-\delta)}e^{0.01\lambda^2}\big) \qquad (2.17)$$

$$\times\big(\|g\|_{L_T^\infty(H^\gamma)}\|f\|_{X_T}+\|f\|_{L_T^\infty(H^\gamma)}\|g\|_{X_T}\big).$$

Clearly, (2.7) is a consequence of the estimates (2.10) and (2.17). $\quad\square$

We now turn to the proof of Lemma 2.2.

Proof of Lemma 2.2. Using Hausdorff-Young's inequality, we find

$$
\int_{\eta_0 t}^{t} e^{sN_0^2} \| 1_{|\xi| \leq 2N_0} |\xi|^{\frac{3}{2}+\delta} \widehat{F}(s,\xi) \|_{L^2_\xi} \, ds
$$

$$
\lesssim \int_{\eta_0 t}^{t} e^{sN_0^2} N_0^{\frac{3}{2}+\delta} \| 1_{|\xi| \leq 2N_0} \|_{L^{\frac{6}{1-4\delta}}} \| \widehat{F}(s,\xi) \|_{L^{\frac{3}{1+2\delta}}} \, ds
$$

$$
\lesssim \int_{\eta_0 t}^{t} e^{sN_0^2} N_0^{\frac{3}{2}+\delta} \cdot N_0^{\frac{1}{2}-2\delta} \| F(s) \|_{L^{\frac{3}{2(1-\delta)}}} \, ds
$$

$$
\lesssim \sup_{0<s\leq T} \left(s^\delta \| F(s) \|_{L^{\frac{3}{2(1-\delta)}}} \right) \cdot t^{-\frac{\delta}{2}} (N_0\sqrt{t})^{-\delta} \cdot \left(e^{N_0^2 t} - 1 \right).
$$

Note that the function $f(x) = x^{-\delta}(e^{x^2} - 1)$ is monotonically increasing in $x > 0$. In particular, for $t \leq T$, one has (recall $2N_0 = N_1 = \lambda T^{-\frac{1}{2}}$)

$$
f(N_0\sqrt{t}) \leq f(N_0\sqrt{T}) \leq 2^\delta \lambda^{-\delta} e^{N_0^2 T}.
$$

This leads to (2.11).

On the other hand for the piece containing $1_{|\xi| \geq 2N_0}$, we have

$$
t^{\frac{\delta}{2}} \| \int_{\eta_0 t}^{t} e^{N_0^2 s} |\xi|^{\frac{3}{2}+\delta} 1_{|\xi| \geq 2N_0} e^{-\frac{1}{10}(t-s)|\xi|^2} \widehat{F}(s,\xi) \, ds \|_{L^2_\xi}
$$

$$
\lesssim t^{\frac{\delta}{2}} \int_{\eta_0 t}^{t} e^{N_0^2 s} (t-s)^{-1+\frac{\delta}{2}} \| F(s,\cdot) \|_{L^{\frac{3}{2(1-\delta)}}} \, ds
$$

$$
\lesssim t^{\frac{\delta}{2}} \int_{\eta_0 t}^{t} e^{N_0^2 s} (t-s)^{-1+\frac{\delta}{2}} s^{-\delta} \, ds \sup_{0<s\leq T} \left(s^\delta \| F(s,\cdot) \|_{L^{\frac{3}{2(1-\delta)}}} \right)
$$

$$
\lesssim \int_{\eta_0}^{1} e^{N_0^2 t\tau} (1-\tau)^{-1+\frac{\delta}{2}} d\tau \cdot \sup_{0<s\leq T} \left(s^\delta \| F(s,\cdot) \|_{L^{\frac{3}{2(1-\delta)}}} \right)
$$

$$
\lesssim \langle N_0^2 T \rangle^{-\frac{\delta}{2}} e^{N_0^2 T} \sup_{0<s\leq T} \left(s^\delta \| F(s,\cdot) \|_{L^{\frac{3}{2(1-\delta)}}} \right).
$$

Note that in the above we adopted the notation $\langle x \rangle = \sqrt{1+x^2}$. Clearly $\langle N_0^2 T \rangle^{-\frac{\delta}{2}} \leq (N_0^2 T)^{-\frac{\delta}{2}}$. Thus (2.12) holds. In the last step above, we used the elementary inequality (for $\Lambda \gg 1$):

$$\int\limits_{\eta_0}^{1} e^{\Lambda\tau}(1-\tau)^{-1+\frac{\delta}{2}}d\tau = e^{\Lambda}\int\limits_{0}^{1-\eta_0} e^{-\Lambda s}s^{-1+\frac{\delta}{2}}ds \lesssim \Lambda^{-\frac{\delta}{2}}e^{\Lambda}.$$

This completes the proof of Lemma 2.2. $\square$

Next we turn to a pedestrian result: the local well-posedness of (GNS) in classical Sobolev spaces. Given the bilinear form $Q(u,v)$ in (1.1), we define $\mathcal{B}$ via

$$\begin{cases} \partial_t \mathcal{B} - \Delta \mathcal{B} = Q(u,v), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \mathcal{B}|_{t=0} = 0. \end{cases} \tag{2.18}$$

In yet other words, $\mathcal{B}$ is related to $Q(u,v)$ by the integral:

$$\mathcal{B} = \int\limits_0^t e^{(t-s)\Delta}Q(u(s),v(s))ds.$$

Clearly, $\mathcal{B}$ is a bilinear form in u and v. This bilinearity is precisely what allows the ensuing *a priori* estimates to directly establish the existence and uniqueness of $\mathcal{B}$.

Proposition 2.2. *Let* $\gamma \in [1/2,\infty[$, $p_\gamma \overset{\text{def}}{=} \begin{cases} 4, & \text{if } \gamma > 1, \\ \frac{8}{3-2\gamma} & \text{if } \gamma \in [1/2,1], \end{cases}$ *and* u,v *belong to* $L^{p_\gamma}([0,T]; H^{\gamma+\frac{2}{p_\gamma}})$. *Then* (2.18) *admits a unique solution* $\mathcal{B} \in C([0,T];H^\gamma) \cap L^2(]0,T[;\dot{H}^{\gamma+1})$. *Moreover, there holds*

$$\|\mathcal{B}\|_{L_T^\infty(H^\gamma)} + \|\nabla\mathcal{B}\|_{L_T^2(H^\gamma)}$$

$$\leq C_\gamma(T)\|u\|_{L_T^{p_\gamma}(H^{\gamma+\frac{2}{p_\gamma}})}\|v\|_{L_T^{p_\gamma}(H^{\gamma+\frac{2}{p_\gamma}})} \quad \text{with} \quad C_\gamma(T) \overset{\text{def}}{=} \begin{cases} CT^{\frac{1}{4}}, & \text{if } \gamma > 1, \\ T^{\frac{1}{2}-\frac{2}{p_\gamma}} & \text{if } \gamma \in [1/2,1], \end{cases} \tag{2.19}$$

and

$$\|\mathcal{B}\|_{L_T^{p_\gamma}(H^{\gamma+\frac{2}{p_\gamma}})} \leq C_\gamma(T)\big(1+T^{\frac{1}{p_\gamma}}\big)\|u\|_{L_T^{p_\gamma}(H^{\gamma+\frac{2}{p_\gamma}})}\|v\|_{L_T^{p_\gamma}(H^{\gamma+\frac{2}{p_\gamma}})}. \tag{2.20}$$

Proof. For simplicity, we just present the *a priori* estimates (2.19) and (2.20). We first get, by taking H^γ inner product of (2.18) with $\mathcal{B}$, that

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{B}\|_{H^\gamma}^2 + \|\nabla\mathcal{B}\|_{H^\gamma}^2 = \big(\langle D\rangle^\gamma Q(u,v), \langle D\rangle^\gamma \mathcal{B}\big)_{L^2}. \tag{2.21}$$

In the case when $\gamma > 1$, we deduce from (1.1) and the product-law in Sobolev spaces (see Theorem 8.3.1 of [18] or [27]), that

$$\left|\left(\langle D\rangle^{\gamma}Q(u,v),\langle D\rangle^{\gamma}\mathcal{B}\right)_{L^2}\right| \lesssim \||D|^{-\frac{1}{2}}Q(u,v)\|_{H^{\gamma}}\||D|^{\frac{1}{2}}\mathcal{B}\|_{H^{\gamma}}$$
$$\lesssim \|u\|_{H^{\gamma+\frac{1}{2}}}\|v\|_{H^{\gamma+\frac{1}{2}}}\|\mathcal{B}\|_{H^{\gamma}}^{\frac{1}{2}}\|\nabla\mathcal{B}\|_{H^{\gamma}}^{\frac{1}{2}}.$$

This implies

$$\int_0^T \left|\left(\langle D\rangle^{\gamma}Q(u,v),\langle D\rangle^{\gamma}\mathcal{B}\right)_{L^2}\right|dt$$
$$\lesssim T^{\frac{1}{4}}\|u\|_{L_T^4(H^{\gamma+\frac{1}{2}})}\|v\|_{L_T^4(H^{\gamma+\frac{1}{2}})}\|\mathcal{B}\|_{L_T^\infty(H^{\gamma})}^{\frac{1}{2}}\|\nabla\mathcal{B}\|_{L_T^2(H^{\gamma})}^{\frac{1}{2}} \tag{2.22}$$
$$\leq CT^{\frac{1}{2}}\|u\|_{L_T^4(H^{\gamma+\frac{1}{2}})}^2\|v\|_{L_T^4(H^{\gamma+\frac{1}{2}})}^2 + \frac{1}{4}\left(\|\mathcal{B}\|_{L_T^\infty(H^{\gamma})}^2 + \|\nabla\mathcal{B}\|_{L_T^2(H^{\gamma})}^2\right).$$

Similarly for $\gamma \in [1/2, 1]$, we have

$$\left|\left(\langle D\rangle^{\gamma}Q(u,v),\langle D\rangle^{\gamma}\mathcal{B}\right)_{L^2}\right| \lesssim \|u\otimes v\|_{H^{\gamma}}\|\nabla\mathcal{B}\|_{H^{\gamma}}$$
$$\lesssim \|u\|_{H^{\gamma+\frac{2}{p_{\gamma}}}}\|v\|_{H^{\gamma+\frac{2}{p_{\gamma}}}}\|\nabla\mathcal{B}\|_{H^{\gamma}}.$$

This yields

$$\int_0^T \left|\left(\langle D\rangle^{\gamma}Q(u,v),\langle D\rangle^{\gamma}\mathcal{B}\right)_{L^2}\right|dt$$
$$\lesssim T^{\frac{1}{2}-\frac{2}{p_{\gamma}}}\|u\|_{L_T^{p_{\gamma}}(H^{\gamma+\frac{2}{p_{\gamma}}})}\|v\|_{L_T^{p_{\gamma}}(H^{\gamma+\frac{2}{p_{\gamma}}})}\|\nabla\mathcal{B}\|_{L_T^2(H^{\gamma})} \tag{2.23}$$
$$\leq CT^{1-\frac{4}{p_{\gamma}}}\|u\|_{L_T^{p_{\gamma}}(H^{\gamma+\frac{2}{p_{\gamma}}})}^2\|v\|_{L_T^{p_{\gamma}}(H^{\gamma+\frac{2}{p_{\gamma}}})}^2 + \frac{1}{2}\|\nabla\mathcal{B}\|_{L_T^2(H^{\gamma})}^2,$$

where we used the fact $\gamma \geq \frac{1}{2}$, so that $p_{\gamma} \geq 4$.

By integrating (2.21) on $[0, T]$ and then plugging the estimate (2.22) or (2.23) into the resulting inequality, we obtain (2.19).

On the other hand, a simple interpolation argument gives

$$\|\mathcal{B}\|_{L_T^{p_{\gamma}}(H^{\gamma+\frac{2}{p_{\gamma}}})} \leq \|\mathcal{B}\|_{L_T^\infty(H^{\gamma})}^{1-\frac{2}{p_{\gamma}}}\|\langle D\rangle\mathcal{B}\|_{L_T^2(H^{\gamma})}^{\frac{2}{p_{\gamma}}}$$
$$\lesssim \|\mathcal{B}\|_{L_T^\infty(H^{\gamma})}^{1-\frac{2}{p_{\gamma}}}\left(T^{\frac{1}{2}}\|\mathcal{B}\|_{L_T^\infty(H^{\gamma})} + \|\nabla\mathcal{B}\|_{L_T^2(H^{\gamma})}\right)^{\frac{2}{p_{\gamma}}}.$$

This together with (2.19) yields (2.20). This concludes the proof of Proposition 2.2. □

We now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. We proceed via the following three steps:

Step 1. The local existence of classical solution.

In view of (2.18), we seek a solution to $u = e^{t\Delta}u_0 + \mathcal{B}(u, u)$ via the following iterative scheme:

$$u_1 \overset{\text{def}}{=} e^{t\Delta}u_0 \quad \text{and} \quad u_{n+1} \overset{\text{def}}{=} e^{t\Delta}u_0 + \mathcal{B}(u_n, u_n). \tag{2.24}$$

Let $p_\gamma > 2$ be the same as in Proposition 2.2. Minkowski inequality yields

$$
\begin{aligned}
\left\| e^{t\Delta}u_0 \right\|_{L_T^{p_\gamma}(H^{\gamma + \frac{2}{p_\gamma}})} &\lesssim \left\| \langle\xi\rangle^\gamma e^{-t|\xi|^2} \widehat{u_0}(\xi) \right\|_{L_T^{p_\gamma}(L_\xi^2)} + \left\| \langle\xi\rangle^\gamma |\xi|^{\frac{2}{p_\gamma}} e^{-t|\xi|^2} \widehat{u_0}(\xi) \right\|_{L_T^{p_\gamma}(L_\xi^2)} \\
&\lesssim T^{\frac{1}{p_\gamma}} \|u_0\|_{H^\gamma} + \left\| \langle\xi\rangle^\gamma |\xi|^{\frac{2}{p_\gamma}} \|e^{-t|\xi|^2}\|_{L_T^{p_\gamma}} \widehat{u_0}(\xi) \right\|_{L_\xi^2} \\
&\leq C\big(1 + T^{\frac{1}{p_\gamma}}\big)\|u_0\|_{H^\gamma}.
\end{aligned}
\tag{2.25}
$$

By (2.20), we have

$$\left\| \mathcal{B}(u_n, u_n) \right\|_{L_T^{p_\gamma}(H^{\gamma + \frac{2}{p_\gamma}})} \leq C_\gamma(T)\big(1 + T^{\frac{1}{p_\gamma}}\big)\|u_n\|_{L_T^{p_\gamma}(H^{\gamma + \frac{2}{p_\gamma}})}^2.$$

In the subcritical regime $\gamma > 1/2$, estimate (2.19) yields $C_\gamma(T) = T^\theta$ with $\theta > 0$. Scaling-wise, the factor T^θ directly reflects the subcritical nature of the nonlinear estimates. Hence we may choose $T_1 > 0$ such that

$$T_1 \overset{\text{def}}{=} \sup\big\{ T \leq 1, \quad 4CC_\gamma(T)\big(1 + T^{\frac{1}{p_\gamma}}\big)^2(1 + \|u_0\|_{H^\gamma}) < 1 \big\}. \tag{2.26}$$

With this definition of T_1, we can find $u \in L_{T_1}^{p_\gamma}(H^{\gamma + \frac{2}{p_\gamma}})$ satisfying (1.10) and

$$\lim_{n\to\infty} \|u_n - u\|_{L_{T_1}^{p_\gamma}(H^{\gamma + \frac{2}{p_\gamma}})} = 0 \quad \text{and} \quad \sup_{n\geq 1} \|u_n\|_{L_{T_1}^{p_\gamma}(H^{\gamma + \frac{2}{p_\gamma}})} \leq 2C\big(1 + T_1^{\frac{1}{p_\gamma}}\big)\|u_0\|_{H^\gamma}. \tag{2.27}$$

By (2.19) and (2.24), we obtain

$$\sup_{n\geq 1}\big(\|u_n\|_{L_{T_1}^\infty(H^\gamma)} + \|\nabla u_n\|_{L_{T_1}^2(H^\gamma)}\big) \leq C\|u_0\|_{H^\gamma}. \tag{2.28}$$

Notice from (2.24) that

$$
\begin{aligned}
u_{n+1} - u &= \mathcal{B}(u_n, u_n) - \mathcal{B}(u, u) \\
&= B(u_n - u, u_n) + \mathcal{B}(u, u_n - u).
\end{aligned}
$$

Using (2.19) and (2.27), we deduce that for $T \leq T_1$,

$$\|(u_{n+1} - u)\|_{L_T^\infty(H^\gamma)} + \|\nabla(u_{n+1} - u)\|_{L_T^2(H^\gamma)}$$

$$\leq C\|u_n - u\|_{L_T^{p_\gamma}(H^{\gamma+\frac{2}{p_\gamma}})} \left(\|u_n\|_{L_T^{p_\gamma}(H^{\gamma+\frac{2}{p_\gamma}})} + \|u\|_{L_T^{p_\gamma}(H^{\gamma+\frac{2}{p_\gamma}})} \right) \qquad (2.29)$$

$$\leq C\|u_0\|_{H^\gamma} \|u_n - u\|_{L_T^{p_\gamma}(H^{\gamma+\frac{2}{p_\gamma}})} \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2. The uniform estimate of $\|u_n\|_{X_T}$.

Using (2.2) and (2.24), we write

$$\widehat{u}_{n+1,\mathrm{h}}(t,\xi) \overset{\text{def}}{=} \widehat{u}_{n+1}(t,\xi) 1_{|\xi| \geq 0.01 N_1}$$

$$= \widehat{u}_{n+1}(t,\xi) 1_{0.01 N_1 \leq |\xi| < 0.1 N_1} + \widehat{e^{t\Delta} u_0} 1_{|\xi| \geq 0.1 N_1} + \mathcal{B}(\widehat{u_n, u_n})(t,\xi) 1_{|\xi| \geq 0.1 N_1}. \qquad (2.30)$$

By (1.11) and (2.5), we get

$$\|u_{n+1}\|_{X_T} \leq C T^{\frac{1}{2}(\gamma-\frac{1}{2})} \lambda^{-(\gamma-\frac{1}{2}-\delta)} \|u_{n+1,\mathrm{h}}\|_{L_T^\infty(H^\gamma)} + \|e^{t\Delta} u_0\|_{X_T}$$

$$+ \Big\| t^{\frac{\delta}{2}} 1_{|\xi| \geq 0.1 N_1} |\xi|^{\delta+\frac{1}{2}} e^{\lambda \frac{t}{\sqrt{T}} |\xi| - \frac{\lambda^2 t}{4T}} \int_0^t e^{-(t-s)|\xi|^2} Q(\widehat{u_n, u_n})(s,\xi) \, ds \Big\|_{L_T^\infty(L_\xi^2)}.$$

Thanks to the condition $\gamma > \frac{1}{2} + 2\delta$, we have

$$\|e^{t\Delta} u_0\|_{X_T} = \| t^{\frac{\delta}{2}} |\xi|^{\frac{1}{2}+\delta} 1_{|\xi| \geq 0.01 N_1} e^{-\frac{\lambda^2 t}{4T} + \lambda \frac{t}{\sqrt{T}} |\xi|} e^{-t|\xi|^2} \widehat{u_0}(\xi) \|_{L_T^\infty(L_\xi^2)}$$

$$\lesssim T^{\frac{1}{2}(\gamma-\frac{1}{2})} \lambda^{-(\gamma-\frac{1}{2}-\delta)} \| 1_{|\xi| \geq 0.01 N_1} |\xi|^\gamma \widehat{u_0}(\xi) \|_{L_\xi^2}$$

$$\lesssim T^{\frac{1}{2}(\gamma-\frac{1}{2})} \lambda^{-(\gamma-\frac{1}{2}-\delta)} \|u_{\mathrm{h}}\|_{L_T^\infty(H^\gamma)},$$

where in the last step we invoke the limit solution u determined by (2.27) (note that by (2.28) $u \in L_T^\infty H^\gamma$). Using Proposition 2.1 for the nonlinear piece, we obtain

$$\|u_{n+1}\|_{X_T} \leq C \Big(T^{\frac{1}{2}(\gamma-\frac{1}{2})} \lambda^{-(\gamma-\frac{1}{2}-\delta)} \big[\big(\|u_{n+1,\mathrm{h}}\|_{L_T^\infty(H^\gamma)} + \|u_{\mathrm{h}}\|_{L_T^\infty(H^\gamma)} \big)$$

$$+ \big(1 + e^{4\eta_0 \lambda^2} \big) T^\delta \|u_n\|_{L_T^\infty(H^\gamma)} \|u_{n,\mathrm{h}}\|_{L_T^\infty(H^\gamma)} \big] + \lambda^{-\delta} e^{\frac{\lambda^2}{4}} \|u_n\|_{X_T}^2$$

$$+ \lambda^{2-\delta} T^{\frac{\delta}{2}} \big(1 + \lambda^{\frac{1}{2}+\delta+\gamma} T^{\frac{1}{2}(\gamma-\frac{1}{2}-\delta)} e^{0.01 \lambda^2} \big) \|u_n\|_{L_T^\infty(H^\gamma)} \|u_n\|_{X_T} \Big).$$

Then we deduce from (2.29) that for arbitrary but fixed small enough constant $\varepsilon > 0$, there exists an integer N so that for any $n \geq$ N, there holds

$$\|u_{n+1}\|_{X_T} \leq C \Big(T^{\frac{1}{2}(\gamma-\frac{1}{2})} \lambda^{-(\gamma-\frac{1}{2}-\delta)} \big[1 + \big(1 + e^{4\eta_0 \lambda^2} \big) T^\delta \|u_0\|_{H^\gamma} \big]$$

$$\times \big(\varepsilon + \|u_{\mathrm{h}}\|_{L_T^\infty(H^\gamma)} \big) + \lambda^{-\delta} e^{\frac{\lambda^2}{4}} \|u_n\|_{X_T}^2 \qquad (2.31)$$

$$+ \lambda^{2-\delta} T^{\frac{\delta}{2}} \big(1 + \lambda^{\frac{1}{2}+\delta+\gamma} T^{\frac{1}{2}(\gamma-\frac{1}{2}-\delta)} e^{0.01 \lambda^2} \big) \|u_0\|_{H^\gamma} \|u_n\|_{X_T} \Big).$$

Before proceeding, we present the following lemma:

Lemma 2.3 (*Uniform high frequency smallness in H^γ*). *Let $\gamma > \frac{1}{2}$ and $u \in C([0,T]; H^\gamma)$ be a mild solution of (GNS) with initial data $u_0 \in H^\gamma$. We denote the Fourier multiplier $\widehat{P_{>J}}(\xi) \overset{\text{def}}{=} 1_{|\xi| \geq 0.01J}$. Then one has*

$$\eta_J^\gamma(T) \overset{\text{def}}{=} \max_{0 \leq t \leq T} \|P_{>J} u(t)\|_{H^\gamma} \to 0, \qquad as \ \ J \to \infty. \tag{2.32}$$

We admit the lemma for the time being and postpone its proof till the end of this section.

We now continue our proof of Theorem 1.2. For $0 < \eta_0 < \frac{\delta}{8(2\gamma-1)}$ and $\eta_J^\gamma(t)$ being given by (1.8), we define

$$\lambda(T) \overset{\text{def}}{=} \sqrt{(2\gamma - 1)\big(|\ln T| + \ln|\ln T|\big) + 3\beta(T)} \quad \text{with}$$
$$\beta(T) \overset{\text{def}}{=} \min\Big\{ \ |\ln\big(\eta_{T^{-\frac{1}{2}}}^\gamma(T)\big)|, \ \frac{1}{2}\big(\gamma - \frac{1}{2}\big)|\ln T| \ \Big\}. \tag{2.33}$$

In the above definition, if $\eta_{T^{-\frac{1}{2}}}^\gamma(T) = 0$, then the expression $|\ln(\eta_{T^{-\frac{1}{2}}}^\gamma)|$ is tacitly defined to be infinity for which $\beta(T) = \frac{1}{2}(\gamma - \frac{1}{2})|\ln T|$. Thus $\beta(T)$ and $\lambda(T)$ are well-defined in all cases. We also define

$$T_2 \overset{\text{def}}{=} \sup\Big\{ \ T \leq T_1, \ e^{4\eta_0 \lambda^2(T)} T^\delta \leq 1,$$
$$C\lambda^{2-\delta}(T) T^{\frac{\delta}{2}} \big(1 + \lambda^{\frac{1}{2} + \delta + \gamma}(T) T^{\frac{1}{2}(\gamma - \frac{1}{2} - \delta)} e^{0.01\lambda^2(T)}\big) \|u_0\|_{H^\gamma} \leq \frac{1}{2} \ \Big\}. \tag{2.34}$$

We observe that due to $\gamma > \frac{1}{2} + \delta$, T_2 is well-defined for δ being sufficiently small.

Then by virtue of (2.31), for any $n \geq N$ and $T \leq T_2$, we achieve

$$\|u_{n+1}\|_{X_T} \leq C\Big(T^{\frac{1}{2}(\gamma - \frac{1}{2})} \lambda^{-(\gamma - \frac{1}{2} - \delta)}(T)\big(1 + \|u_0\|_{H^\gamma}\big)\big(\varepsilon + \|u_h\|_{L_t^\infty(H^\gamma)}\big)$$
$$+ \lambda^{-\delta}(T) e^{\frac{\lambda^2(T)}{4}} \|u_n\|_{X_T}^2\Big) + \frac{1}{2}\|u_n\|_{X_T}. \tag{2.35}$$

Let us compare $\|u_{n+1}\|_{X_T}$ with Z, which is determined by

$$2Z = C\Big(T^{\frac{1}{2}(\gamma - \frac{1}{2})} \lambda^{-(\gamma - \frac{1}{2} - \delta)}(T)\big(1 + \|u_0\|_{H^\gamma}\big)\big(\varepsilon + \|u_h\|_{L_T^\infty(H^\gamma)}\big) + \lambda^{-\delta}(T) e^{\frac{\lambda^2(T)}{4}} Z^2\Big).$$

In particular, thanks to Lemma 2.3 and (2.33), for any sufficiently small positive constant c_0, which will be determined later on, we can shrink $t_0 \leq T_2$ to be so small that

$$t_0 \overset{\text{def}}{=} \sup\big\{ T \leq T_2, C^2 T^{\frac{1}{2}(\gamma - \frac{1}{2})} \lambda^{-(\gamma - \frac{1}{2})}(T)\big(1 + \|u_0\|_{H^\gamma}\big)\big(\varepsilon + \|u_h\|_{L_T^\infty(H^\gamma)}\big) e^{\frac{\lambda^2(T)}{4}} \leq c_0\big}. \tag{2.36}$$

We notice that for $\lambda(T)$ defined by (2.33), t_0 defined by above can be reached.

Then we deduce that for $n \geq N$ and $T \leq t_0$,

$$\|u_{n+1}\|_{X_T} \leq C T^{\frac{1}{2}(\gamma - \frac{1}{2})} \lambda^{-(\gamma - \frac{1}{2} - \delta)}(T) \big(1 + \|u_0\|_{H^\gamma}\big)\big(\varepsilon + \|u_h\|_{L_T^\infty(H^\gamma)}\big). \qquad (2.37)$$

Step 3. The convergence[3] of $\{t^{\frac{\delta}{2}} \mathfrak{T}(u_n)\}$ in $L^\infty([0,t_0]; \dot{H}^{\frac{1}{2}+\delta})$.

Recall (1.11) and (2.3). We deduce from (2.2) and Lemma 2.1 that for $T \leq t_0$,

$$
\begin{aligned}
\big\|t^{\frac{\delta}{2}} \mathfrak{T}(u_{n+1} - u_n)\big\|_{L_T^\infty(\dot{H}^{\frac{1}{2}+\delta})} \leq & \big\|t^{\frac{\delta}{2}} \mathfrak{T}\big((u_{n+1} - u_n)_1\big)\big\|_{L_T^\infty(\dot{H}^{\frac{1}{2}+\delta})} + \|u_{n+1} - u_n\|_{X_T} \\
\leq & C\|u_{n+1} - u_n\|_{L_T^\infty(H^\gamma)} + \|u_{n+1} - u_n\|_{X_T}.
\end{aligned}
\qquad (2.38)
$$

By (2.30), we have

$$
\begin{aligned}
\|u_{n+1} - u_n\|_{X_T} \leq & \Big\|t^{\frac{\delta}{2}} |\xi|^{\delta + \frac{1}{2}} 1_{0.01 N_1 \leq |\xi| < 0.1 N_1} e^{\lambda \frac{t}{\sqrt{T}} |\xi| - \frac{\lambda^2 t}{4T}} (\widehat{u}_{n+1} - \widehat{u}_n)\Big\|_{L_T^\infty(L_\xi^2)} \\
& + \Big\|t^{\frac{\delta}{2}} 1_{|\xi| \geq 0.1 N_1} |\xi|^{\delta + \frac{1}{2}} e^{\lambda \frac{t}{\sqrt{T}} |\xi| - \frac{\lambda^2 t}{4T}} \int_0^t e^{-(t-s)|\xi|^2} \mathcal{F}\Big(Q(u_n - u_{n-1}, u_n) \\
& \qquad\qquad\qquad + Q(u_{n-1}, u_n - u_{n-1})\Big)(s,\xi) \, ds\Big\|_{L_T^\infty(L_\xi^2)}.
\end{aligned}
$$

Using (2.5) and Proposition 2.1, we get

$$
\begin{aligned}
\|u_{n+1} - u_n\|_{X_T} \leq & C\Big(T^{\frac{1}{2}(\gamma - \frac{1}{2})} \lambda^{-(\gamma - \frac{1}{2} - \delta)}(T) \big[\|u_{n+1} - u_n\|_{L_T^\infty(H^\gamma)} \\
& + T^\delta \big(\|u_n\|_{L_T^\infty(H^\gamma)} + \|u_{n-1}\|_{L_T^\infty(H^\gamma)}\big) \|u_n - u_{n-1}\|_{L_T^\infty(H^\gamma)}\big] \\
& + \lambda^{2-\delta}(T) T^{\frac{\delta}{2}} \big(\|u_n\|_{X_T} + \|u_{n-1}\|_{X_T}\big) \|u_n - u_{n-1}\|_{L_T^\infty(H^\gamma)} \\
& + \|u_n - u_{n-1}\|_{X_T} \big[\lambda^{-\delta}(T) e^{\frac{\lambda^2(T)}{4}} \big(\|u_n\|_{X_T} + \|u_{n-1}\|_{X_T}\big) \\
& \qquad\qquad + \lambda^{2-\delta} T^{\frac{\delta}{2}} \big(\|u_n\|_{L_T^\infty(H^\gamma)} + \|u_{n-1}\|_{L_T^\infty(H^\gamma)}\big)\big]\Big).
\end{aligned}
$$

It follows from (2.36) and (2.37) that for $T \leq t_0$

$$
\begin{aligned}
\lambda^{-\delta} e^{\frac{\lambda^2(T)}{4}} \big(\|u_n\|_{X_T} + \|u_{n-1}\|_{X_T}\big) \leq & 2C T^{\frac{1}{2}(\gamma - \frac{1}{2})} \lambda^{-(\gamma - \frac{1}{2})}(T) \\
& \times \big(1 + \|u_0\|_{H^\gamma}\big)\big(\varepsilon + \|u_h\|_{L_T^\infty(H^\gamma)}\big) e^{\frac{\lambda^2(T)}{4}} \leq \frac{2}{C} c_0.
\end{aligned}
$$

Then if we take c_0 to be so small that $c_0 \leq \frac{1}{8}$, we deduce from (2.28) that for $T \leq t_0$,

---

[3] A conceptually simpler approach is to obtain a contraction in $C_t^0 L_x^2$ for the "un-weighted" iterates themselves, and then appeal to the weak closedness of the relevant Banach spaces to handle the higher norms.

$$\|u_{n+1} - u_n\|_{X_T} \leq C\big(\|u_{n+1} - u_n\|_{L_T^\infty(H^\gamma)} + \|u_0\|_{H^\gamma}\|u_n - u_{n-1}\|_{L_T^\infty(H^\gamma)}\big)$$
$$+ \big(\frac{1}{4} + C\lambda_\varepsilon^{2-\delta}(T)T^{\frac{\delta}{2}}\|u_0\|_{H^\gamma}\big)\|u_n - u_{n-1}\|_{X_T}.$$

By (2.34), we infer for $T \leq t_0$

$$\|u_{n+1} - u_n\|_{X_T} \leq C\big(\|u_{n+1} - u_n\|_{L_T^\infty(H^\gamma)} + \|u_n - u_{n-1}\|_{L_T^\infty(H^\gamma)}\big) + \frac{1}{2}\|u_n - u_{n-1}\|_{X_T}.$$

Plugging the above estimate into (2.38), we find for $T \leq t_0$,

$$\big\|t^{\frac{\delta}{2}}\mathfrak{T}(u_{n+1} - u_n)\big\|_{L_T^\infty(\dot{H}^{\frac{1}{2}+\delta})} \leq \frac{1}{2}\big\|t^{\frac{\delta}{2}}\mathfrak{T}(u_n - u_{n-1})\big\|_{L_T^\infty(\dot{H}^{\frac{1}{2}+\delta})}$$
$$+ C\big(\|u_{n+1} - u_n\|_{L_T^\infty(H^\gamma)} + \|u_n - u_{n-1}\|_{L_T^\infty(H^\gamma)}\big).$$
$$(2.39)$$

Hence $\big\{t^{\frac{\delta}{2}}\mathfrak{T}(u_n)\big\}$ is a Cauchy sequence in $L_T^\infty(\dot{H}^{\frac{1}{2}+\delta})$ for any $T \leq t_0$. As a result, it follows that for any $T \leq t_0$,

$$\lim_{n\to\infty}\big\|t^{\frac{\delta}{2}}\mathfrak{T}(u_{n+1} - u)\big\|_{L_T^\infty(\dot{H}^{\frac{1}{2}+\delta})} = 0 \quad \text{and} \quad \big\|t^{\frac{\delta}{2}}e^{-\frac{\lambda^2 t}{4T}}e^{\frac{\lambda t}{\sqrt{T}}|D|}u(t)\big\|_{L_T^\infty(\dot{H}^{\frac{1}{2}+\delta})} \leq C. \quad (2.40)$$

In particular, by taking $t = T$ in (2.40), we obtain

$$\big\|e^{\lambda(T)\sqrt{T}|D|}u(T)\big\|_{\dot{H}^{\frac{1}{2}+\delta}} \leq CT^{-\frac{\delta}{2}}e^{\frac{\lambda^2(T)}{4}} \quad \text{for any } T \leq t_0.$$

This together with (1.8) yields (1.7), completing the proof of Theorem 1.2. □

We now complete the proof of Lemma 2.3.

Proof of Lemma 2.3. We first observe that the linear part satisfies

$$\sup_{t\geq 0}\big\|P_{>J}e^{t\Delta}u_0\big\|_{H^\gamma} \leq \|P_{>J}u_0\|_{H^\gamma} \to 0, \qquad \text{as } J \to \infty.$$

For the nonlinear part, we first deal with the case $\gamma \in]1/2, 3/2[$. Let $J \geq 1$ which later will tend to infinity. Take sufficiently small constant $\epsilon > 0$ such that $0 < \epsilon < \gamma - \frac{1}{2}$. Using the product-law in Sobolev spaces (cf. Theorem 8.3.1 of [18] or [27]), we have

$$\big\|\int_0^t P_{>J}\langle D\rangle^\gamma e^{(t-s)\Delta}Q(u,u)(s)\,ds\big\|_{L^2}$$

$$\lesssim J^{-\epsilon}\int_0^t \big\||D|^{1+\epsilon+\gamma}P_{>J}e^{(t-s)\Delta}(u \otimes u)(s)\big\|_{L^2}\,ds$$

$$\lesssim J^{-\epsilon} \int_0^t \left\| |D|^{\frac{5}{2}-\gamma+\epsilon} P_{>J} e^{(t-s)\Delta} |D|^{2\gamma-\frac{3}{2}} (u \otimes u)(s) \right\|_{L^2} ds$$

$$\lesssim J^{-\epsilon} \int_0^t (t-s)^{-\frac{1}{2}(\frac{5}{2}-\gamma+\epsilon)} \|u(s)\|_{H^\gamma}^2 ds$$

$$\lesssim J^{-\epsilon} t^{\frac{1}{2}(\gamma-\frac{1}{2}-\epsilon)} \|u\|_{L_t^\infty(H^\gamma)}^2 \to 0, \qquad \text{as } J \to \infty.$$

Note that in the very last step, we appeal to the fact that $\gamma > \frac{1}{2} + \epsilon$.

For $\gamma = \frac{3}{2}$, we take $\epsilon \in]0,1[$. By a slight modification of the above argument, we have

$$\left\| \int_0^t P_{>J} \langle D \rangle^\gamma e^{(t-s)\Delta} Q(u,u)(s) \, ds \right\|_{L^2}$$

$$\lesssim J^{-\epsilon} \int_0^t \left\| |D|^{1+\epsilon+\frac{3}{2}} P_{>J} e^{(t-s)\Delta} (u \otimes u)(s) \right\|_{L^2} ds$$

$$\lesssim J^{-\epsilon} \int_0^t \left\| |D|^{1+2\epsilon} P_{>J} e^{(t-s)\Delta} \langle D \rangle^{\frac{3}{2}-\epsilon} (u \otimes u)(s) \right\|_{L^2} ds$$

$$\lesssim J^{-\epsilon} \int_0^t (t-s)^{-\frac{1}{2}(1+2\epsilon)} \|u(s)\|_{H^{\frac{3}{2}}}^2 ds \lesssim J^{-\epsilon} t^{\frac{1}{2}-\epsilon} \|u\|_{L_t^\infty(H^{\frac{3}{2}})}^2 \to 0, \qquad \text{as } J \to \infty.$$

Finally for $\gamma > \frac{3}{2}$, we note that $H^\gamma(\mathbb{R}^3)$ is an algebra. Clearly

$$\left\| \int_0^t P_{>J} \langle D \rangle^\gamma e^{(t-s)\Delta} Q(u,u)(s) \, ds \right\|_{L^2}$$

$$\lesssim J^{-\epsilon} \int_0^t \left\| |D|^{1+\epsilon+\gamma} P_{>J} e^{(t-s)\Delta} (u \otimes u)(s) \right\|_{L^2} ds$$

$$\lesssim J^{-\epsilon} \int_0^t (t-s)^{-\frac{1}{2}(1+\epsilon)} \|u(s)\|_{H^\gamma}^2 ds \lesssim J^{-\epsilon} t^{\frac{1}{2}(1-\epsilon)} \|u\|_{L_t^\infty(H^\gamma)}^2 \to 0, \qquad \text{as } J \to \infty.$$

Collecting the above estimates, we conclude the proof of (2.32). $\square$

3. The critical case: $\gamma = \frac{1}{2}$

In this section we tackle the critical case $\gamma = 1/2$ of Theorem 1.2. One notable feature of the critical case is that we must exploit fully the frequency localization. This is already

manifested in the following uniform high frequency smallness lemma. Albeit this is more or less standard, we include a proof for the sake of completeness.

Proposition 3.1 *(Uniform high frequency smallness in* $\dot{H}^{\frac{1}{2}}$*). Let* $u_0 \in \dot{H}^{\frac{1}{2}}$*. There exists a positive time* $T = T(u_0)$ *so that the system* (GNS) *with initial data* u_0 *has a unique solution* $u \in C([0,T]; \dot{H}^{\frac{1}{2}}) \cap L^2(]0,T[; \dot{H}^{\frac{3}{2}})$ *and*

$$
\|u\|_{L^\infty_T(\dot{H}^{\frac{1}{2}})} + \|\nabla u\|_{L^2_T(\dot{H}^{\frac{1}{2}})} + \|t^{\frac{m+\delta}{2}} u\|_{L^\infty_T(\dot{H}^{m+\delta+\frac{1}{2}})}
$$

$$
+ \|t^{\frac{m+\delta}{2}} \nabla u\|_{L^2_T(\dot{H}^{m+\delta+\frac{1}{2}})} \leq C\|u_0\|_{\dot{H}^{\frac{1}{2}}} \quad \textit{for any integer } m \geq 0 \quad \textit{and} \quad \delta \in]0,1[.
$$

$$(3.1)$$

Furthermore there holds

$$
\zeta_J^{\frac{1}{2}}(T) \overset{\text{def}}{=} \max_{0 \leq t \leq T} \|1_{|\xi| \geq J} |\xi|^{\frac{1}{2}} \widehat{u}(t,\xi)\|_{L^2_\xi} \to 0, \quad \textit{as } J \to \infty. \tag{3.2}
$$

Remark 3.1. It is possible to obtain all polynomial smoothing estimates in (3.1) at one stroke. We sketch the argument as follows. The key is to estimate the following Z-norm:

$$
\|u\|_{Z_T} \overset{\text{def}}{=} \|t^{\frac{1}{8}} |D|^{\frac{3}{4}} v\|_{L^\infty_T(L^2)}, \quad \text{with} \quad \widehat{v}(t,\xi) \overset{\text{def}}{=} e^{\frac{\sqrt{t}}{2}|\xi|} \widehat{u}(t,\xi). \tag{3.3}
$$

For the linear part, it is not difficult to check that

$$
\|t^{\frac{1}{8}} |D|^{\frac{3}{4}} e^{\frac{\sqrt{t}}{2}|D|} e^{t\Delta} u_0\|_{L^\infty(\mathbb{R}^+; L^2)} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{2}}};
$$

$$
\lim_{t \to 0^+} t^{\frac{1}{8}} \||D|^{\frac{3}{4}} e^{\frac{\sqrt{t}}{2}|D|} e^{t\Delta} u_0\|_{L^2} = 0. \tag{3.4}
$$

On the other hand, for the nonlinear part, we first get

$$
t^{\frac{1}{8}} \int_0^t \left\| e^{\frac{\sqrt{t}}{2}|D|} |D|^{\frac{3}{4}} e^{(t-s)\Delta} Q(u(s), u(s)) \right\|_{L^2} ds
$$

$$
\leq t^{\frac{1}{8}} \int_0^t \left\| |\xi|^{\frac{7}{4}} e^{\frac{1}{2}(\sqrt{t} - \sqrt{s})|\xi|} e^{-(t-s)|\xi|^2} |\widehat{v}(s)| * |\widehat{v}(s)| \right\|_{L^2_\xi} ds
$$

$$
\lesssim t^{\frac{1}{8}} \int_0^t \left\| |\xi|^{\frac{7}{4}} e^{-\frac{1}{2}(t-s)|\xi|^2} |\widehat{v}(s)| * |\widehat{v}(s)| \right\|_{L^2_\xi} ds
$$

$$
\lesssim t^{\frac{1}{8}} \int_0^t (t-s)^{-\frac{7}{8}} \left\| |\widehat{v}(s)| * |\widehat{v}(s)| \right\|_{L^2_\xi} ds
$$

$$\lesssim t^{\frac{1}{8}} \int_0^t (t-s)^{-\frac{7}{8}} \big\| \mathcal{F}_{\xi \to x}(|\widehat{v}(s,\xi)|) \big\|_{L^4}^2 \, ds.$$

Applying Sobolev $H^{\frac{3}{4}}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$, we obtain

$$t^{\frac{1}{8}} \int_0^t \big\| e^{\frac{\sqrt{t}}{2}|D|} |D|^{\frac{3}{4}} e^{(t-s)\Delta} Q(u(s), u(s)) \big\|_{L^2} \, ds \lesssim t^{\frac{1}{8}} \int_0^t (t-s)^{-\frac{7}{8}} \big\| |D|^{\frac{3}{4}} \mathcal{F}_{\xi \to x}(|\widehat{v}(s,\xi)|) \big\|_{L^2}^2 \, ds$$

$$\lesssim t^{\frac{1}{8}} \int_0^t (t-s)^{-\frac{7}{8}} s^{-\frac{1}{4}} \, ds \|u\|_{Z_T}^2$$

$$\lesssim \|u\|_{Z_T}^2.$$

Upon choosing a sufficiently small T, this together with (3.4) yields a unique local solution u on $[0,T]$ to (1.10) satisfying $\|u\|_{Z_T} \lesssim \|u_0\|_{\dot{H}^{\frac{1}{2}}}$ and enjoying all polynomial smoothing estimates.

Proof of Proposition 3.1. We first get, by a similar derivation of (2.25), that

$$\|e^{t\Delta} u_0\|_{L^4(\mathbb{R}^+; \dot{H}^1)} = \big\| |\xi| e^{-t|\xi|^2} \widehat{u}_0(\xi) \big\|_{L^4(\mathbb{R}^+; L^2_\xi)}$$

$$\leq \big\| |\xi| \| e^{-t|\xi|^2} \|_{L^4_t(\mathbb{R}^+)} \widehat{u}_0(\xi) \big\|_{L^2_\xi} \leq C \|u_0\|_{\dot{H}^{\frac{1}{2}}}.$$

Thus

$$\lim_{T \to 0} \|e^{t\Delta} u_0\|_{L^4_T(\dot{H}^1)} = 0. \tag{3.5}$$

For $\mathcal{B}$ being determined by (2.18), we deduce from (2.20) that

$$\|\mathcal{B}\|_{L^4_T(\dot{H}^1)} \leq C_0 \|u\|_{L^4_T(\dot{H}^1)} \|v\|_{L^4_T(\dot{H}^1)} \quad \text{if } T \leq 1.$$

Take $T(u_0) > 0$ sufficiently small such that

$$4C_0 \|e^{t\Delta} u_0\|_{L^4_T(\dot{H}^1)} < 1.$$

The usual fixed point theorem ensures that (1.10) admits a unique solution u in $L^4(]0,T[; \dot{H}^1)$. Moreover, it follows from (2.19) that

$$\|u\|_{L^\infty_T(\dot{H}^{\frac{1}{2}})} + \|\nabla u\|_{L^2_T(\dot{H}^{\frac{1}{2}})} \leq C(\|u_0\|_{\dot{H}^{\frac{1}{2}}}) \quad \text{and}$$

$$\|u\|_{L^q_T(\dot{H}^{\frac{1}{2}+\frac{2}{q}})} \leq \|u\|_{L^\infty_T(\dot{H}^{\frac{1}{2}})}^{1-\frac{2}{q}} \|u\|_{L^2_T(\dot{H}^{\frac{3}{2}})}^{\frac{2}{q}} \leq C(\|u_0\|_{\dot{H}^{\frac{1}{2}}}) \quad \text{for any } q \in [2,\infty]. \tag{3.6}$$

For any $p \in]1, \infty[$, we choose $q \in]2, \infty[$ with $p > \frac{q}{2}$. Then we get, by applying the product-law in Sobolev spaces, that

$$\|\mathcal{B}(u,u)(t)\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}} \lesssim \int_0^t \left\| |\xi|^{\frac{1}{2}+\frac{2}{p}} e^{-(t-s)|\xi|^2} |\xi| \widehat{u \otimes u}(s) \right\|_{L^2} ds$$

$$\lesssim \int_0^t \left\| |\xi|^{2+\frac{2}{p}-\frac{4}{q}} e^{-(t-s)|\xi|^2} |\xi|^{\frac{4}{q}-\frac{1}{2}} \widehat{u \otimes u}(s) \right\|_{L^2} ds$$

$$\lesssim \int_0^t (t-s)^{-1-\frac{1}{p}+\frac{2}{q}} \|u(s)\|^2_{\dot{H}^{\frac{1}{2}+\frac{2}{q}}} ds.$$

By (3.6) and Hardy-Littlewood-Sobolev inequality, we obtain

$$\|\mathcal{B}(u,u)\|_{L_T^p(\dot{H}^{\frac{1}{2}+\frac{2}{p}})} \lesssim \left\| \int_0^\infty |t-s|^{-1-\frac{1}{p}+\frac{2}{q}} \chi_{[0,T]}(s) \|u(s)\|^2_{\dot{H}^{\frac{1}{2}+\frac{2}{q}}} ds \right\|_{L^p}$$

$$\lesssim \|u\|^2_{L_T^q(\dot{H}^{\frac{1}{2}+\frac{2}{q}})} \leq C(\|u_0\|_{\dot{H}^{\frac{1}{2}}}).$$

A simple interpolation argument yields

$$\|\mathcal{B}(u,u)\|_{L_T^2(\dot{B}^{\frac{3}{2}}_{2,1})} \lesssim \|\mathcal{B}(u,u)\|^{\frac{1}{2}}_{L_T^4(\dot{H}^1)} \|\mathcal{B}(u,u)\|^{\frac{1}{2}}_{L_T^{\frac{4}{3}}(\dot{H}^2)} \leq C(\|u_0\|_{H^{\frac{1}{2}}}), \tag{3.7}$$

where $\|a\|_{\dot{B}^{\frac{3}{2}}_{2,1}}$ denotes the homogeneous Besov norm of a in the space $\dot{B}^{\frac{3}{2}}_{2,1}$ (see Definition 2.15 of [2]; see also Lemma 2.7 of [27] and the discussion therein for more general interpolation inequalities).

Clearly,

$$\|e^{t\Delta} u_0\|_{L^2(\mathbb{R}^+; L^\infty)} = \left(\int_0^\infty \|t^{\frac{1}{2}} e^{t\Delta} u_0\|^2_{L^\infty} \frac{dt}{t} \right)^{\frac{1}{2}} = \|u_0\|_{\dot{B}^{-1}_{\infty,2}} \leq C \|u_0\|_{\dot{H}^{\frac{1}{2}}}.$$

This together with (3.7) ensures that

$$\|u\|_{L_T^2(L^\infty)} \leq C(\|u_0\|_{\dot{H}^{\frac{1}{2}}}). \tag{3.8}$$

Next we turn to time-weighted energy estimates. For simplicity, we just present the *a priori* estimate. Fix $\delta \in]0, 1[$. Taking $\dot{H}^{m+\delta+\frac{1}{2}}$ inner product of (GNS) with u, we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{\dot{H}^{m+\delta+\frac{1}{2}}} + \|\nabla u\|^2_{\dot{H}^{m+\delta+\frac{1}{2}}} &= \left(|D|^{m+\delta+\frac{1}{2}}Q(u,u), |D|^{m+\delta+\frac{1}{2}}u\right)_{L^2} \\
&\lesssim \|u\otimes u\|_{\dot{H}^{m+\delta+\frac{1}{2}}}\|\nabla u\|_{\dot{H}^{m+\delta+\frac{1}{2}}} \\
&\leq C\|u\|^2_{L^\infty}\|u\|^2_{\dot{H}^{m+\delta+\frac{1}{2}}} + \frac{1}{2}\|\nabla u\|^2_{\dot{H}^{m+\delta+\frac{1}{2}}}.
\end{aligned}
\tag{3.9}
$$

In the above we used the simple estimate that

$$
\|u\otimes u\|_{\dot{H}^{m+\delta+\frac{1}{2}}} \leq C\|u\|_{L^\infty}\|u\|_{\dot{H}^{m+\delta+\frac{1}{2}}}.
$$

Multiplying (3.9) by $t^{m+\delta}$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|t^{\frac{m+\delta}{2}}u(t)\|^2_{\dot{H}^{m+\delta+\frac{1}{2}}} + \|t^{\frac{m+\delta}{2}}\nabla u\|^2_{\dot{H}^{m+\delta+\frac{1}{2}}} \\
&\leq (m+\delta)\|t^{\frac{m-1+\delta}{2}}u(t)\|^2_{\dot{H}^{m+\delta+\frac{1}{2}}} + C\|u\|^2_{L^\infty}\|t^{\frac{m+\delta}{2}}u(t)\|^2_{\dot{H}^{m+\delta+\frac{1}{2}}}.
\end{aligned}
$$

A simple Gronwall argument yields

$$
\begin{aligned}
\|t^{\frac{m+\delta}{2}}u\|^2_{L^\infty_t(\dot{H}^{m+\delta+\frac{1}{2}})} + \|t^{\frac{m+\delta}{2}}\nabla u\|^2_{L^2_t(\dot{H}^{m+\delta+\frac{1}{2}})} &\leq \Big(\|t^{\frac{m-1+\delta}{2}}\nabla u(t)\|^2_{L^2_t(\dot{H}^{m+\delta-\frac{1}{2}})} \\
&\quad + (m+\delta)\|t^{\frac{m-1+\delta}{2}}u(t)\|^2_{L^2_t(\dot{H}^{m+\delta-\frac{1}{2}})}\Big)\exp\big(C\|u\|^2_{L^2_t(L^\infty)}\big).
\end{aligned}
\tag{3.10}
$$

To complete the inductive proof of the time-weighted estimate in (3.1), we require the following lemma. Its proof is deferred until after that of Proposition 3.1.

Lemma 3.1. *Under the assumptions of Proposition 3.1, for any $t \leq T(u_0)$, one has*

$$
\|t^{\frac{\delta}{2}}u\|^2_{L^\infty_t(\dot{H}^{\delta+\frac{1}{2}})} + \|t^{\frac{\delta}{2}}\nabla u\|^2_{L^2_t(\dot{H}^{\delta+\frac{1}{2}})} \leq C(\|u_0\|_{\dot{H}^{\frac{1}{2}}}).
\tag{3.11}
$$

Finally, we present the proof of (3.2). Due to $u \in C([0,T];\dot{H}^{\frac{1}{2}})$, for any $\epsilon > 0$, we can choose $\tau_0 > 0$ so small that

$$
\max_{0 \leq t \leq \tau_0} \|u(t) - u_0\|_{\dot{H}^{\frac{1}{2}}} \leq 0.001\epsilon.
$$

This implies for any $J \geq 1$ that

$$
\max_{0 \leq t \leq \tau_0} \|1_{|\xi|\geq J}|\xi|^{\frac{1}{2}}(\widehat{u}(t,\xi) - \widehat{u_0}(\xi))\|_{L^2_\xi} \leq 0.01\epsilon.
$$

Taking $J_0 \geq 1$ sufficiently large such that

$$
\|1_{|\xi|\geq J_0}|\xi|^{\frac{1}{2}}\widehat{u_0}(\xi)\|_{L^2_\xi} \leq 0.01\epsilon,
$$

we obtain for any $J \geq J_0$,

$$\max_{0 \le t \le \tau_0} \|1_{|\xi| \ge J} |\xi|^{\frac{1}{2}} \widehat{u}(t, \xi)\|_{L^2_\xi} \le 0.02\epsilon. \tag{3.12}$$

On the other hand, it follows from (3.1) that $t^{\frac{3}{4}} u \in L^\infty([0, T]; \dot{H}^2)$. Thus

$$\max_{\tau_0 \le t \le T} \|P_{>J} u(t, \cdot)\|_{\dot{H}^{\frac{1}{2}}} \le C \tau_0^{-\frac{3}{4}} J^{-\frac{3}{2}} \to 0 \qquad \text{as } J \to \infty.$$

In particular, we can take $J_1 \ge 1$ sufficiently large such that

$$\max_{\tau_0 \le t \le T_0} \|1_{|\xi| \ge J_1} |\xi|^{\frac{1}{2}} \widehat{u}(t, \xi)\|_{L^2_\xi} \le 0.1\epsilon.$$

Together with (3.12), we obtain for $J \ge \max\{J_0, J_1\}$:

$$\max_{0 \le t \le T} \|1_{|\xi| \ge J} |\xi|^{\frac{1}{2}} \widehat{u}(t, \xi)\|_{L^2_\xi} \le \epsilon.$$

This leads to (3.2), and we complete the proof of Proposition 3.1. □

We now turn to the proof of Lemma 3.1.

Proof of Lemma 3.1. We first observe that

$$
\begin{aligned}
\|t^{\frac{\delta-1}{2}} e^{t\Delta} u_0\|^2_{L^2(\mathbb{R}^+; \dot{H}^{\delta+\frac{1}{2}})} &= \int_{\mathbb{R}^3} |\xi|^{1+2\delta} \int_0^\infty t^{\delta-1} e^{-2t|\xi|^2} \, dt |\widehat{u_0}(\xi)|^2 \, d\xi \\
&= \int_{\mathbb{R}^3} \int_0^\infty \tau^{\delta-1} e^{-2\tau} \, d\tau |\xi| |\widehat{u_0}(\xi)|^2 \, d\xi = C \|u_0\|^2_{\dot{H}^{\frac{1}{2}}}.
\end{aligned}
\tag{3.13}
$$

This will be used in the nonlinear estimate below.

By taking the $\dot{H}^{\delta-\frac{1}{2}}$ inner product of (2.18) with $u = v$ and then multiplying the resulting inequality by $t^{\delta-1}$, we obtain that

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \big\| t^{\frac{\delta-1}{2}} \mathcal{B}(u, u)(t) \big\|^2_{\dot{H}^{\delta-\frac{1}{2}}} + \big\| t^{\frac{\delta-1}{2}} \mathcal{B}(u, u)(t) \big\|^2_{\dot{H}^{\delta+\frac{1}{2}}} + (1-\delta) \big\| t^{\frac{\delta}{2}-1} \mathcal{B}(u, u)(t) \big\|^2_{\dot{H}^{\delta-\frac{1}{2}}} \\
&= t^{\delta-1} \big(Q(u, u), \mathcal{B}(u, u) \big)_{\dot{H}^{\delta-\frac{1}{2}}} \\
&\le C t^{\frac{\delta-1}{2}} \big\| (e^{t\Delta} u_0 + \mathcal{B}(u, u)) \otimes (e^{t\Delta} u_0 + \mathcal{B}(u, u)) \big\|_{\dot{H}^{\delta-\frac{1}{2}}} \big\| t^{\frac{\delta-1}{2}} \mathcal{B}(u, u)(t) \big\|_{\dot{H}^{\delta+\frac{1}{2}}} \\
&\le C t^{\delta-1} \big\| (e^{t\Delta} u_0 + \mathcal{B}(u, u)) \otimes (e^{t\Delta} u_0 + \mathcal{B}(u, u)) \big\|^2_{\dot{H}^{\delta-\frac{1}{2}}} + \frac{1}{2} \big\| t^{\frac{\delta-1}{2}} \mathcal{B}(u, u)(t) \big\|^2_{\dot{H}^{\delta+\frac{1}{2}}}.
\end{aligned}
$$

By the product-law in Sobolev spaces, we infer

$$
\begin{aligned}
&\frac{d}{dt} \big\| t^{\frac{\delta-1}{2}} \mathcal{B}(u, u)(t) \big\|^2_{\dot{H}^{\delta-\frac{1}{2}}} + \big\| t^{\frac{\delta-1}{2}} \mathcal{B}(u, u)(t) \big\|^2_{\dot{H}^{\delta+\frac{1}{2}}} + (1-\delta) \big\| t^{\frac{\delta}{2}-1} \mathcal{B}(u, u)(t) \big\|^2_{\dot{H}^{\delta-\frac{1}{2}}} \\
&\lesssim \big\| t^{\frac{\delta-1}{2}} e^{t\Delta} u_0 \big\|^2_{\dot{H}^{\delta+\frac{1}{2}}} \big(\|e^{t\Delta} u_0\|^2_{\dot{H}^{\frac{1}{2}}} + \|\mathcal{B}(u, u)\|^2_{\dot{H}^{\frac{1}{2}}} \big) + \|\mathcal{B}(u, u)\|^2_{\dot{B}^{\frac{3}{2}}} \big\| t^{\frac{\delta-1}{2}} \mathcal{B}(u, u)(t) \big\|^2_{\dot{H}^{\delta-\frac{1}{2}}}.
\end{aligned}
$$

By applying Gronwall's inequality and using (3.6), (3.7), we find for $t \leq T(u_0)$,

$$
\begin{aligned}
\left\| t^{\frac{\delta-1}{2}} \mathcal{B}(u,u)(t) \right\|^2_{L^2_t(\dot{H}^{\delta+\frac{1}{2}})} \leq & C\big(\|e^{t\Delta} u_0\|^2_{L^\infty_t(\dot{H}^{\frac{1}{2}})} + \|\mathcal{B}(u,u)\|^2_{L^\infty_t(\dot{H}^{\frac{1}{2}})} \big) \\
& \times \big\| t^{\frac{\delta-1}{2}} e^{t\Delta} u_0 \big\|^2_{L^2_t(\dot{H}^{\delta+\frac{1}{2}})} \exp\big(C \|\mathcal{B}(u,u)\|^2_{L^2_t(\dot{B}^{\frac{3}{2}})} \big) \\
\leq & C(\|u_0\|_{\dot{H}^{\frac{1}{2}}}).
\end{aligned}
$$

Together with (3.13), this yields

$$
\big\| t^{\frac{\delta-1}{2}} u \big\|_{L^2_t(\dot{H}^{\delta+\frac{1}{2}})} \leq C(\|u_0\|_{\dot{H}^{\frac{1}{2}}}). \tag{3.14}
$$

We remark that estimate of this type was first proposed by Chemin and Planchon in [7] for the classical 3-D Navier-Stokes system. The avid reader may view it as a natural L^2_t version of the classical Kato spaces (see [20]).

On the other hand, by taking $\dot{H}^{\delta+\frac{1}{2}}$ inner product of (GNS) with u and using the product-law in Sobolev spaces, we get

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{\dot{H}^{\delta+\frac{1}{2}}} + \|\nabla u\|^2_{\dot{H}^{\delta+\frac{1}{2}}} = & (Q(u,u), u)_{\dot{H}^{\delta+\frac{1}{2}}} \\
\lesssim & \|u \otimes u\|_{\dot{H}^{\delta+\frac{1}{2}}} \|\nabla u\|_{\dot{H}^{\delta+\frac{1}{2}}} \\
\lesssim & \|u\|_{L^\infty} \|u\|_{\dot{H}^{\delta+\frac{1}{2}}} \|\nabla u\|_{\dot{H}^{\delta+\frac{1}{2}}}.
\end{aligned}
$$

Multiplying t^δ to the above inequality, we obtain

$$
\frac{d}{dt} \|t^{\frac{\delta}{2}} u(t)\|^2_{\dot{H}^{\delta+\frac{1}{2}}} + \|t^{\frac{\delta}{2}} \nabla u\|^2_{\dot{H}^{\delta+\frac{1}{2}}} \leq \delta \|t^{\frac{\delta-1}{2}} u(t)\|^2_{\dot{H}^{\delta+\frac{1}{2}}} + C\|u\|^2_{L^\infty} \|t^{\frac{\delta}{2}} u\|^2_{\dot{H}^{\delta+\frac{1}{2}}}.
$$

A simple Gronwall argument yields

$$
\begin{aligned}
& \|t^{\frac{\delta}{2}} u\|^2_{L^\infty_t(\dot{H}^{\delta+\frac{1}{2}})} + \|t^{\frac{\delta}{2}} \nabla u\|^2_{L^2_t(\dot{H}^{\delta+\frac{1}{2}})} \\
& \leq C \|t^{\frac{\delta-1}{2}} u\|^2_{L^2_t(\dot{H}^{\delta+\frac{1}{2}})} \exp\big(C\|u\|^2_{L^2_t(L^\infty)} \big) \quad \text{for } t \leq T(u_0).
\end{aligned}
$$

This together with (3.8) and (3.14) imply (3.11), completing the proof of Lemma 3.1. $\quad\square$

Analogous to the norm $\|\cdot\|_{X_T}$ defined in (1.11) for the subcritical case, we define a corresponding norm for the critical case: for a sufficiently small $\delta > 0$,

$$
\|u\|_{Y_T} \overset{\text{def}}{=} \big\| t^{\frac{\delta}{2}} |\xi|^{\delta+\frac{1}{2}} 1_{|\xi| \geq T^{-\frac{1}{4}}} e^{-\frac{\lambda^2 t}{4T} + \lambda \frac{t}{\sqrt{T}} |\xi|} \widehat{u}(t,\xi) \big\|_{L^\infty_T(L^2_\xi)}. \tag{3.15}
$$

Note here we introduce the special cut-off $1_{|\xi| \geq T^{-\frac{1}{4}}}$ to break the critical scaling. In what follows, we shall focus on the estimate of the Y_T-norm of u.

Proposition 3.2. *Let $u_0 \in \dot{H}^{\frac{1}{2}}$. There exists a sufficiently small positive constant t_1 so that the system (GNS) with initial data u_0 admits a unique local solution u on $[0, t_1]$ and*

$$\|u\|_{Y_T} \le C e^{10^{-4}\lambda^2(T)} \|u_{\mathrm{h}}\|_{L_T^\infty(\dot{H}^{\frac{1}{2}})} \quad \text{for any } T \le t_1, \tag{3.16}$$

where $\lambda(T)$ is defined by (1.12) and u_{h} is defined by (3.17) below.

Proof. For simplicity, we just present the *a priori* estimates. We take $M \overset{\text{def}}{=} T^{-\frac{1}{4}}$ and split the solution u of (GNS) as

$$
\begin{aligned}
u = u_{\mathrm{l}} + u_{\mathrm{h}} \quad \text{with} \quad \widehat{u_{\mathrm{l}}} \overset{\text{def}}{=} \underbrace{\widehat{u} \cdot 1_{|\xi|<M}} \quad \text{and} \quad \widehat{u_{\mathrm{h}}} \overset{\text{def}}{=} \underbrace{\widehat{u} \cdot 1_{|\xi|\ge M}}, \\
\widehat{\mathcal{S}(u)}(t,\xi) \overset{\text{def}}{=} e^{-\frac{\lambda^2 t}{4T} + \lambda \frac{t}{\sqrt{T}}|\xi|} \widehat{u}(t,\xi).
\end{aligned}
\tag{3.17}
$$

We first observe from Proposition 3.1 that for all $0 < t \le T \le 1$:

$$
\begin{aligned}
t^{\frac{\delta}{2}} e^{\frac{\lambda^2 t}{4T}} \||D|^{\delta+\frac{1}{2}}\mathcal{S}(u_{\mathrm{l}})(t)\|_{L^2} &\lesssim t^{\frac{\delta}{2}} \||\xi|^{\delta+\frac{1}{2}} e^{\lambda\frac{t}{\sqrt{T}}|\xi|} 1_{|\xi|<T^{-\frac{1}{4}}} \widehat{u}(t,\xi)\|_{L_\xi^2} \\
&\lesssim t^{\frac{\delta}{2}} T^{-\frac{\delta}{4}} e^{\lambda T^{\frac{1}{4}}} \||\xi|^{\frac{1}{2}} \widehat{u}(t,\xi)\|_{L_\xi^2} \\
&\lesssim T^{\frac{\delta}{4}} e^{\lambda T^{\frac{1}{4}}} C(\|u_0\|_{\dot{H}^{\frac{1}{2}}}).
\end{aligned}
\tag{3.18}
$$

Note that due to our low frequency cut-off $1_{|\xi|\le T^{-\frac{1}{4}}}$, there is a saving of $T^{\frac{\delta}{4}}$ in the low frequency estimate (3.18).

In view of (3.15), the Y_T-norm of u is just $\|t^{\frac{\delta}{2}}|D|^{\delta+\frac{1}{2}}\mathcal{S}(u_{\mathrm{h}})\|_{L_T^\infty(L^2)}$. To estimate $\|u\|_{Y_T}$, we first notice that

$$
\begin{aligned}
&\left\|t^{\frac{\delta}{2}} 1_{M\le|\xi|\le 3M} |\xi|^{\delta+\frac{1}{2}} e^{-\frac{\lambda^2 t}{4T} + \lambda\frac{t}{\sqrt{T}}|\xi|} \widehat{u}\right\|_{L_T^\infty(L_\xi^2)} \\
&\le T^{\frac{\delta}{2}} M^\delta \left\|e^{-t\left(\frac{\lambda}{2\sqrt{T}}-|\xi|\right)^2} e^{9tM^2} |\xi|^{\frac{1}{2}} 1_{|\xi|\ge M} \widehat{u}\right\|_{L_T^\infty(L_\xi^2)} \\
&\lesssim T^{\frac{\delta}{4}} \|u_{\mathrm{h}}\|_{L_t^\infty(H^{\frac{1}{2}})}.
\end{aligned}
\tag{3.19}
$$

To complete the estimate of $\|u\|_{Y_T}$, it remains for us to estimate the term

$$\left\|1_{|\xi|\ge 3M} t^{\frac{\delta}{2}} |\xi|^{\delta+\frac{1}{2}} e^{-\frac{\lambda^2 t}{4T} + \lambda\frac{t}{\sqrt{T}}|\xi|} \widehat{u}\right\|_{L_T^\infty(L^2)}.$$

For this, as in the proof of Theorem 1.2, we shall turn to nonlinear estimates. Note that, thanks to the judicious frequency cut-off $1_{|\xi|\ge 3M}$, there will be no low-low interactions in the nonlinear estimate below.

Similar to the proof of Proposition 2.1, for sufficiently small $\eta_0 > 0$ (the needed smallness will become clear in the course of the proof), we split the integral $\int_0^t = \int_0^{\eta_0 t} + \int_{\eta_0 t}^t$ and estimate each pieces separately.

Step 1. The estimate of the short-time piece $\int_0^{\eta_0 t}$.

Recall that $N_1 = \lambda T^{-\frac{1}{2}}$. We write

$$\left\| t^{\frac{\delta}{2}} |\xi|^{\frac{1}{2}+\delta} 1_{|\xi|\geq 3M} \int_0^{\eta_0 t} e^{-(t-s)|\xi|^2} e^{\lambda \frac{t}{\sqrt{T}}|\xi| - \frac{\lambda^2 t}{4T}} \widehat{Q(u,u)}(s,\xi)\,ds \right\|_{L_T^\infty(L_\xi^2)}$$

$$\lesssim \left\| t^{\frac{\delta}{2}} 1_{|\xi|<2N_1} \int_0^{\eta_0 t} |\xi|^{\frac{3}{2}+\delta} e^{s|\xi|^2} 1_{|\xi|\geq 3M} \widehat{u\otimes u}(s,\xi)\,ds \right\|_{L_T^\infty(L_\xi^2)} \tag{3.20}$$

$$+ \left\| t^{\frac{\delta}{2}} 1_{|\xi|\geq 2N_1} \int_0^{\eta_0 t} |\xi|^{\frac{3}{2}+\delta} e^{-\frac{1}{10}t|\xi|^2} \widehat{u\otimes u}(s,\xi)\,ds \right\|_{L_T^\infty(L_\xi^2)}.$$

We first deduce from Hausdorff-Young's inequality and (2.9) that

$$t^{\frac{\delta}{2}} \int_0^{\eta_0 t} \left\| 1_{|\xi|<2N_1} |\xi|^{\frac{3}{2}+\delta} e^{s|\xi|^2} 1_{|\xi|\geq 3M} \widehat{u\otimes u}(s,\xi) \right\|_{L_\xi^2}\,ds$$

$$\lesssim t^{\frac{\delta}{2}} \int_0^{\eta_0 t} e^{4sN_1^2} N_1^{\frac{3}{2}+\delta} \left\| 1_{|\xi|\leq 2N_1} \right\|_{L_\xi^6} \left\| 1_{|\xi|\geq 3M} \widehat{u\otimes u}(s,\xi) \right\|_{L_\xi^3}\,ds$$

$$\lesssim t^{\frac{\delta}{2}} \int_0^{\eta_0 t} e^{4sN_1^2} N_1^{\frac{3}{2}+\delta} \cdot N_1^{\frac{1}{2}} \left(\|u_l \otimes u_h(s)\|_{L^{\frac{3}{2}}} + \|u_h \otimes u_h(s)\|_{L^{\frac{3}{2}}} \right)\,ds$$

$$\lesssim (N_1^2 T)^{1+\frac{\delta}{2}} e^{4\eta_0 T N_1^2} \|u\|_{L_T^\infty(L^3)} \|u_h\|_{L_T^\infty(L^3)}$$

$$\lesssim e^{5\eta_0 \lambda^2} \|u\|_{L_T^\infty(\dot{H}^{\frac{1}{2}})} \|u_h\|_{L_T^\infty(\dot{H}^{\frac{1}{2}})}.$$

Along the same line, we compute

$$t^{\frac{\delta}{2}} \left\| \int_0^{\eta_0 t} |\xi|^{\frac{3}{2}+\delta} 1_{|\xi|\geq 2N_1} e^{-\frac{t}{10}|\xi|^2} \widehat{u\otimes u}(s,\xi)\,ds \right\|_{L_\xi^2}$$

$$\lesssim t^{\frac{\delta}{2}} \int_0^{\eta_0 t} \left\| |\xi|^{\frac{3}{2}+\delta} e^{-\frac{t}{10}|\xi|^2} \right\|_{L_\xi^6} \left\| 1_{|\xi|\geq 2N_1} \widehat{u\otimes u}(s,\xi) \right\|_{L_\xi^3}\,ds$$

$$\lesssim t^{-1} \int_0^{\eta_0 t} \left(\|u_l \otimes u_h(s)\|_{L^{\frac{3}{2}}} + \|u_h \otimes u_h(s)\|_{L^{\frac{3}{2}}} \right) ds$$

$$\lesssim \|u\|_{L_T^\infty(\dot{H}^{\frac{1}{2}})} \|u_h\|_{L_T^\infty(\dot{H}^{\frac{1}{2}})}.$$

Plugging the above estimates into (3.20), we obtain

$$\left\| t^{\frac{\delta}{2}} |\xi|^{\frac{1}{2}+\delta} 1_{|\xi| \geq 3M} \int_0^{\eta_0 t} e^{-(t-s)|\xi|^2} e^{\lambda \frac{t}{\sqrt{T}} |\xi| - \frac{\lambda^2 t}{4T}} \widehat{Q(u,u)}(s,\xi)\, ds \right\|_{L_T^\infty(L_\xi^2)} \tag{3.21}$$

$$\lesssim e^{5\eta_0 \lambda^2} \|u\|_{L_T^\infty(\dot{H}^{\frac{1}{2}})} \|u_h\|_{L_T^\infty(\dot{H}^{\frac{1}{2}})}.$$

Step 2. The estimate of the piece $\int_{\eta_0 t}^t$.

In view of (1.1) and (3.17), we have

$$t^{\frac{\delta}{2}} \Big| |\xi|^{\frac{1}{2}+\delta} 1_{|\xi| \geq 3M} \int_{\eta_0 t}^t e^{-(t-s)|\xi|^2} e^{\lambda \frac{t}{\sqrt{T}} |\xi| - \frac{\lambda^2 t}{4T}} \widehat{Q(u,u)}(s,\xi)\, ds \Big|$$

$$\lesssim t^{\frac{\delta}{2}} |\xi|^{\frac{3}{2}+\delta} \int_{\eta_0 t}^t e^{-(t-s)|\xi|^2} e^{\lambda \frac{t-s}{\sqrt{T}} |\xi| + \frac{\lambda^2 s}{2T} - \frac{\lambda^2 t}{4T}} 1_{|\xi| \geq 3M} \Big| \widehat{\mathcal{S}(u) \otimes \mathcal{S}(u)}(s,\xi) \Big|\, ds. \tag{3.22}$$

By frequency localization (2.9), we find

$$1_{|\xi| \geq 3M} \widehat{\mathcal{S}(u) \otimes \mathcal{S}(u)} = 1_{|\xi| \geq 3M} \widehat{\mathcal{S}(u_l) \otimes \mathcal{S}(u_h)} + 1_{|\xi| \geq 3M} \widehat{\mathcal{S}(u_h) \otimes \mathcal{S}(u_l)}$$

$$+ 1_{|\xi| \geq 3M} \widehat{\mathcal{S}(u_h) \otimes \mathcal{S}(u_h)}.$$

We first handle the contribution due to $\mathcal{S}(u_h) \otimes \mathcal{S}(u_h)$. Indeed, it is easy to observe that

$$\left\| t^{\frac{\delta}{2}} |\xi|^{\frac{3}{2}+\delta} \int_{\eta_0 t}^t e^{-(t-s)|\xi|^2} e^{\lambda \frac{t-s}{\sqrt{T}} |\xi| + \frac{\lambda^2 s}{2T} - \frac{\lambda^2 t}{4T}} \widehat{\mathcal{S}(u_h) \otimes \mathcal{S}(u_h)}(s,\xi)\, ds \right\|_{L_T^\infty(L_\xi^2)}$$

$$\lesssim \left\| t^{\frac{\delta}{2}} \int_{\eta_0 t}^t |\xi|^{\frac{3}{2}+\delta} e^{-(t-s)\left(|\xi| - \frac{\lambda}{2\sqrt{T}}\right)^2 + \frac{\lambda^2 s}{4T}} \widehat{\mathcal{S}(u_h) \otimes \mathcal{S}(u_h)}(s,\xi)\, ds \right\|_{L_T^\infty(L_\xi^2)}$$

$$\lesssim \left\| t^{\frac{\delta}{2}} \int_{\eta_0 t}^t |\xi|^{\frac{3}{2}+\delta} 1_{|\xi| \leq N_1} e^{\frac{\lambda^2 s}{4T}} \widehat{\mathcal{S}(u_h) \otimes \mathcal{S}(u_h)}(s,\xi)\, ds \right\|_{L_T^\infty(L_\xi^2)}$$

$$+ \left\| t^{\frac{\delta}{2}} \int_{\eta_0 t}^t |\xi|^{\frac{3}{2}+\delta} 1_{|\xi| \geq N_1} e^{-\frac{1}{10}(t-s)|\xi|^2 + \frac{\lambda^2 s}{4T}} \widehat{\mathcal{S}(u_h) \otimes \mathcal{S}(u_h)}(s,\xi)\, ds \right\|_{L_T^\infty(L_\xi^2)}.$$

By Lemma 2.2 and noting $N_1 \geq 1$, we get

$$\left\| t^{\frac{\delta}{2}} |\xi|^{\frac{3}{2}+\delta} \int_{\eta_0 t}^{t} e^{-(t-s)|\xi|^2} e^{\lambda \frac{t-s}{\sqrt{T}} |\xi| + \frac{\lambda^2 s}{2T} - \frac{\lambda^2 t}{4T}} \widehat{\mathcal{S}(u_{\mathrm{h}}) \otimes \mathcal{S}(u_{\mathrm{h}})}(s, \xi) \, ds \right\|_{L_T^{\infty}(L_\xi^2)}$$

$$\lesssim \lambda^{-\delta} e^{\frac{\lambda^2}{4}} \cdot \sup_{0 < s \leq T} \left(s^{\delta} \| \mathcal{S}(u_{\mathrm{h}})(s) \mathcal{S}(u_{\mathrm{h}})(s) \|_{L_T^{\infty}(L^{\frac{3}{2(1-\delta)}})} \right)$$

$$\lesssim \lambda^{-\delta} e^{\frac{\lambda^2}{4}} \cdot \sup_{0 < s \leq T} \left(s^{\delta} \| |D|^{\frac{1}{2}+\delta} \mathcal{S}(u_{\mathrm{h}})(s) \|_{L_T^{\infty}(L^2)}^2 \right)$$

$$\lesssim \lambda^{-\delta} e^{\frac{\lambda^2}{4}} \| u \|_{Y_T}^2.$$

Note that in the last step above, we applied the Sobolev embedding $\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3) \hookrightarrow L^{\frac{3}{(1-\delta)}}(\mathbb{R}^3)$.

Similarly, we have

$$\left\| t^{\frac{\delta}{2}} |\xi|^{\frac{3}{2}+\delta} \int_{\eta_0 t}^{t} e^{-(t-s)|\xi|^2} e^{\lambda \frac{t-s}{\sqrt{T}} |\xi| + \frac{\lambda^2 s}{2T} - \frac{\lambda^2 t}{4T}} \widehat{\mathcal{S}(u_{\mathrm{l}}) \otimes \mathcal{S}(u_{\mathrm{h}})}(s, \xi) \, ds \right\|_{L_T^{\infty}(L_\xi^2)}$$

$$\lesssim \left\| t^{\frac{\delta}{2}} \int_{\eta_0 t}^{t} |\xi|^{\frac{3}{2}+\delta} e^{-(t-s)\left(|\xi| - \frac{\lambda}{2\sqrt{T}}\right)^2 + \frac{\lambda^2 s}{4T}} \widehat{\mathcal{S}(u_{\mathrm{l}}) \otimes \mathcal{S}(u_{\mathrm{h}})}(s, \xi) \, ds \right\|_{L_T^{\infty}(L_\xi^2)}$$

$$\lesssim \left\| t^{\frac{\delta}{2}} \int_{\eta_0 t}^{t} |\xi|^{\frac{3}{2}+\delta} \mathbf{1}_{|\xi| \leq N_1} e^{\frac{\lambda^2 s}{4T}} \widehat{\mathcal{S}(u_{\mathrm{l}}) \otimes \mathcal{S}(u_{\mathrm{h}})}(s, \xi) \, ds \right\|_{L_T^{\infty}(L_\xi^2)}$$

$$+ \left\| t^{\frac{\delta}{2}} \int_{\eta_0 t}^{t} |\xi|^{\frac{3}{2}+\delta} \mathbf{1}_{|\xi| \geq N_1} e^{-\frac{1}{10}(t-s)|\xi|^2 + \frac{\lambda^2 s}{4T}} \widehat{\mathcal{S}(u_{\mathrm{l}}) \otimes \mathcal{S}(u_{\mathrm{h}})}(s, \xi) \, ds \right\|_{L_T^{\infty}(L_\xi^2)}.$$

By Lemma 2.2 and a similar derivation as (2.16), we get

$$\left\| t^{\frac{\delta}{2}} |\xi|^{\frac{3}{2}+\delta} \int_{\eta_0 t}^{t} e^{-(t-s)|\xi|^2} e^{\lambda \frac{t-s}{\sqrt{T}} |\xi| + \frac{\lambda^2 s}{2T} - \frac{\lambda^2 t}{4T}} \widehat{\mathcal{S}(u_{\mathrm{l}}) \otimes \mathcal{S}(u_{\mathrm{h}})}(s, \xi) \, ds \right\|_{L_T^{\infty}(L_\xi^2)}$$

$$\lesssim \left(\lambda^{2-\delta} + 1 \right) \cdot \sup_{0 < s \leq T} \left(s^{\delta} e^{\frac{\lambda^2 s}{4T}} \| \mathcal{S}(u_{\mathrm{l}})(s) \mathcal{S}(u_{\mathrm{h}})(s) \|_{L_T^{\infty}(L^{\frac{3}{2(1-\delta)}})} \right)$$

$$\lesssim \lambda^{2-\delta} \cdot \sup_{0 < s \leq T} \left(s^{\frac{\delta}{2}} e^{\frac{\lambda^2 s}{4T}} \| |D|^{\frac{1}{2}+\delta} \mathcal{S}(u_{\mathrm{l}})(s) \|_{L^2} \cdot s^{\frac{\delta}{2}} \| |D|^{\frac{1}{2}+\delta} \mathcal{S}(u_{\mathrm{h}})(s) \|_{L^2} \right)$$

$$\lesssim \lambda^{2-\delta} T^{\frac{\delta}{4}} e^{\lambda T^{\frac{1}{4}}} \| u_0 \|_{\dot{H}^{\frac{1}{2}}} \| u \|_{Y_T}, \qquad \text{(by (3.18))}.$$

Plugging the above estimates into (3.22), we obtain

$$
\left\| t^{\frac{\delta}{2}} |\xi|^{\frac{1}{2}+\delta} 1_{|\xi|\geq 3M} \int_{\eta_0 t}^{t} e^{-(t-s)|\xi|^2} e^{\lambda \frac{t}{\sqrt{T}}|\xi| - \frac{\lambda^2 t}{4T}} \widehat{Q(u,u)}(s,\xi)\,ds \right\|_{L_T^\infty(L_\xi^2)}
$$

$$
\lesssim \lambda^{-\delta} e^{\frac{\lambda^2}{4}} \|u\|_{Y_T}^2 + \lambda^{2-\delta} T^{\frac{\delta}{4}} e^{\lambda T^{\frac{1}{4}}} \|u_0\|_{\dot H^{\frac{1}{2}}} \|u\|_{Y_T}.
\tag{3.23}
$$

Step 3. The estimate of $\|u\|_{Y_T}$.

We now turn to the estimate of (3.16). Recall the definitions (1.10) and (3.15). Collecting the estimates (3.19), (3.21) and (3.23), we get

$$
\|u\|_{Y_T} \leq \|e^{t\Delta} u_0\|_{Y_T} + C_1 \big(T^{\frac{\delta}{4}} \|u_{\mathrm h}\|_{L_t^\infty(\dot H^{\frac{1}{2}})} + e^{5\eta_0 \lambda^2} \|u_0\|_{\dot H^{\frac{1}{2}}} \|u_{\mathrm h}\|_{L_t^\infty(\dot H^{\frac{1}{2}})}
$$

$$
+ \lambda^{-\delta} e^{\frac{\lambda^2}{4}} \|u\|_{Y_T}^2 + T^{\frac{\delta}{4}} \lambda^{2-\delta} e^{\lambda T^{\frac{1}{4}}} \|u_0\|_{\dot H^{\frac{1}{2}}} \|u\|_{Y_T} \big).
\tag{3.24}
$$

To see the smallness of the linear term $\|e^{t\Delta} u_0\|_{Y_T}$, we observe that for all $0 < t \leq T$:

$$
\|e^{t\Delta} u_0\|_{Y_T} = \left\| t^{\frac{\delta}{2}} |\xi|^{\frac{1}{2}+\delta} 1_{|\xi| \geq T^{-\frac{1}{4}}} e^{-\frac{\lambda^2 t}{4T} + \lambda \frac{t}{\sqrt{T}}|\xi|} e^{-t|\xi|^2} \widehat{u_0}(\xi) \right\|_{L_T^\infty(L_\xi^2)}
$$

$$
\lesssim \left\| 1_{|\xi| \geq 2N_1} |\xi|^{\frac{1}{2}} t^{\frac{\delta}{2}} |\xi|^\delta e^{-\frac{1}{10} t |\xi|^2} \widehat{u_0}(\xi) \right\|_{L_T^\infty(L_\xi^2)}
$$

$$
+ \left\| t^{\frac{\delta}{2}} |\xi|^{\frac{1}{2}+\delta} 1_{|\xi| \leq 2N_1} 1_{|\xi| \geq T^{-\frac{1}{4}}} \widehat{u_0}(\xi) \right\|_{L_T^\infty(L_\xi^2)}
$$

$$
\lesssim \|u_{\mathrm h}\|_{L_t^\infty(\dot H^{\frac{1}{2}})} + \lambda^\delta \|u_{\mathrm h}\|_{L_t^\infty(\dot H^{\frac{1}{2}})} \leq C_1 \lambda^\delta \|u_{\mathrm h}\|_{L_T^\infty(\dot H^{\frac{1}{2}})}.
$$

Now take $\lambda = \lambda(T)$ as in (1.12). Choose $\mathcal{T} > 0$ sufficiently small such that

$$
C_1 \|u_0\|_{H^{\frac{1}{2}}} T^{\frac{\delta}{4}} \cdot \lambda^{2-\delta}(T) e^{\lambda(T) T^{\frac{1}{4}}} \leq \frac{1}{2}, \qquad \forall\, 0 < T \leq \mathcal{T}.
\tag{3.25}
$$

By (3.24), we have for any $T \leq \mathcal{T}$,

$$
\|u\|_{Y_T} \leq C_2 \big(e^{5\eta_0 \lambda^2(T)} \|u_{\mathrm h}\|_{L_T^\infty(\dot H^{\frac{1}{2}})} + \lambda^{-\delta}(T) e^{\frac{\lambda^2(T)}{4}} \|u\|_{Y_T}^2 \big).
\tag{3.26}
$$

We further shrink $0 < t_1 \leq \mathcal{T}$ to be so small that

$$
4 C_2^2 \|u_{\mathrm h}\|_{L_T^\infty(\dot H^{\frac{1}{2}})} e^{5\eta_0 \lambda^2(T)} \cdot \lambda^{-\delta}(T) e^{\frac{\lambda^2(T)}{4}} \leq \frac{1}{2}, \qquad \forall\, 0 < T \leq t_1.
\tag{3.27}
$$

Note that, thanks to the definition of $\lambda(T)$ given by (1.12) and (3.2), the existence of t_1 is not an issue.

It follows from (3.26) and (3.27) that for any $T \leq t_1$,

$$
\|u\|_{Y_T} \leq 2 C_2 e^{5\eta_0 \lambda^2(T)} \|u_{\mathrm h}\|_{L_T^\infty(\dot H^{\frac{1}{2}})}.
\tag{3.28}
$$

Taking $\eta_0 = 10^{-5}$, we conclude the proof of (3.16) and also the proof of Proposition 3.2. $\quad\square$

We now complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Again for simplicity, we just present the *a priori* estimates. In view of (3.17), we get, by summing up (3.16) and (3.18), that for any $T \le t_1$,

$$
\begin{aligned}
\left\| t^{\frac{\delta}{2}} \mathcal{S}(u) \right\|_{L_T^\infty(\dot{H}^{\frac{1}{2}+\delta})} \le & \left\| t^{\frac{\delta}{2}} \mathcal{S}(u_{\mathrm{l}}) \right\|_{L_T^\infty(\dot{H}^{\frac{1}{2}+\delta})} + \|u\|_{Y_T} \\
\le & C(\|u_0\|_{\dot{H}^{\frac{1}{2}}}) T^{\frac{\delta}{4}} e^{-\frac{\lambda^2(T)}{4}} + C e^{10^{-4}\lambda^2(T)} \|u_{\mathrm{h}}\|_{L_T^\infty(\dot{H}^{\frac{1}{2}})}.
\end{aligned}
$$

Taking $t = T$, we get for $T \le t_1$ (as determined by Proposition 3.2) that

$$
T^{\frac{\delta}{2}} e^{-\frac{\lambda^2(T)}{4}} \left\| e^{\lambda(T)\sqrt{T}|D|} u(T) \right\|_{\dot{H}^{\frac{1}{2}+\delta}} \le C(\|u_0\|_{\dot{H}^{\frac{1}{2}}}) T^{\frac{\delta}{4}} e^{-\frac{\lambda^2(T)}{4}} + C e^{10^{-4}\lambda^2(T)} \|u_{\mathrm{h}}\|_{L_T^\infty(\dot{H}^{\frac{1}{2}})}.
$$

This readily leads to (1.13). We thus complete the proof of Theorem 1.3. $\quad\square$

Acknowledgments

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