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On the Euler–Poincaré Equation with Non-Zero Dispersion

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Abstract

We consider the Euler–Poincaré equation on $\mathbb{R}^d$, $d \geqq 2$. For a large class of smooth initial data we prove that the corresponding solution blows up in finite time. This settles an open problem raised by Chae and Liu (Commun Math Phys 314:671–687, 2012). Our analysis exhibits some new concentration mechanisms and hidden monotonicity formulas associated with the Euler–Poincaré flow. In particular we show an abundance of blowups emanating from smooth initial data with certain sign properties. No size restrictions are imposed on the data. We also showcase a class of initial data for which the corresponding solution exists globally in time.

1. Introduction

We consider the following Euler–Poincaré equation on $\mathbb{R}^d$, $d \geqq 2$:

$$\begin{cases} \partial_t m + (u \cdot \nabla)m + (\nabla u)^T m + (\mathrm{div} u)m = 0, \quad t > 0, \ x \in \mathbb{R}^d; \\ m = (1 - \alpha\Delta)u; \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

Here, $u = (u_1, \ldots, u_d) : \mathbb{R}^d \to \mathbb{R}^d$ represents the velocity and $m = (m_1, \ldots, m_d) : \mathbb{R}^d \to \mathbb{R}^d$ denotes the momentum. The parameter $\alpha > 0$ in the second equation of (1.1) corresponds to the square of the length scale; it is sometimes called the dispersion parameter in the literature. The notation $(\nabla u)^{\mathrm{T}}$ denotes the transpose of the matrix ∇u. To avoid any confusion it is useful to recast equation (1.1) in the component-wise form as

$$\partial_t m_i + u_j \partial_j m_i + (\partial_i u_j) m_j + (\partial_j u_j) m_i = 0. \tag{1.2}$$

Here and throughout the rest of this paper we shall use the Einstein summation convention. By using tensor notation, one can combine the second and the last term in (1.2) and write it more compactly as

$$\partial_t m + \nabla \cdot (m \otimes u) + (\nabla u)^T m = 0. \tag{1.3}$$

The last term in (1.3) is not in conservative form. Following Chae and Liu [1] (see formula (1)–(4) on page 673 therein), one can introduce a stress-tensor T_{ij}

$$T_{ij} = m_i u_j + \frac{1}{2}\delta_{ij}|u|^2 - \alpha \partial_i u \cdot \partial_j u + \frac{1}{2}\alpha \delta_{ij}|\nabla u|^2$$

and rewrite (1.3) as

$$\partial_t m_i + \partial_j T_{ij} = 0. \tag{1.4}$$

By the second equation in (1.1), we have

$$\begin{aligned} m_i u_j &= u_i u_j - \alpha(\partial_{kk} u_i) u_j \\ &= u_i u_j - \alpha \partial_k ((\partial_k u_i) u_j) + \alpha(\partial_k u_i \partial_k u_j). \end{aligned}$$

Therefore, the tensor T_{ij} can be rewritten as

$$\begin{aligned} T_{ij} = u_i u_j &- \alpha \partial_k ((\partial_k u_i) u_j) + \alpha(\partial_k u_i \partial_k u_j) \\ &+ \frac{1}{2}\delta_{ij}|u|^2 - \alpha \partial_i u \cdot \partial_j u + \frac{1}{2}\alpha \delta_{ij}|\nabla u|^2. \end{aligned} \tag{1.5}$$

Roughly speaking, the above expressions show that the tensor T is of the form

$$T = O(|u|^2 + |\partial u|^2 + \partial(u \partial u)).$$

Such a decomposition is very useful in deriving low frequency L^p estimates later (see Proposition 1.1). For smooth solutions with enough spatial decay, there are two natural conservation laws,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} m \mathrm{d}x = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} (|u|^2 + \alpha|\nabla u|^2)\,\mathrm{d}x = 0. \end{aligned} \tag{1.6}$$

We shall need only the second one for later constructions.

The Euler–Poincaré equations were first introduced by Holm, Marsden, and Ratiu in [4,5]. In one dimension ($d = 1$) the Euler–Poincaré equations reduce to the Camassa–Holm equations of the form

$$\partial_t m + u \partial_x m + 2\partial_x u m = 0, \ \ m = (1 - \alpha \partial_{xx}) u.$$

The well-posedness of local and global weak solutions of Camassa-Holm equations have been intensively studied (see [8] and references therein). In two dimensions, the Euler–Poincaré equation is known as the averaged template matching equation in the computer vision literature [2,3,6]. For the applications of Euler–Poincaré equations in computational anatomy, see [7,9]. The rigorous analysis of the Euler–Poincaré equations was initiated by Chae and Liu [1] who established a fairly complete wellposedness theory for both weak and strong solutions. We summarize some of their main results (relevant to our context) as follows [here α is the dispersion parameter in the second equation of (1.1)]:

- Let $\alpha \geqq 0$ and $u_0 \in H^k(\mathbb{R}^d)$ with $k > \frac{d}{2} + 3$.[1] Then, there exist $T = T(\|u_0\|_{H^k}) > 0$ and a unique classical solution $u = u(x,t)$ to (1.1) in the space $C([0,T), H^k(\mathbb{R}^d))$.
- Let $0 < T^* \leqq +\infty$ be the maximal lifespan corresponding to the solution $u \in C_t^0 H^k$. If $T^* < \infty$, then

$$\limsup_{t \longrightarrow T^*} \|u(t)\|_{H^k} = \infty \Longleftrightarrow \int_0^{T^*} \|S(t)\|_{\dot{B}^0_{\infty,\infty}} \,\mathrm{d}t = \infty. \tag{1.7}$$

  Here, $S = (S_{ij})$ is the deformation tensor of u with $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. See (1.17) for the definition of the homogeneous Besov norm $\|\cdot\|_{\dot{B}^0_{\infty,\infty}}$.
- Let $\alpha = 0$. Let $u_0 \in H^k(\mathbb{R}^d)$, $k > \frac{d}{2} + 2$, and let it have reflection symmetry with respect to the origin, that is,

$$u_0(x) = -u_0(-x), \quad \forall\, x \in \mathbb{R}^d.$$

  If $\mathrm{div} u_0(0) < 0$, then the corresponding classical solution blows up in finite time.

Note that the Chae–Liu blowup result stated above is valid only for $\alpha = 0$, in which case the Euler–Poincaré equation reduces to a version of a high-dimensional Burgers system. The main idea of Chae–Liu is to consider the evolution of $\mathrm{div} u$ at the origin. Namely, by using reflection symmetry and (1.2), one obtains

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{div} u(0,t)) &= -2\left(\sum_{i,j=1}^{d} (\partial_i u_j + \partial_j u_i)\right)^2 - (\mathrm{div} u(0,t))^2 \\ &\leqq -(\mathrm{div} u(0,t))^2, \end{aligned}$$

and blowup follows from the assumption $\mathrm{div} u_0(0) < 0$. Unfortunately, this elegant argument does not work for the non-degenerate case $\alpha > 0$ due to some extra high order terms which do not enjoy any monotonicity properties. Thus Chae and Liu raised the following

Problem. *For the Euler–Poincaré system* (1.1) ($\alpha > 0$)*, do there exist finite time blowups from smooth initial data?*

[1] For $\alpha = 0$ one only needs $k > \frac{d}{2} + 1$, since the corresponding system is a symmetric hyperbolic system of conservation laws with a convex entropy, see Theorem 1 in [1] for more details.

The main purpose of this paper is to settle the above problem in the affirmative. Since we are mainly interested in the case $\alpha > 0$, the actual value of α will play no role in our analysis, so we shall simply set $\alpha = 1$ throughout the rest of this paper. We start by considering a special class of *radial* flows invariant under the Euler–Poincaré dynamics. More precisely, let $m = \nabla\phi$ where ϕ is a radial scalar-valued function.[2] By (1.2), and noting that $\partial_j m_i = \partial_i m_j$ for any i, j, we have

$$\begin{aligned} -\partial_t m_i &= m_i \partial_j u_j + u_j \partial_j m_i + \partial_i u_j m_j \\ &= m_i \partial_j u_j + u_j \partial_i m_j + \partial_i u_j m_j \\ &= m_i (\nabla \cdot u) + \partial_i (m \cdot u). \end{aligned} \tag{1.8}$$

Therefore, the radial function ϕ satisfies

$$\begin{aligned} -\partial_t \phi'(r,t) = &-\phi(r,t)\phi'(r,t) + \phi'(r,t)((1-\Delta)^{-1}\phi)(r,t) \\ &+ \Big((1-\Delta)^{-1}\nabla\phi \cdot \nabla\phi\Big)', \quad r = |x| > 0, \end{aligned} \tag{1.9}$$

with initial data $\phi(r,0) = \phi_0(r)$. Here and throughout the rest of this paper, we will slightly abuse the notation and denote any radial function f on $\mathbb{R}^d$ as $f(x) = f(|x|) = f(r)$ whenever there is no confusion. We also use the notation $f' = f'(r)$ to denote the radial derivative. Assuming ϕ (and its derivatives) decays sufficiently fast at infinity, we may integrate (1.9) on the slab $[r, \infty)$ and obtain

$$\begin{aligned} \partial_t \phi(r,t) = &\frac{1}{2}\phi(r,t)^2 + \int_r^\infty \phi'(s,t)((1-\Delta)^{-1}\phi)(s,t)\,\mathrm{d}s \\ &- \Big((1-\Delta)^{-1}\nabla\phi \cdot \nabla\phi\Big)(r,t). \end{aligned} \tag{1.10}$$

At the cost of a nonlocal integration, equation (1.10) greatly simplifies the analysis and will be our main object of study in this paper. We begin with a simple proposition that in some sense justifies the validity of the equation (1.10). To allow some generality, we also include the result for dimension $d = 1$ which is needed for some results in Section 2 (see Theorem 2.1).

Proposition 1.1. *Let the dimension $d \geqq 1$. Assume initially $m_0 = \nabla\phi_0$, where ϕ_0 is a radial function on $\mathbb{R}^d$ and $\phi_0 \in H^k$ for some $k > \frac{d}{2} + 2$. If $d = 1, 2$, we also assume that $\phi_0 \in \dot{B}^0_{1,\infty}(\mathbb{R}^d)$. Then for any $t > 0$, the solution $m(t) = (1-\Delta)u(t)$ can be written as $m(t) = \nabla\phi(t)$, where $\phi(t)$ is radial*[3] *and $\phi(t) \in H^k(\mathbb{R}^d)$ for $d \geqq 3$, $\phi(t) \in H^k(\mathbb{R}^d) \cap \dot{B}^0_{1,\infty}(\mathbb{R}^d)$ for $d = 1, 2$. Each $\phi(t,r)$ solves (1.10) in the classical sense. Moreover, we have the growth estimate*

[2] By using the derivation below, it is not difficult to check that if initially $m_0 = \nabla\phi_0$ and ϕ_0 is a smooth radial function, then for any $t > 0$ we can write $m(t) = \nabla\phi(t)$ with $\phi(t)$ being radial and smooth, as well. The radial assumption here is essential. In the general case one cannot expect that irrotational flows are preserved in time.

[3] For $d = 1$, ϕ is radial means that ϕ is an even function on $\mathbb{R}$.

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leqq B_0 \cdot (1+t)^{\frac{2}{3}}, \ \forall t \geqq 0, \ \text{if } d=1, \tag{1.11}$$

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leqq B_1 \cdot (1+t)^{\frac{1}{2}}, \ \forall t \geqq 0, \ \text{if } d=2, \tag{1.12}$$

$$\|\phi(t,\cdot)\|_{L^2(\mathbb{R}^d)} \leqq B_2 \cdot (1+t), \ \forall t \geqq 0, \ \text{if } d \geqq 3. \tag{1.13}$$

Here, $B_0 > 0$, $B_1 > 0$, $B_2 > 0$ are some constants depending only on the initial data ϕ_0.

With Proposition 1.1 in hand, we can control the low frequency part of the solution and express the blowup/continuation (1.7) in terms of the scalar function ϕ alone. Thus

Lemma 1.2. *Let ϕ_0 be radial. If $d \geqq 3$, we assume that $\phi_0 \in H^k(\mathbb{R}^d)$ for some $k > \frac{d}{2}+2$. If $d=2$, we assume that $\phi_0 \in H^k(\mathbb{R}^2) \cap \dot{B}^0_{1,\infty}(\mathbb{R}^2)$ for some $k \geqq 4$. Let u be the maximal lifespan solution corresponding to initial data $u_0 = (1-\Delta)^{-1} m_0 = (1-\Delta)^{-1}\nabla\phi_0$. If the maximal lifespan $T^* < \infty$, then*

$$\limsup_{t \longrightarrow T^*} \|u(t)\|_{H^k} = \infty \Longleftrightarrow \int_0^{T^*} \|\phi(t)\|_{L^\infty(\mathbb{R}^d)}\, \mathrm{d}t = \infty.$$

We shall omit the proof of Lemma 1.2 since it follows directly from (1.7), (1.12)–(1.13), and the embedding $L^\infty \hookrightarrow \dot{B}^0_{\infty,\infty}$.

We now state our main results. Apart from regularity assumptions, the first result says that if initially $\phi_0(0) \geqq 0$, then the corresponding solution blows up in finite time. It is a bit surprising in that such a local condition dictates the whole nonlocal Euler–Poincaré dynamics.

Theorem 1.3. *Let the dimension $d \geqq 2$. Let ϕ_0 be a radial real-valued function on $\mathbb{R}^d$ such that $\phi_0 \in H^k(\mathbb{R}^d)$ for some $k > \frac{d}{2} + 2$. For $d = 2$ we also assume that $\phi_0 \in \dot{B}^0_{1,\infty}(\mathbb{R}^2)$. Let the initial velocity $u_0 = (1-\Delta)^{-1} m_0 = (1-\Delta)^{-1}\nabla\phi_0$. If $\phi_0(0) \geqq 0$ and ϕ_0 is not identically zero, then the corresponding solution blows up in finite time.*

The next result deals with the opposite scenario, $\phi_0(0) < 0$. Under the assumption that $\phi_0(r)$ is monotonically increasing, we show that the corresponding solution exists globally in time. In some sense it reveals the nonlinear depletion mechanism hidden in the Euler–Poincaré dynamics.

Theorem 1.4. (Global regularity for a class of non-positive monotone data) *Let the dimension $d \geqq 2$. Let ϕ_0 be a radial real-valued function on $\mathbb{R}^d$ such that $\phi_0 \in H^k(\mathbb{R}^d)$ for some $k > \frac{d}{2}+2$. For $d = 2$ we also assume that $\phi_0 \in \dot{B}^0_{1,\infty}(\mathbb{R}^2)$. If $\phi_0(0) \leqq 0$ and ϕ_0 is monotonically increasing on $[0,\infty)$ (that is, $\phi_0'(r) \geqq 0$ for any $0 \leqq r < \infty$), then the corresponding solution $u(t) = (1-\Delta)^{-1}\nabla\phi(t)$ exists globally in time. Moreover, for any $t > 0$, $\phi(t,\cdot)$ is monotonically increasing on $[0,\infty)$.*

We have the following corollary, which computes the asymptotics of $\phi(0,t)$ as $t \to \infty$. To allow some generality we state it as a conditional result in that we assume that the corresponding solution exists globally in time.

Corollary 1.5. (Asymptotics of $\phi(0,t)$) *Let the dimension $d \geqq 2$. Let ϕ_0 be a radial real-valued function on $\mathbb{R}^d$ such that $\phi_0 \in H^k(\mathbb{R}^d)$ for some $k > \frac{d}{2} + 2$. For $d = 2$ we also assume that $\phi_0 \in \dot{B}^0_{1,\infty}(\mathbb{R}^2)$. Assume that $\phi_0(0) < 0$, $u_0 = (1-\Delta)^{-1}\nabla\phi_0$ and the corresponding solution $u(t) = (1-\Delta)^{-1}\nabla\phi(t)$ exists globally on $[0,\infty)$. Then $\phi(0,t)$ is strictly monotonically increasing in t and $\frac{d}{dt}\phi(0,t) > 0$ for any $t \geqq 0$. There are some constants $C_1 > 0$, $C_2 > 0$ such that for $d \geqq 3$*

$$0 < -\phi(0,t) < \frac{C_1}{1+t}, \quad \forall t > 0; \tag{1.14}$$

and for $d = 2$

$$0 < -\phi(0,t) < \frac{C_2}{\log(10+t)}, \quad \forall t > 0. \tag{1.15}$$

In particular, $\lim_{t\to\infty}\phi(0,t) = 0$.

Remark 1.6. The decay rates in (1.14)–(1.15) are probably not optimal. It is an interesting question to study the long time behavior of global solutions to such systems with no damping or dissipation.

It is very tempting to conjecture that the single condition $\phi_0(0) < 0$ may yield global wellposedness. Our last result rules out this possibility. We exhibit a family of smooth *negative* initial data for which the corresponding solution blows up in finite time. In particular, the initial data ϕ_0 will satisfy $\phi_0(0) < 0$.

Theorem 1.7. *There exists a family $\mathcal{A}$ of smooth initial data such that the following hold:*

- *For each $\phi_0 \in \mathcal{A}$, we have $\phi_0(x) < 0$ for any $x \in \mathbb{R}^d$.*
- *The corresponding solution $u(t) = (1-\Delta)^{-1}\nabla\phi(t)$ blows up in finite time. Moreover, $\phi(0,t)$ is a monotonically increasing function of t for each t within the lifespan of the solution.*

We conclude the introduction by setting up some

Notations. For any two quantities X and Y, we shall write $X \lesssim Y$ if $X \leqq CY$ for some harmless constant $C > 0$. Similarly, we define $X \gtrsim Y$. We write $X \sim Y$ if both $X \lesssim Y$ and $X \gtrsim Y$ hold.

We will need to use the Littlewood–Paley frequency projection operators. Let $\varphi(\xi)$ be a smooth bump function supported in the ball $|\xi| \leqq 2$ and equal to one on the ball $|\xi| \leqq 1$. For each dyadic number $N \in 2^{\mathbb{Z}}$, we define the Littlewood–Paley operators

$$\begin{aligned}
\widehat{P_{\leq N} f}(\xi) &:= \varphi(\xi/N)\hat{f}(\xi), \\
\widehat{P_{>N} f}(\xi) &:= [1-\varphi(\xi/N)]\hat{f}(\xi), \\
\widehat{P_N f}(\xi) &:= [\varphi(\xi/N) - \varphi(2\xi/N)]\hat{f}(\xi).
\end{aligned} \tag{1.16}$$

Similarly, we can define $P_{<N}$, $P_{\geqq N}$, and $P_{M<\cdot\leqq N} := P_{\leqq N} - P_{\leqq M}$, whenever M and N are dyadic numbers.

We recall the following standard Bernstein inequality: for any $1 \leqq p < q \leqq \infty$,

$$\|P_N f\|_{L^q(\mathbb{R}^d)} \lesssim N^{d\left(\frac{1}{p}-\frac{1}{q}\right)} \|P_N f\|_{L^p(\mathbb{R}^d)}.$$

Here, P_N can be replaced by $P_{<N}$ or $P_{\leqq N}$.

For any $1 \leqq p \leqq \infty$, the homogeneous Besov norm $\dot{B}^0_{p,\infty}$ is defined as

$$\|f\|_{\dot{B}^0_{p,\infty}} = \sup_{M\in 2^{\mathbb{Z}}} \|P_M f\|_{L^p(\mathbb{R}^d)}. \tag{1.17}$$

We need the following interpolation inequalities on $\mathbb{R}$ and $\mathbb{R}^2$:

$$\|f\|_{L^2(\mathbb{R})} \lesssim \|f\|^{\frac{2}{3}}_{\dot{B}^0_{1,\infty}(\mathbb{R})} \cdot \|\nabla f\|^{\frac{1}{3}}_{L^2(\mathbb{R})}, \tag{1.18}$$

$$\|f\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|^{\frac{1}{2}}_{\dot{B}^0_{1,\infty}(\mathbb{R}^2)} \cdot \|\nabla f\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}. \tag{1.19}$$

The proof of (1.18)–(1.19) is a standard exercise in Littlewood–Paley calculus. We sketch the proof of the second inequality (1.19) here for the sake of completeness. The first one is proved similarly.

Proof of (1.19). Let $N_0 > 0$ be a dyadic number whose value will be chosen later. Then by Bernstein, we have

$$\begin{aligned}\|f\|^2_{L^2(\mathbb{R}^2)} &\lesssim \sum_{N<N_0} N^2 \|P_N f\|^2_{L^1(\mathbb{R}^2)} + \sum_{N\geqq N_0} N^{-2} \cdot \|\nabla P_N f\|^2_{L^2(\mathbb{R}^2)} \\ &\lesssim N_0^2 \|f\|^2_{\dot{B}^0_{1,\infty}(\mathbb{R}^2)} + N_0^{-2} \|\nabla f\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Choosing $N_0 \sim \frac{\|\nabla f\|_2}{\|f\|_{\dot{B}^0_{1,\infty}}}$ then yields the result. □

2. Proof of Proposition 1.1 and Some Intermediate Results

In this section we first give the proof of Proposition 1.1. After that, we shall deduce several weak blowup results, some of which have certain concentration and/or size restrictions on the initial data. However, the proofs of these results are somewhat simpler and they serve to illustrate the main difficulties in proving the sharp result Theorem 1.3.

Proof of Proposition 1.1. Since $m = (1-\Delta)u = \nabla\phi$, we have $u = (1-\Delta)^{-1}\nabla\phi$. By using (1.6), we obtain

$$\|(1-\Delta)^{-1}\nabla\phi(t)\|_2 + \sum_{i,j=1}^{d} \|(1-\Delta)^{-1}\partial_i\partial_j\phi(t)\|_2 \lesssim 1, \quad \forall t \geqq 0. \tag{2.20}$$

From (2.20), we have

$$\|P_{\geqq 1}\phi(t)\|_2 \lesssim 1, \quad \forall t \geqq 0.$$

By using the local theory worked out in [1], we have control of $\|u(t)\|_{H^k}$. Since $u = (1-\Delta)^{-1}\nabla\phi$, by Bernstein we have $\|P_{\geqq 1}\phi\|_2 \lesssim \|\nabla u\|_2$. Then we only need to estimate $\|P_{<1}\phi(t)\|_2$. By (1.4), we have

$$m_i(t) = m_i(0) - \sum_{j=1}^{d} \int_0^t (\partial_j T_{ij})(\tau)\,\mathrm{d}\tau.$$

Therefore, by (1.5) and Bernstein,

$$\begin{aligned}
\|P_{<1}\phi(t)\|_2 &\lesssim \|P_{<1}\Delta^{-1}\nabla\cdot m(t)\|_2 \\
&\lesssim \|\phi(0)\|_2 + \sum_{j=1}^{d}\int_0^t \|P_{<1}\Delta^{-1}\partial_i\partial_j T_{ij}(\tau)\|_2\,\mathrm{d}\tau \\
&\lesssim \|\phi(0)\|_2 + \int_0^t \left(\|u(\tau)\|_2^2 + \|\nabla u(\tau)\|_2^2\right)\mathrm{d}\tau \\
&\lesssim \|\phi(0)\|_2 + C_1 t, \quad \forall t \geqq 0, \qquad (2.21)
\end{aligned}$$

where $C_1 > 0$ depends on $\|u_0\|_{H^1}$, and we have used the conservation law (1.6). Note that the derivation here is valid for all dimensions $d \geqq 1$. Hence, the estimate (1.13) follows.

Similarly, by using the fact that

$$\sup_{N\in 2^{\mathbb{Z}}} \|P_N \Delta^{-1}\partial_i\partial_j\|_{L^1\to L^1} < \infty,$$

we obtain in the case $d = 1, 2$,

$$\|P_{<1}\phi(t)\|_{\dot{B}^0_{1,\infty}(\mathbb{R}^d)} \leqq C_2(1+t), \quad \forall t \geqq 0,$$

where $C_2 > 0$ depends only on ϕ_0. The growth estimates (1.11)–(1.12) then follows from the above estimate, the conservation law $\|\nabla P_{<1}\phi\|_2 \lesssim \|P_{<1}(1-\Delta)u\|_2 \lesssim \|u\|_2 \lesssim 1$, and the interpolation inequalities (1.18)–(1.19) (applied to $f = P_{<1}\phi$).

Finally, we need to justify (1.10). In particular, we need to show that the integral $\int_r^\infty (\phi')(s,t)((1-\Delta)^{-1}\phi)(s,t)\,\mathrm{d}s$ converges. Indeed, this follows from the estimate

$$\begin{aligned}
\int_r^\infty |\phi'|\cdot|(1-\Delta)^{-1}\phi|\,\mathrm{d}s &\lesssim \left\| \frac{|\nabla\phi|\cdot(1-\Delta)^{-1}\phi}{|x|^{d-1}} \right\|_{L^1(\mathbb{R}^d)} \\
&\lesssim \|\nabla\phi\|_\infty \cdot \|(1-\Delta)^{-1}\phi\|_\infty + \|\nabla\phi\|_2\cdot\|(1-\Delta)^{-1}\phi\|_2 \\
&< \infty.
\end{aligned}$$

Since $\phi \in H^k$, ϕ is a smooth function. Since the above integral converges, it follows that (1.10) holds in the classical sense. □

We now formulate a simple blowup result, which requires three rather restrictive conditions on the initial ϕ_0: positivity, monotonicity and sufficient concentration at the spatial origin. Due to these simplifying assumptions, the proof is much simpler compared to that of our main theorem 1.3 in later sections. Note that the case dimension $d = 1$ is covered here which cannot be handled by Theorem 1.3.

Theorem 2.1. *Let the dimension $d \geqq 1$. Assume that ϕ_0 is a radial*[4] *real-valued function on $\mathbb{R}^d$ and $\phi_0 \in H^k(\mathbb{R}^d)$ for some $k > \frac{d}{2} + 2$. Assume that $\phi_0'(r) \leqq 0$ for any $r > 0$ and ϕ_0 is not identically zero. There exists a constant $C > 0$ such that if*

$$\phi_0(0) \geqq C\|\phi_0\|_{L_x^2(\mathbb{R}^d)}$$

and the initial velocity $u_0 = (1-\Delta)^{-1}m_0 = (1-\Delta)^{-1}\nabla\phi_0$, then the corresponding solution blows up in finite time.

Proof of Theorem 2.1. Note that by assumption we have that ϕ_0 attains its global maximum at $r = 0$ and $\phi_0(0) > 0$. We first show that for any $t > 0$ within the lifespan of the solution, we have $\phi'(r,t) \leqq 0$ for any $r > 0$. Indeed, by (1.9), we have

$$\begin{aligned} -\partial_t\phi'(r,t) &= -\phi(r,t)\phi'(r,t) + \phi'(r,t)((1-\Delta)^{-1}\phi)(r,t) \\ &\quad + \Big(((1-\Delta)^{-1}\phi)' \cdot \phi'\Big)'. \end{aligned}$$

Set $g(r,t) = \phi'(r,t)$, then by using the above equation and grouping the coefficients, we see that

$$\partial_t g(r,t) + a_1(r,t)g(r,t) + a_2(r,t)\partial_r g(r,t) = 0, \quad \forall r \geqq 0,$$

where a_1, a_2 are some smooth functions. Since $g(r,0) = \phi_0'(r) \leqq 0$, a simple method of characteristics argument then yields immediately that $g(r,t) \leqq 0$, $\forall r \geqq 0$. Hence $\phi'(r,t) \leqq 0$, for any $r \geqq 0$.

Now set $r = 0$ in (1.10), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) = \frac{1}{2}\phi(0,t)^2 + \int_0^\infty \phi'(s,t)((1-\Delta)^{-1}\phi)(s,t)\,\mathrm{d}s. \tag{2.22}$$

By using an argument similar to the derivation of (2.21) (here we are treating all dimensions $d \geqq 1$), we have for all $t \geqq 0$,

$$\begin{aligned} \|\phi(t)\|_{L^2(\mathbb{R}^d)} &\lesssim \|P_{<1}\phi(t)\|_{L^2(\mathbb{R}^d)} + \|P_{\geqq 1}\phi(t)\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \|\phi_0\|_{L^2(\mathbb{R}^d)} + \|u_0\|_{H^1(\mathbb{R}^d)}t + \|u_0\|_{H^1(\mathbb{R}^d)} \\ &\lesssim \|\phi_0\|_{L^2(\mathbb{R}^d)}(1+t), \end{aligned} \tag{2.23}$$

where we have used the relation $u_0 = (1-\Delta)^{-1}\nabla\phi_0$.

[4] In one dimension, we simply require that ϕ_0 be an even function.

Since $\phi'(r,t) \leqq 0$ for any $r \geqq 0$, we have $\|\phi(t)\|_\infty = \phi(0,t)$. Therefore, by (2.23), we have

$$\begin{aligned} \|(1-\Delta)^{-1}\phi\|_\infty &\leqq C\|\phi(t)\|_2 + \frac{1}{100}\|\phi(t)\|_\infty \\ &\leqq C\|\phi_0\|_2 + \frac{1}{100}\phi(t,0). \end{aligned} \tag{2.24}$$

Plugging (2.24) into (2.22) and using the fact that $(1-\Delta)^{-1}\phi \geqq 0$, $\phi' \leqq 0$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) \geqq \frac{1}{4}\phi(0,t)^2 - C\|\phi_0\|_2(1+t)\cdot\phi(0,t). \tag{2.25}$$

Clearly, for $\phi_0(0) > 0$ sufficiently large (compared to $\|\phi_0\|_2$), $\phi(0,t)$ will blow up in finite time. □

Our next result refines Theorem 2.1 in that it removes the size assumption on ϕ_0. For some technical reasons (see Lemma 2.5), it treats only dimensions $d \geqq 3$.

Theorem 2.2. *Let the dimension $d \geqq 3$. Assume that ϕ_0 is a radial real-valued function on $\mathbb{R}^d$ and $\phi_0 \in H^k(\mathbb{R}^d)$ for some $k > \frac{d}{2} + 2$. Assume that $\phi_0'(r) \leqq 0$ for any $r > 0$ and that ϕ_0 is not identically zero. If the initial velocity $u_0 = (1-\Delta)^{-1}m_0 = (1-\Delta)^{-1}\nabla\phi_0$, then the corresponding solution blows up in finite time.*

The proof of Theorem 2.2 relies on the following lemma, which can be regarded as a type of Poincaré inequality.

Lemma 2.3. *Let the dimension $d \geqq 1$. For any $C_1 > 0$, $1 \leqq p < \infty$, there is a constant $\varepsilon_0 > 0$ depending only on C_1, p and the dimension d such that the following holds:*

Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is a function (not necessarily radial) such that

$$0 \leqq f(x) \leqq f(0), \qquad \forall x \in \mathbb{R}^d.$$

If $\|f\|_{L^p_x} \leqq C_1|f(0)| < \infty$, then

$$\left|\left(\frac{\Delta}{1-\Delta}f\right)(0)\right| \geqq \varepsilon_0|f(0)|. \tag{2.26}$$

Proof of Lemma 2.3. Without loss of generality, we may assume that $f(0) = 1$. Denote the Bessel potential $K(x) = \mathcal{F}^{-1}((1+|\xi|^2)^{-1})(x)$. Note that K is a positive radial function on $\mathbb{R}^d$ and that $K \in L^1_x \cap L^q_x$ for any $1 < q < \frac{d}{d-2}$ (for $d = 2$ we have $K \in L^1_x \cap L^q_x$ for any $q < \infty$, and for $d = 1$ we have $K \in L^1_x \cap L^\infty_x$). Then,

$$\begin{aligned} \left(\frac{-\Delta}{1-\Delta}f\right)(0) &= f(0) - (\frac{1}{1-\Delta}f)(0) \\ &= \int_{\mathbb{R}^d} K(y)(f(0)-f(y))\,\mathrm{d}y. \end{aligned}$$

Assume the bound (2.26) is not true. Then there exists a sequence of nonnegative functions f_n such that $f_n(0) = 1$, $\|f_n\|_{L_x^\infty} \leqq 1$, $\|f_n\|_{L_x^p} \leqq C_1$ and

$$\int_{\mathbb{R}^d} K(y)(1 - f_n(y))\,\mathrm{d}y \to 0. \tag{2.27}$$

Now take a number $r > p$ sufficiently large such that $K \in L_x^{\frac{r}{r-1}}$. Obviously, $\|f_n\|_{L_x^r} \leqq C_2 < \infty$ for some constant $C_2 > 0$ independent of n. By passing to a subsequence in n, if necessary, we have $f_n \rightharpoonup g$ weakly in L_x^r for some $g \in L_x^r$. Furthermore, we have $\|g\|_{L_x^\infty} \leqq 1$. By (2.27) and the fact that $K \in L_x^{\frac{r}{r-1}}$, we then obtain

$$\int_{\mathbb{R}^d} K(y)(1 - g(y))\,\mathrm{d}y = 0,$$

which implies $g(y) = 1$ for almost everywhere $y \in \mathbb{R}^d$. This clearly contradicts the fact that $g \in L_x^r$. The lemma is proved. □

Remark 2.4. It is also possible to give a constructive proof of Lemma 2.3. For example, in the three-dimensional case, we have $K(x) = \mathcal{F}^{-1}((1+|\xi|^2)^{-1})(x) = \frac{1}{4\pi}\frac{\mathrm{e}^{-|x|}}{|x|}$. Let $p' = \frac{p}{p-1}$ be the usual Hölder conjugate of p. Let $R > 0$ be a number whose value will be chosen later. Then we have

$$\begin{aligned}\int_{\mathbb{R}^3} K(y)f(y)\,\mathrm{d}y &= \int_{|y|<R} K(y)f(y)\,\mathrm{d}y + \int_{|y|>R} K(y)f(y)\,\mathrm{d}y\\ &\leqq \|f\|_{L^\infty}\int_0^R r\mathrm{e}^{-r}\,\mathrm{d}r + \left(\int_{|y|>R} K^{p'}\,\mathrm{d}y\right)^{1/p'}\|f\|_{L^p}\\ &\leqq \left(1-(R+1)\mathrm{e}^{-R}\right)\|f\|_{L^\infty}\\ &\quad + \left(\frac{1}{(4\pi)^{p'-1}}\int_R^\infty \mathrm{e}^{-p'r}r^{2-p'}\,\mathrm{d}r\right)^{1/p'}\|f\|_{L^p}.\end{aligned} \tag{2.28}$$

To estimate (2.28), we compute (note that $p' > 1$, and assume that $R > 1$)

$$\begin{aligned}\int_R^\infty \mathrm{e}^{-p'r}r^{2-p'}\,\mathrm{d}r &\leqq R^{1-p'}\mathrm{e}^{(1-p')R}\int_R^\infty \mathrm{e}^{-r}r\,\mathrm{d}r\\ &= R^{1-p'}\mathrm{e}^{(1-p')R}(R+1)\mathrm{e}^{-R}\\ &\leqq CR^{2-p'}\mathrm{e}^{-p'R}.\end{aligned}$$

Plugging the above estimate into (2.28), we have

$$\int_{\mathbb{R}^3} K(y)f(y)\,\mathrm{d}y \leqq \left(1-(R+1)\mathrm{e}^{-R}\right)\|f\|_{L^\infty} + CR^{(2-p')/p'}\mathrm{e}^{-R}\|f\|_{L^p}.$$

Since $p' > 1$, we have $(2-p')/p' < 1$. If there is a constant C_1 such that $\|f\|_{L^p} \leqq C_1\|f\|_{L^\infty}$ holds, then we can always choose R big enough to obtain

$$\int_{\mathbb{R}^3} K(y)f(y)\,\mathrm{d}y \leqq (1-\varepsilon_0)\|f\|_{L^\infty}.$$

This then leads to (2.26).

For the proof of Theorem 2.2, we need a slightly modified version of Lemma 2.3. Note the dimension restriction $d \geqq 3$ and see also Remark 2.6 below.

Lemma 2.5. *Let the dimension $d \geqq 3$. For any $C_1 > 0$, $1 \leqq p < \infty$, there is a constant $\varepsilon_0 > 0$ depending only on C_1, p and the dimension d such that the following holds:*

Suppose $f : \mathbb{R}^d \to \mathbb{R}$ satisfies

$$0 \leqq f(x) \leqq f(0), \quad \forall\, x \in \mathbb{R}^d.$$

If $f \in L_x^2(\mathbb{R}^d)$ and

$$\left\| \frac{|\nabla|}{1+|\nabla|} f \right\|_{L_x^2} \leqq C_1 |f(0)| < \infty,$$

then

$$\left| \left(\frac{\Delta}{1-\Delta} f \right)(0) \right| \geqq \varepsilon_0 |f(0)|. \tag{2.29}$$

Remark 2.6. We stress that the dimension restriction $d \geqq 3$ is necessary in Lemma 2.5. In dimensions $d = 1, 2$, there exist counterexamples which are made of approximating sequences of the constant functions. To see this, let $t > 0$ and define

$$f_t(x) = \mathrm{e}^{-t|x|^2}, \quad x \in \mathbb{R}^d.$$

Then obviously $f_t(0) = 1$ and

$$\hat{f}_t(\xi) = \text{const} \cdot t^{-\frac{d}{2}} \mathrm{e}^{-\frac{|\xi|^2}{4t}}, \quad \xi \in \mathbb{R}^d.$$

When $d = 1, 2$, it is not difficult to check that

$$\begin{aligned} &\left\| \frac{|\nabla|}{1+|\nabla|} f_t(x) \right\|_{L_x^2}^2 \\ &\lesssim \int_{\mathbb{R}^d} \frac{|\xi|^2}{1+|\xi|^2} \cdot t^{-d} \mathrm{e}^{-\frac{|\xi|^2}{2t}} \, \mathrm{d}\xi \\ &\lesssim t^{1-\frac{d}{2}} \lesssim 1, \quad \text{as } t \to 0. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\left| \left(\frac{\Delta}{1-\Delta} f_t \right)(0) \right| \\ &\lesssim \int_{\mathbb{R}^d} \frac{|\xi|^2}{1+|\xi|^2} \cdot t^{-\frac{d}{2}} \mathrm{e}^{-\frac{|\xi|^2}{4t}} \, \mathrm{d}\xi \\ &\lesssim t^2 \to 0, \quad \text{as } t \to 0. \end{aligned}$$

Obviously, (2.29) cannot hold in this case.

Proof of Lemma 2.5. Again we will argue by contradiction. Assume (2.29) does not hold. Then we can find a sequence of nonnegative functions $f_n \in L^2_x(\mathbb{R}^d)$ with $\|f_n\|_\infty = f_n(0) = 1$ such that

$$\left\| \frac{|\nabla|}{1+|\nabla|} f_n \right\|_{L^2_x} \leqq C_1, \tag{2.30}$$

and

$$\int_{\mathbb{R}^d} K(y)(1 - f_n(y))\,\mathrm{d}y \to 0, \quad \text{as } n \to \infty. \tag{2.31}$$

By (2.30) and by passing to a subsequence in n if necessary, we can find $g \in L^2_x(\mathbb{R}^d)$ such that

$$\frac{|\nabla|}{1+|\nabla|} f_n \rightharpoonup g, \quad \text{weak in } L^2_x(\mathbb{R}^d), \text{ as } n \to \infty.$$

Now, for any $\phi \in \mathcal{S}(\mathbb{R}^d)$, observe that

$$\frac{1+|\nabla|}{|\nabla|}\phi \in L^2_x(\mathbb{R}^d), \quad \text{for } d \geqq 3.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^d} f_n \phi\,\mathrm{d}x &= \int_{\mathbb{R}^d} \frac{|\nabla|}{1+|\nabla|} f_n \cdot \frac{1+|\nabla|}{|\nabla|}\phi\,\mathrm{d}x \\ &\to \int_{\mathbb{R}^d} g \cdot \frac{1+|\nabla|}{|\nabla|}\phi\,\mathrm{d}x \\ &=: T(\phi), \quad \text{as } n \to \infty. \end{aligned} \tag{2.32}$$

Since $0 \leqq f_n \leqq 1$ and

$$T(\phi) = \lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \phi\,\mathrm{d}x,$$

it follows that for $\phi \geqq 0$, we have $T(\phi) \geqq 0$. Therefore, by the Riesz representation theorem, we have

$$T(\phi) = \int_{\mathbb{R}^d} \phi\,\mathrm{d}\mu,$$

for some non-negative Borel measure $d\mu$. Now since

$$\left| \int_{\mathbb{R}^d} f_n \phi\,\mathrm{d}x \right| \leqq \|\phi\|_{L^1_x(\mathbb{R}^d)},$$

we get

$$\left| \int_{\mathbb{R}^d} \phi\,\mathrm{d}\mu \right| \leqq \|\phi\|_{L^1_x(\mathbb{R}^d)}, \quad \forall \phi \in \mathcal{S}(\mathbb{R}^d).$$

Therefore, in a standard way, we can extend $d\mu \in (L^1_x)^* = L^\infty_x$. Hence, for some $f_\infty \in L^\infty_x(\mathbb{R}^d)$ with $0 \leqq f_n(x) \leqq 1$, a.e. $x \in \mathbb{R}^d$, we have

$$T(\phi) = \int_{\mathbb{R}^d} \phi(x) f_\infty(x)\, \mathrm{d}x.$$

By a density argument, we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \phi \, \mathrm{d}x = \int_{\mathbb{R}^d} f_\infty \phi \, \mathrm{d}x, \quad \forall \phi \in L^1_x(\mathbb{R}^d).$$

In particular,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} K(x) f_n(x)\, \mathrm{d}x = \int_{\mathbb{R}^d} K(x) f_\infty(x)\, \mathrm{d}x.$$

Therefore, by (2.31)

$$\int_{\mathbb{R}^d} K(x)(1 - f_\infty(x))\, \mathrm{d}x = 0,$$

and obviously $f_\infty(x) = 1$ a.e. $x \in \mathbb{R}^d$.

Plugging this back into (2.32), we obtain for any $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} g \cdot \frac{1 + |\nabla|}{|\nabla|} \phi \, \mathrm{d}x = \int_{\mathbb{R}^d} \phi \, \mathrm{d}x,$$

or on the Fourier side,

$$\int_{\mathbb{R}^d} \hat{g}(\xi) \cdot \frac{1 + |\xi|}{|\xi|} \hat{\phi}(\xi)\, \mathrm{d}\xi = \hat{\phi}(0).$$

From this and the fact that $\hat{g} \in L^2$, it follows easily that $\hat{g}(\xi) = 0$ a.e. $\xi \in \mathbb{R}^d$. This is obviously a contradiction. □

We are now ready to complete the

Proof of Theorem 2.2. Denote $g = (1 - \Delta)^{-1}\phi$. Set $r = 0$ in (1.10), which we then rewrite as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) &= \int_0^\infty ((1-\Delta)g)' \, \Delta g \, \mathrm{d}r \\
&= \int_0^\infty g' \Delta g \, \mathrm{d}r - \int_0^\infty (\Delta g)' \Delta g \, \mathrm{d}r \\
&= \int_0^\infty g' \left(g'' + \frac{d-1}{r} g' \right) \mathrm{d}r + \frac{1}{2} \left((\Delta g)(0,t) \right)^2 .
\end{aligned} \tag{2.33}$$

Note that g is a radial function, so $g'(0,t) = 0$, so we have

$$\int_0^\infty g' g'' \, \mathrm{d}r = 0.$$

Therefore, we obtain from (2.33) the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) \geqq \frac{1}{2}\left((\Delta g)(0,t)\right)^2$$
$$= \frac{1}{2}\left(\left(\frac{\Delta}{1-\Delta}\phi\right)(0,t)\right)^2. \tag{2.34}$$

By using the energy conservation (1.6) and the relation $u = (1-\Delta)^{-1}\nabla\phi$, we have

$$\left\| \frac{|\nabla|}{1+|\nabla|}\phi(\cdot,t) \right\|_{L^2_x} \leqq C_3 < \infty,$$

where C_3 is some constant independent of t.

Note that $\phi(0,t) \geqq \phi(0,0) > 0$. Since we assume the dimension $d \geqq 3$, by Lemma 2.5, we have

$$\left| \left(\frac{\Delta}{1-\Delta}\phi\right)(0,t) \right| \geqq \varepsilon_0 \phi(0,t),$$

where $\varepsilon_0 > 0$ is independent of t.

Plugging this estimate into (3.35), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) \geqq \frac{1}{2}\varepsilon_0^2 \phi(0,t)^2,$$

which clearly implies that $\phi(0,t)$ must blow up in finite time. □

3. Proof of Main Theorems

3.1. Proof of Theorem 1.3

By (2.33) and the fact

$$\int_0^\infty g' g''\, \mathrm{d}r = 0,$$

we obtain the following identity,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) = (d-1)\int_0^\infty \frac{(g')^2}{r}\,\mathrm{d}r + \frac{1}{2}\left((\Delta g)(0,t)\right)^2$$
$$= (d-1)\int_0^\infty \frac{(g')^2}{r}\,\mathrm{d}r + \frac{1}{2}\left(\left(\frac{\Delta}{1-\Delta}\phi\right)(0,t)\right)^2$$
$$= (d-1)\int_0^\infty \frac{(g')^2}{r}\,\mathrm{d}r + \frac{1}{2}\left(\phi(0,t) - g(0,t)\right)^2. \tag{3.35}$$

Since $\phi_0(0) \geqq 0$ and ϕ_0 is not identically zero, we have that, for all $t \geqq t_0$,

$$\phi(0,t) \geqq A_1, \tag{3.36}$$

where $t_0 > 0$ is any fixed time and A_1 is a constant depending on ϕ_0 and t_0.

Now let $R > 1$ be a parameter whose value will be specified later. Note that by the Fundamental Theorem of Calculus, we have

$$|g(0,t) - g(R,t)| \leqq \int_0^R |g'| dr \leqq \left(\int_0^R \frac{(g')^2}{r} \,\mathrm{d}r \right)^{\frac{1}{2}} \cdot R. \tag{3.37}$$

Then clearly, for dimensions $d \geqq 2$,

$$(3.35) \geqq \frac{1}{100R^2} \left(|\phi(0,t) - g(0,t)| + R \left(\int_0^R \frac{(g')^2}{r} \,\mathrm{d}r \right)^{\frac{1}{2}} \right)^2 \tag{3.38}$$

$$\geqq \frac{1}{100R^2} \left(|\phi(0,t) - g(0,t)| + |g(0,t) - g(R,t)| \right)^2 \tag{3.39}$$

$$\geqq \frac{1}{100R^2} \left(\phi(0,t) - g(R,t) \right)^2 . \tag{3.40}$$

Now we discuss two cases. Consider first the case of dimension $d \geqq 3$. By radial Sobolev embedding and energy conservation (1.6), we have

$$|g(R,t)| \leqq C_d \|\nabla g\|_2 \cdot R^{-\frac{d-2}{2}} \leqq C_d \|u_0\|_{H^1} \cdot R^{-\frac{d-2}{2}}, \tag{3.41}$$

where C_d is constant depending only on the dimension d, and $u_0 = (1-\Delta)^{-1}\nabla\phi_0$ is the initial velocity. By (3.41), we can choose R sufficiently large such that

$$|g(R,t)| \leqq \frac{1}{100} A_1, \tag{3.42}$$

where A_1 was defined in (3.36). Therefore, by (3.40), (3.41), and (3.42), we get for all $t > t_0$,

$$\phi(0,t) - g(R,t) \geqq \frac{1}{2}\phi(0,t).$$

Plugging this estimate into (3.40), we obtain for $t > t_0$, and some constant $\varepsilon_0 > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) \geqq \frac{1}{2}\varepsilon_0 \phi(0,t)^2,$$

which together with (3.36) clearly implies that $\phi(0,t)$ must blow up in finite time.

This finishes the case $d \geqq 3$. Now we turn to the case of dimension $d = 2$. We shall choose for each $g(t)$ the time-dependent parameter $R(t) = R_0(1+t)^{\frac{1}{2}}$, where R_0 will be taken sufficiently large. By (1.12) and radial Sobolev embedding, we have

$$|g(R(t),t)| \leqq C \cdot \|\phi(t)\|_2^{\frac{1}{2}} \cdot (R(t))^{-\frac{1}{2}} \leqq C \cdot B_1 \cdot R_0^{-\frac{1}{2}}.$$

Choosing R_0 sufficiently large gives us (3.42) and, consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) \geqq C \cdot \frac{1}{1+t}\phi(0,t)^2.$$

Integrating the above ODE on the interval $[t_0, \tau)$ with $\tau > t_0$, we get

$$-\frac{1}{\phi(0,\tau)} + \frac{1}{\phi(0,t_0)} \geqq \text{const} \cdot \log(1+\tau).$$

This implies that $\frac{1}{\phi(0,\tau)}$ becomes negative in finite time which obviously contradicts (3.36).

3.2. *Proof of Theorem 1.4*

By repeating an argument similar to the beginning part of the proof of Theorem 2.1, we have $\phi'(r,t) \geqq 0$ for any $r > 0$. Set $\phi = -\psi$. Then by (2.22), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(0,t) = -\frac{\psi(0,t)^2}{2} - \int_0^\infty \psi'(r,t)((1-\Delta)^{-1}\psi)(r,t)\,\mathrm{d}r.$$

By a derivation similar to (2.23), we then have for any $t > 0$,

$$\|\psi(t)\|_{L^2(\mathbb{R}^d)} = \|\phi(t)\|_{L^2(\mathbb{R}^d)} \leqq C \cdot (1+t),$$

where $C > 0$ depends only on ϕ_0.

Therefore, in place of (2.25), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(0,t) \leqq -\frac{\psi(0,t)^2}{4} + C \cdot (1+t) \cdot \psi(0,t).$$

Since $\psi(0,0) \geqq 0$, this clearly shows that $\psi(0,t)$ is bounded for all $t > 0$. By using the blowup criteria Lemma 1.2, we conclude that the corresponding classical solution exists for all time $t > 0$.

3.3. *Proof of Corollary 1.5*

The monotonicity of $\phi(0,t)$ follows directly from the proof of Theorem 1.3 [see (3.35)]. In particular, we know that $\phi(0,t) < 0$ for any $t \geqq 0$ (otherwise the corresponding solution will blow up). It remains to establish the estimates (1.14)–(1.15). By using the same argument as in the proof of Theorem 1.3, we obtain the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) \geqq \varepsilon_0\phi(0,t)^2, \quad \text{if } d \geqq 3,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(0,t) \geqq \frac{\varepsilon_1}{1+t}\phi(0,t)^2, \quad \text{if } d = 2,$$

where $\varepsilon_0 > 0$, $\varepsilon_1 > 0$ are some constants. Integrating the above inequality in time gives us the desired results.

3.4. Proof of Theorem 1.7

Let $\psi_0 \in H^\infty(\mathbb{R}^d)$ be a smooth radial function such that $\psi_0(0) = 0$ and

$$\begin{cases} \psi_0'(x) \leqq 0, & |x| \leqq c_1, \\ \psi_0'(x) > 0, & |x| > c_2, \\ \psi_0(x) < 0, & c_1/2 < |x| < 2c_2. \end{cases} \tag{3.43}$$

Here, $0 < c_1 < c_2 < \infty$ are arbitrary constants.

By local wellposedness theory, there exists $T_0 > 0$ and a smooth solution $\psi = \psi(x,t)$ to (1.1) ($m = \nabla\psi$) in the space $C([-T_0, T_0], H^k)$ for any $k \geqq 0$.

We make the following

Claim. There exists $t_0 > 0$ sufficiently small, such that $\psi(x, -t_0) < 0$ for any $x \in \mathbb{R}^d$.

We now assume the claim is true and complete the proof of the theorem. Take $\phi_0(x) := \psi(x, -t_0)$ for $x \in \mathbb{R}^d$. We shall show that ϕ_0 is the desired initial data leading to blowups. Denote the solution corresponding to the data ϕ_0 as $\phi = \phi(x,t)$. It is obvious that $\phi(x,t) = \psi(x, t - t_0)$ for any $t \geqq 0$. In particular, we have $\phi(0, t_0) = 0$. By using Theorem 1.3, it follows easily that ϕ must blow up at some $t > t_0$. Therefore, ϕ_0 is the desired initial data.

It remains for us to prove the claim. Write $\psi = \psi(r,t) = \psi(x,t)$, where $r = |x|$. Note that $\psi \in C^\infty([0,\infty))$ as a function of r. We can perform an even extension and regard $\psi \in C^\infty(\mathbb{R})$.

By (1.9), we have

$$\begin{aligned} -\partial_t\psi' &= \Big(-\psi + (1-\Delta)^{-1}\psi + \big((1-\Delta)^{-1}\psi\big)''\Big)\psi' + ((1-\Delta)^{-1}\psi)'\cdot\psi'' \\ &=: c(r,t)\psi' + b(r,t)\cdot\psi''. \end{aligned}$$

Here, $c = c(r,t)$, $b = b(r,t)$ are both C^∞-smooth functions for $-T_0 \leqq t \leqq T_0$ and $r \in \mathbb{R}$. Note that c is an even function and b is an odd function. Also, for some constant $B > 0$,

$$\sup_{|t|\leqq T_0} (\|b(t,\cdot)\|_\infty + \|\partial_r b(t,\cdot)\|_\infty) \leqq B < \infty. \tag{3.44}$$

Denote $f(r,t) = \psi'(r,t)$; then $f(r,t)$ satisfies the transport equation

$$\partial_t f + b\partial_r f + cf = 0. \tag{3.45}$$

Introduce the characteristic lines

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} z(t,\alpha) = b(z(t,\alpha), t), \\ z(0,\alpha) = \alpha \in \mathbb{R}. \end{cases}$$

For each $-T_0 \leqq t \leqq T_0$, the map $\alpha \to z(t, \alpha)$ is a smooth diffeomorphism. Furthermore, we have the obvious estimate

$$|z(t, \alpha) - \alpha| \leqq tB, \tag{3.46}$$

where $B > 0$ is the same constant as in (3.44). By integrating (3.45) along the characteristic line, we have

$$f(z(t, \alpha), t) = f(\alpha, 0) \exp\left(-\int_0^t c(z(\alpha, s), s)\, \mathrm{d}s\right), \quad \forall \alpha \in \mathbb{R},\ t \in [-T_0, T_0]. \tag{3.47}$$

Now, take t_1 sufficiently small such that (see (3.43) for the definition of the constant c_1)

$$t_1 \leqq \min\left\{\frac{c_1}{8B},\ T_0\right\}.$$

By (3.46), if $|t| \leqq t_1$ and $|z(t, \alpha)| \leqq \frac{c_1}{2}$, then obviously $|\alpha| \leqq c_1$. By (3.47), (3.43), we conclude that

$$\psi'(r, t) = f(r, t) \leqq 0, \quad \forall\, |t| \leqq t_1,\ r \leqq \frac{c_1}{2}. \tag{3.48}$$

By using a similar argument, we also obtain

$$\psi'(r, t) > 0, \quad \forall\, |t| \leqq t_1,\ r \geqq 2c_2, \tag{3.49}$$

By (3.35) and the fact that $\psi_0(0) = 0$, we obtain $\psi(0, t) < 0$ for all $t \in [-T_0, 0)$. It follows from (3.48) that

$$\psi(r, t) < 0, \quad \forall -t_1 \leqq t < 0,\ r \leqq \frac{c_1}{2}. \tag{3.50}$$

Similarly, using the fact that $\psi(\infty, t) = 0$ and (3.49), we obtain

$$\psi(r, t) < 0, \quad \forall -t_1 \leqq t < 0,\ r \geqq 2c_2. \tag{3.51}$$

By (3.43) and smoothness of the local solution, there exists some $t_2 > 0$ sufficiently small such that

$$\psi(r, t) < 0, \quad \forall\ |t| \leqq t_2,\ \frac{c_1}{2} \leqq r \leqq 2c_2.$$

Now, obviously, the claim follows if we take $t_0 = \min\{t_1, t_2\}$.

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