



On Kato–Ponce and fractional Leibniz

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Abstract. We show that in the Kato–Ponce inequality $\|J^s(fg) - fJ^s g\|_p \lesssim \|\partial f\|_\infty \|J^{s-1}g\|_p + \|J^s f\|_p \|g\|_\infty$, the $J^s f$ term on the right-hand side can be replaced by $J^{s-1}\partial f$. This solves a question raised in Kato–Ponce [14]. We propose a new fractional Leibniz rule for $D^s = (-\Delta)^{s/2}$ and similar operators, generalizing the Kenig–Ponce–Vega estimate [15] to all $s > 0$. We also prove a family of generalized and refined Kato–Ponce type inequalities which include many commutator estimates as special cases. To showcase the sharpness of the estimates at various endpoint cases, we construct several counterexamples. In particular, we show that in the original Kato–Ponce inequality, the L^∞ -norm on the right-hand side cannot be replaced by the weaker BMO norm. Some divergence-free counterexamples are also included.

1. Introduction

Let $J^s = (1 - \Delta)^{s/2}$, $s \in \mathbb{R}$. In [14], Kato and Ponce proved the following fundamental commutator estimate:

$$(1.1) \quad \|J^s(fg) - fJ^s g\|_p \lesssim_{s,p,d} \|J^s f\|_p \|g\|_\infty + \|\partial f\|_\infty \|J^{s-1}g\|_p,$$

where $s > 0$, $1 < p < \infty$, $\partial = (\partial_1, \dots, \partial_d)$ (occasionally we also denote it as ∇) and $f, g \in \mathcal{S}(\mathbb{R}^d)$. On page 892 of [14] (see Remark 1.1(c) therein), they conjectured that the $J^s f$ term on the right-hand side can be replaced by $J^{s-1}\partial f$. The first purpose of this paper is to confirm that this is indeed the case.

Theorem 1.1. *Let $s > 0$, $1 < p < \infty$. Then for any $f, g \in \mathcal{S}(\mathbb{R}^d)$,*

$$(1.2) \quad \|J^s(fg) - fJ^s g\|_p \lesssim_{s,p,d} \|J^{s-1}\partial f\|_p \|g\|_\infty + \|\partial f\|_\infty \|J^{s-1}g\|_p.$$

Furthermore, for $0 < s \leq 1$,

$$\|J^s(fg) - fJ^s g\|_p \lesssim_{s,p,d} \|J^{s-1}\partial f\|_p \|g\|_\infty.$$

Mathematics Subject Classification (2010): 35Q35.

Keywords: Kato–Ponce, fractional Leibniz.

More general results are available. See Theorem 1.9 in the later part of this introduction.

Denote $D^s = (-\Delta)^{s/2}$. In [15], Kenig, Ponce and Vega (KPV) proved the fundamental estimate:

$$(1.3) \quad \|D^s(fg) - fD^s g - gD^s f\|_p \lesssim_{s,s_1,s_2,p,p_1,p_2,d} \|D^{s_1} f\|_{p_1} \|D^{s_2} g\|_{p_2},$$

where $s = s_1 + s_2$, $0 < s, s_1, s_2 < 1$, $1/p = 1/p_1 + 1/p_2$ and $1 < p, p_1, p_2 < \infty$. A natural question is to investigate what is the natural formulation/generalisation of the KPV estimate when $s \geq 1$. The second purpose of this paper is to solve this problem. Our theorem below establishes a new fractional Leibniz rule for any D^s , $s > 0$. It includes various end-point situations.

Theorem 1.2. Case 1: $1 < p < \infty$.

Let $s > 0$ and $1 < p < \infty$. Then for any $s_1, s_2 \geq 0$ with $s_1 + s_2 = s$, and any $f, g \in \mathcal{S}(\mathbb{R}^d)$, the following hold:

(1) If $1 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$, then

$$(1.4) \quad \begin{aligned} & \|D^s(fg) - \sum_{|\alpha| \leq s_1} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{|\beta| \leq s_2} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f\|_p \\ & \lesssim_{s,s_1,s_2,p,p_1,p_2,d} \|D^{s_1} f\|_{p_1} \|D^{s_2} g\|_{p_2}. \end{aligned}$$

(2) If $p_1 = p$, $p_2 = \infty$, then

$$(1.5) \quad \begin{aligned} & \|D^s(fg) - \sum_{|\alpha| < s_1} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{|\beta| \leq s_2} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f\|_p \\ & \lesssim_{s,s_1,s_2,p,d} \|D^{s_1} f\|_p \|D^{s_2} g\|_{\text{BMO}}. \end{aligned}$$

(3) If $p_1 = \infty$, $p_2 = p$, then

$$(1.6) \quad \|D^s(fg) - \sum_{|\alpha| \leq s_1} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{|\beta| < s_2} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f\|_p \lesssim_{s,s_1,s_2,p,d} \|D^{s_1} f\|_{\text{BMO}} \|D^{s_2} g\|_p.$$

In the above we adopt the usual multi-index notation, namely $\alpha = (\alpha_1, \dots, \alpha_d)$, $\partial^\alpha = \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$, $|\alpha| = \sum_{j=1}^d \alpha_j$ and $\alpha! = \alpha_1! \dots \alpha_d!$. The operator $D^{s,\alpha}$ is defined via Fourier transform¹ as

$$\widehat{D^{s,\alpha} g}(\xi) = \widehat{D^{s,\alpha}}(\xi) \hat{g}(\xi), \quad \text{and} \quad \widehat{D^{s,\alpha}}(\xi) = i^{-|\alpha|} \partial_\xi^\alpha (|\xi|^s).$$

Case 2: $1/2 < p \leq 1$.

If $1/2 < p \leq 1$, $s > d/p - d$ or $s \in 2\mathbb{N}$, then for any $1 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$, any $s_1, s_2 \geq 0$ with $s_1 + s_2 = s$,

$$\begin{aligned} & \|D^s(fg) - \sum_{|\alpha| \leq s_1} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{|\beta| \leq s_2} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f\|_p \\ & \lesssim_{s,s_1,s_2,p,p_1,p_2,d} \|D^{s_1} f\|_{p_1} \|D^{s_2} g\|_{p_2}. \end{aligned}$$

¹ The precise form of the Fourier transform does not matter. But see (2.1) for the definition used in this paper.

Remark 1.3. As usual, empty summation (such as $\sum_{0 \leq |\alpha| < 0}$) is defined as zero. Let $0 < s, s_1, s_2 < 1$ in (1.4); then

$$\|D^s(fg) - fD^s g - gD^s f\|_p \lesssim_{s, s_1, s_2, p_1, p_2, d} \|D^{s_1} f\|_{p_1} \|D^{s_2} g\|_{p_2},$$

i.e., we recover the estimate (1.3). Let $s_1 = 0, s_2 = s, 0 < s < 1$ in (1.5); then we get

$$\|D^s(fg) - gD^s f\|_p \lesssim \|f\|_p \|D^s g\|_{\text{BMO}} \lesssim \|f\|_p \|D^s g\|_\infty, \quad 1 < p < \infty.$$

Similarly, let $s_1 = s, 0 < s \leq 1$ in (1.5); then we get

$$(1.7) \quad \|D^s(fg) - fD^s g - gD^s f\|_p \lesssim_{s, p, d} \|D^s f\|_p \|g\|_{\text{BMO}}.$$

Thus for $0 < s \leq 1, 1 < p < \infty$,

$$(1.8) \quad \|D^s(fg) - fD^s g\|_p \lesssim_{s, p, d} \|D^s f\|_p \|g\|_\infty.$$

The inequality (1.8) for $0 < s < 1, 1 < p < \infty$ was proved in [15] (see also Problem 2.7, Problem 2.8 on page 77 of [19]). Let us also point it out that the estimate (1.7) suggests that, due to the presence of the term $gD^s f$, the L^∞ norm on the right-hand side of (1.8) is sharp and cannot be replaced by the weaker BMO norm in general. See Corollary 7.4 for more definitive and precise statements.

Remark. If we slightly abuse our notation and denote $D^{s, \alpha}$ as a fractional differentiation operator $\tilde{D}^{s-|\alpha|}$ (i.e., of order $s - |\alpha|$), then Theorem 1.2 roughly says that (suppressing constant coefficients)

$$\begin{aligned} D^s(fg) &\sim fD^s g + \partial f \tilde{D}^{s-1} g + \dots + \partial^{[s_1]} f \tilde{D}^{s-[s_1]} g \\ &\quad + gD^s f + \partial g \tilde{D}^{s-1} f + \dots + \partial^{[s_2]} g \tilde{D}^{s-[s_2]} f + O(\|D^{s_1} f\|_{p_1} \cdot \|D^{s_2} g\|_{p_2}). \end{aligned}$$

In yet other words, neglecting error terms, the nonlocal operator D^s can be effectively regarded as a local operator obeying a generalized Leibniz rule.

Theorem 1.2 actually holds for more general differential (and also pseudo-differential) operators. For example, for $s > 0$ suppose A^s is a differential operator such that its symbol $\widehat{A^s}(\xi)$ is a homogeneous function of degree s and $\widehat{A^s}(\xi) \in C^\infty(\mathbb{S}^{d-1})$ (for example: $\widehat{A^s}(\xi) = i|\xi|^{s-1}\xi_1$). Then we have the following corollary. We shall omit the proof since it will be essentially a repetition of the proof of Theorem 1.2.

Corollary 1.4. *Let $1 < p < \infty$ and $s > 0$. Then for any $s_1, s_2 \geq 0$ with $s_1 + s_2 = s$, and any $f, g \in \mathcal{S}(\mathbb{R}^d)$, the following hold:*

(1) *If $1 < p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$, then*

$$(1.9) \quad \left\| A^s(fg) - \sum_{|\alpha| \leq s_1} \frac{1}{\alpha!} \partial^\alpha f A^{s, \alpha} g - \sum_{|\beta| \leq s_2} \frac{1}{\beta!} \partial^\beta g A^{s, \beta} f \right\|_p \lesssim_{A^s, s, s_1, s_2, p, p_1, p_2, d} \|D^{s_1} f\|_{p_1} \|D^{s_2} g\|_{p_2}.$$

(2) If $p_1 = p$, $p_2 = \infty$, then

$$(1.10) \quad \left\| A^s(fg) - \sum_{|\alpha| < s_1} \frac{1}{\alpha!} \partial^\alpha f A^{s,\alpha} g - \sum_{|\beta| \leq s_2} \frac{1}{\beta!} \partial^\beta g A^{s,\beta} f \right\|_p \\ \lesssim_{A^s, s, s_1, s_2, p, d} \|D^{s_1} f\|_p \|D^{s_2} g\|_{\text{BMO}}.$$

(3) If $p_1 = \infty$, $p_2 = p$, then

$$\left\| A^s(fg) - \sum_{|\alpha| \leq s_1} \frac{1}{\alpha!} \partial^\alpha f A^{s,\alpha} g - \sum_{|\beta| < s_2} \frac{1}{\beta!} \partial^\beta g A^{s,\beta} f \right\|_p \\ \lesssim_{A^s, s, s_1, s_2, p, d} \|D^{s_1} f\|_{\text{BMO}} \|D^{s_2} g\|_p.$$

In the above, the operator $A^{s,\alpha}$ is defined via the Fourier transform as

$$\widehat{A^{s,\alpha} g}(\xi) = i^{-|\alpha|} \partial_\xi^\alpha (\widehat{A^s}(\xi)) \hat{g}(\xi).$$

Remark. A further interesting problem is to identify the explicit dependence on the operators A^s in the implied constants. This is useful in some problems connected with a family of operators rather than a single operator.

Remark. One should note that the error terms on the right-hand side of the above inequalities involve D^s rather than A^s . In particular for $0 < s < 1$, we have the following commonly used ones:

$$\|A^s(fg) - fA^s g - gA^s f\|_p \lesssim \|f\|_{p_1} \|D^s g\|_{p_2}, \quad 1 < p_1, p_2 < \infty, \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}; \\ \|A^s(fg) - gA^s f\|_p \lesssim \|f\|_p \|D^s g\|_{\text{BMO}}, \quad 1 < p < \infty; \\ \|A^s(fg) - fA^s g - gA^s f\|_p \lesssim \|f\|_{\text{BMO}} \|D^s g\|_p, \quad 1 < p < \infty.$$

Also for $1 \leq s < 2$,

$$\|A^s(fg) - fA^s g - gA^s f - \nabla g \cdot A^{s,\nabla} f\|_p \lesssim \|f\|_{p_1} \|D^s g\|_{p_2}, \quad 1 < p_1, p_2, p < \infty, \\ \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}; \\ \|A^s(fg) - gA^s f - \nabla g \cdot A^{s,\nabla} f\|_p \lesssim \|f\|_p \|D^s g\|_{\text{BMO}}, \quad 1 < p < \infty; \\ \|A^s(fg) - fA^s g - gA^s f - \nabla g \cdot A^{s,\nabla} f\|_p \lesssim \|f\|_{\text{BMO}} \|D^s g\|_p, \quad 1 < p < \infty,$$

where $\widehat{A^{s,\nabla}}(\xi) = -i \nabla_\xi (\widehat{A^s}(\xi))$.

Remark. At this point it is useful to point out the explicit connection with the classical Leibniz rule and do a sanity check of our formulae. Let $\gamma = (\gamma_1, \dots, \gamma_d)$ be a multi-index. Recall that the classic Leibniz formula for a differential operator ∂^γ takes the form

$$(1.11) \quad \partial^\gamma(fg) = \sum_{\alpha \leq \gamma} \frac{\gamma!}{\alpha!(\gamma - \alpha)!} \partial^\alpha f \partial^{\gamma - \alpha} g.$$

Set $s = |\gamma|$ and denote $A^s = \partial^\gamma$. Then easy to check that (in our notation)

$$\begin{aligned}\widehat{A^{s,\alpha}}(\xi) &= i^{-|\alpha|} \partial_\xi^\alpha (i^{|\gamma|} \xi^\gamma) \\ &= i^{|\gamma|-|\alpha|} \frac{\gamma!}{(\gamma-\alpha)!} \xi^{\gamma-\alpha} = \frac{\gamma!}{(\gamma-\alpha)!} i^{|\gamma-\alpha|} \xi^{\gamma-\alpha}.\end{aligned}$$

Thus $A^{s,\alpha} = \frac{\gamma!}{(\gamma-\alpha)!} \partial^{\gamma-\alpha}$. Clearly (1.9) then takes the form

$$(1.12) \quad \left\| \partial^\gamma(fg) - \sum_{|\alpha| \leq s_1} \frac{\gamma!}{\alpha!(\gamma-\alpha)!} \partial^\alpha f \partial^{\gamma-\alpha} g - \sum_{|\beta| \leq s_2} \frac{\gamma!}{\beta!(\gamma-\beta)!} \partial^\beta g \partial^{\gamma-\beta} f \right\|_p \lesssim \|D^{s_1} f\|_{p_1} \|D^{s_2} g\|_{p_2}.$$

Note that (1.12) captures essentially the main terms in (1.11). In this sense the formula (1.12) provides a “natural” generalization of the classical Leibniz formula (1.11) to the fractional setting.

Remark. One need not worry about the possibility that $\partial^\alpha f \partial^{\gamma-\alpha} g$ may coincide with the terms $\partial^\beta g \partial^{\gamma-\beta} f$. This is because due to the constraint $|\alpha| \leq s_1$, $|\beta| \leq s_2$, such two terms possibly coincide only when $|\alpha| = s_1$, $|\beta| = s_2$ and s_1, s_2 are both integers. But in this case $\partial^\alpha f \partial^{\gamma-\alpha} g$ can be easily bounded by $\|D^{s_1} f\|_{p_1} \|D^{s_2} g\|_{p_2}$ and thus can be included in the error term on the right-hand side .

Remark. One may wonder what is the origin for the appearance of the operators $D^{s,\alpha}$ and $A^{s,\alpha}$ in the new Leibniz rule. To see this, consider the symbol $\sigma(\xi, \eta) = |\xi + \eta|^s$ corresponding to the operator D^s , which on the Fourier side reads

$$\sigma_D(f, g) = \frac{1}{(2\pi)^{2d}} \int \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta.$$

Our new fractional Leibniz rule amounts to the justification of a Taylor expansion:

$$\sigma(\xi, \eta) = \sum_{|\alpha| \leq s_1} \frac{(\partial_\xi^\alpha \sigma)(0, \eta)}{\alpha!} \xi^\alpha + \sum_{|\beta| \leq s_2} \frac{(\partial_\eta^\beta \sigma)(\xi, 0)}{\beta!} \eta^\beta + \sigma_{\text{err}},$$

where (in the non-endpoint situation $1 < p_1, p_2 < \infty$)

$$\|\sigma_{\text{err}}(f, g)\|_p \lesssim \|D^{s_1} f\|_{p_1} \|D^{s_2} g\|_{p_2}.$$

One may further write

$$(\partial_\xi^\alpha \sigma)(0, \eta) \xi^\alpha = i^{-\alpha} (\partial_\xi^\alpha \sigma)(0, \eta) \cdot (i\xi)^\alpha$$

and clearly the first factor accords with the $D^{s,\alpha}$ operator mentioned earlier. In short summary, our new Leibniz rule is simply a separable Taylor expansion of the symbol $\sigma(\xi, \eta)$ up to $O(D^{s_1} f D^{s_2} g)$ in an appropriate sense!

In the following remarks, we discuss a few applications of the new Leibniz rule.

Remark 1.5. Let $m \geq 1$ be an integer and recall $\partial = (\partial_1, \dots, \partial_d)$ on \mathbb{R}^d . For any integer $n \geq 1$, denote

$$\|\partial^n f\|_p = \sum_{|\alpha|=n} \|\partial^\alpha f\|_p.$$

The classical Kato–Ponce inequality for the usual differential operator ∂^m (without loss of generality, one may assume $m \geq 3$) is

$$\sum_{|\gamma|=m} \|\partial^\gamma(fg) - f\partial^\gamma g\|_p \lesssim_{p,d} \|\partial^m f\|_p \|g\|_\infty + \|\partial f\|_\infty \|\partial^{m-1} g\|_p, \quad 1 < p < \infty.$$

The proof of the above inequality, roughly speaking, is a two-step procedure.

Step 1: Leibniz. One writes

$$\partial^\gamma(fg) - f\partial^\gamma g = g\partial^\gamma f + \sum_{|\alpha|=1} \binom{\gamma}{\alpha} \partial^\alpha f \partial^{\gamma-\alpha} g + \sum_{2 \leq |\alpha| \leq m-1, \alpha+\beta=\gamma} \binom{\gamma}{\alpha} \partial^\alpha f \partial^\beta g.$$

Step 2: (Gagliardo–Nirenberg) Interpolation. For $2 \leq |\alpha| \leq m-1$, by using²

$$\begin{aligned} \|\partial^\alpha f\|_{\frac{p(m-1)}{|\alpha|-1}} &\lesssim \|\partial^m f\|_p^{\frac{|\alpha|-1}{m-1}} \|\partial f\|_\infty^{\frac{m-|\alpha|}{m-1}}, \\ \|\partial^\beta g\|_{\frac{p(m-1)}{|\beta|}} &\lesssim \|\partial^{m-1} g\|_p^{\frac{|\beta|}{m-1}} \|g\|_\infty^{1-\frac{|\beta|}{m-1}}, \end{aligned}$$

we get

$$\|\partial^\alpha f \partial^\beta g\|_p \lesssim \|\partial^\alpha f\|_{\frac{p(m-1)}{|\alpha|-1}} \|\partial^\beta g\|_{\frac{p(m-1)}{|\beta|}} \lesssim \|\partial^m f\|_p \|g\|_\infty + \|\partial f\|_\infty \|\partial^{m-1} g\|_p.$$

Thanks to the new Leibniz rule, we can effectively regard the nonlocal operator D^s as the local one and “revive” the above classical proof of Kato–Ponce to work for the nonlocal case. Indeed consider the case $s > 1$ and $1 < p < \infty$; by using Theorem 1.2 with $s_1 = s$, $s_2 = 0$, we get

$$\|D^s(fg) - \sum_{|\alpha|<s} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - gD^s f\|_p \lesssim \|g\|_{\text{BMO}} \|D^s f\|_p.$$

We then have

$$\|D^s(fg) - fD^s g\|_p \lesssim \|\partial f\|_\infty \|D^{s-1} g\|_p + \|g\|_\infty \|D^s f\|_p + \sum_{2 \leq |\alpha| < s} \|\partial^\alpha f D^{s,\alpha} g\|_p.$$

Now observe that if $1 < s \leq 2$, the above summation in α is not present. For $s > 2$, by using

$$\begin{aligned} \|\partial^\alpha f\|_{\frac{p(s-1)}{|\alpha|-1}} &\lesssim \|D^s f\|_p^{\frac{|\alpha|-1}{s-1}} \|\partial f\|_\infty^{\frac{s-|\alpha|}{s-1}}, \\ \|D^{s,\alpha} g\|_{\frac{p(s-1)}{s-|\alpha|}} &\lesssim \|D^{s-1} g\|_p^{\frac{s-|\alpha|}{s-1}} \|g\|_\infty^{\frac{|\alpha|}{s-1}}, \end{aligned}$$

²See Lemma 2.10 for a general proof of the interpolation inequalities.

we get the desired inequality:

$$\|D^s(fg) - fD^s g\|_p \lesssim \|\partial f\|_\infty \|D^{s-1}g\|_p + \|g\|_\infty \|D^s f\|_p.$$

Remark 1.6. In [21], Ye considered the 2D incompressible Bénard equation in which the main unknowns are the velocity $u: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and the temperature $\theta: \mathbb{R}^2 \rightarrow \mathbb{R}$. Define $\mathcal{R}_1 = (-\Delta)^{-1/2}\partial_{x_1}$. Ye proved the commutator estimate

$$(1.13) \quad \|[\mathcal{R}_1, u \cdot \nabla]\theta\|_{L^p(\mathbb{R}^2)} \lesssim_{p,p_1,p_2} \|\nabla u\|_{L^{p_1}(\mathbb{R}^2)} \|\theta\|_{L^{p_2}(\mathbb{R}^2)}, \quad \text{if } \nabla \cdot u = 0,$$

where $1 < p, p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$.

We now explain how to deduce (1.13) from Corollary 1.4. For $j = 1, 2$, define $A_j = \mathcal{R}_1 \partial_j$. Set $s = s_1 = 1$, $s_2 = 0$ in (1.9) and we get

$$\left\| A_j(u_j \theta) - u_j A_j \theta - \sum_{|\alpha|=1} \partial_\alpha u_j A_j^\alpha \theta - \theta A_j u_j \right\|_p \lesssim_{p,p_1,p_2} \|Du_j\|_{p_1} \|\theta\|_{p_2},$$

where

$$\widehat{A_j^\alpha}(\xi) = -i \partial_\xi^\alpha (|\xi|^{-1} \cdot (i\xi_1) \cdot (i\xi_j)).$$

It is easy to check that $\|A_j^\alpha \theta\|_{p_2} \lesssim_{p_2} \|\theta\|_{p_2}$, and we get

$$\left\| \sum_{j=1}^2 (A_j(u_j \theta) - u_j A_j \theta) \right\|_p \lesssim_{p,p_1,p_2} \|Du\|_{p_1} \|\theta\|_{p_2} \lesssim_{p,p_1,p_2} \|\nabla u\|_{p_1} \|\theta\|_{p_2}.$$

The estimate (1.13) then easily follows. Note that one actually does not need to use the divergence-free condition $\nabla \cdot u = 0$ since

$$\|(-\Delta)^{-1/2}\partial_{x_1}((\nabla \cdot u)\theta)\|_p \lesssim_{p,p_1,p_2} \|\nabla u\|_{p_1} \|\theta\|_{p_2}.$$

Remark 1.7. In [8], Fefferman, McCormick, Robinson and Rodrigo (FMRR) considered a class of non-resistive MHD equations and proved a new Kato–Ponce type inequality:

$$(1.14) \quad \|D^s((u \cdot \nabla)B) - (u \cdot \nabla)(D^s B)\|_{L^2(\mathbb{R}^d)} \lesssim_{s,d} \|\nabla u\|_{H^s(\mathbb{R}^d)} \|B\|_{H^s(\mathbb{R}^d)},$$

where $s > d/2$, $u = (u_1, \dots, u_d)$, $B = (B_1, \dots, B_d)$, ∇u , $B \in H^s(\mathbb{R}^d)$. The condition $s > d/2$ is critical for L^∞ -embedding. For dimension $d = 2$, $s = d/2 = 1$, they exhibited a pair of divergence-free $u \in H^2(\mathbb{R}^2)$, $B \in H^1(\mathbb{R}^2)$, such that

$$\partial_k((u \cdot \nabla)B) - (u \cdot \nabla)(\partial_k B) = ((\partial_k u) \cdot \nabla)B \notin L^2(\mathbb{R}^2).$$

In this paper, Corollary 1.4 can be used to generalize the FMRR inequality (1.14) to all $1 < p < \infty$, $s > d/p$. It is also possible to give some refined inequalities for the borderline case $s = d/p$ and construct divergence-free counterexamples for the nonlocal operator $D^{d/p}$ for all $1 < p < \infty$. See Corollary 5.4, Remark 5.5 and Remark 5.6 in Section 5 for more details.

Remark 1.8. In [6], Chae, Constantin, Córdoba, Gancedo and Wu considered several generalized surface quasi-geostrophic models with singular velocities. One of the models considered therein is the following:

$$\begin{cases} \partial_t \theta + v \cdot \nabla \theta = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^2; \\ v = D^{-1+\gamma} \nabla^\perp \theta = (-D^{-1+\gamma} \partial_2 \theta, D^{-1+\gamma} \partial_1 \theta), & 0 < \gamma < 1; \\ \theta(0, x) = \theta_0(x). \end{cases}$$

Note that v scales as $D^\gamma \theta$, which is quite singular for $0 < \gamma < 1$, and this renders the local wellposedness a very nontrivial problem. For $\theta_0 \in H^m(\mathbb{R}^2)$ with $m \geq 4$ being an integer, they proved local wellposedness by using skew-symmetry of the operator $D^{-1+\gamma} \nabla^\perp$ (to rewrite the nonlinear term in terms of a commutator) and a commutator estimate of the form (see Proposition 2.1 therein)

$$\begin{aligned} \sum_{j=1}^2 \|D^s \partial_j(gf) - gD^s \partial_j f\|_{L^2(\mathbb{R}^2)} &\lesssim_s \|D^s f\|_2 \|\widehat{Dg}(\eta)\|_{L^1_\eta} + \|f\|_2 \|\widehat{D^{1+s}g}(\eta)\|_{L^1_\eta} \\ &\lesssim_{s,\epsilon} \|D^s f\|_2 \|g\|_{H^{2+\epsilon}} + \|f\|_2 \|g\|_{H^{2+s+\epsilon}}, \end{aligned}$$

where $s \in \mathbb{R}$ and $\epsilon > 0$. We now show how to use our new Leibniz rule to obtain a more refined result, namely sharp local wellposedness in H^s for any $s > 2 + \gamma$. Indeed by taking $f = D^{\gamma-1} \nabla^\perp \theta$, $g = \nabla \theta$, $s_1 = 1$, $s_2 = s - 1$, $p = p_2 = 2$, $p_1 = \infty$ in Theorem 1.2, we get

$$\left\| D^s(fg) - \sum_{|\alpha| \leq 1} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{|\beta| < s-1} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f \right\|_2 \lesssim \|Df\|_{\text{BMO}} \|D^{s-1}g\|_2 \lesssim \|\theta\|_{H^s}^2.$$

Now consider the contribution of each summand (in either α or β) separately.

- $\alpha = 0$. Obviously $fD^s g = D^{\gamma-1} \nabla^\perp \theta \cdot \nabla D^s \theta$ and

$$\int_{\mathbb{R}^2} (D^{\gamma-1} \nabla^\perp \theta \cdot \nabla D^s \theta) D^s \theta \, dx = 0$$

by using integration by parts.

- $|\alpha| = 1$. In this case

$$\begin{aligned} \|\partial^\alpha f D^{s,\alpha} g\|_2 &= \|\partial^\alpha D^{\gamma-1} \nabla^\perp \theta \cdot D^{s,\alpha} \nabla \theta\|_2 \\ &\lesssim \|\partial^\alpha D^{\gamma-1} \nabla^\perp \theta\|_\infty \cdot \|D^{s,\alpha} \nabla \theta\|_2 \lesssim \|\theta\|_{H^s} \cdot \|\theta\|_{H^s}. \end{aligned}$$

- $\beta = 0$. Observe that $gD^s f = \nabla \theta \cdot D^{\gamma-1} \nabla^\perp D^s \theta$, and

$$\int (\nabla \theta \cdot D^{\gamma-1} \nabla^\perp D^s \theta) D^s \theta \, dx = - \int D^{\gamma-1} \nabla^\perp \cdot (D^s \theta \nabla \theta) D^s \theta \, dx.$$

We then write (this is the elegant trick used in [6])

$$\begin{aligned} \int (\nabla \theta \cdot D^{\gamma-1} \nabla^\perp D^s \theta) D^s \theta \, dx &= -\frac{1}{2} \int D^s \theta (D^{\gamma-1} \nabla^\perp \cdot (D^s \theta \nabla \theta) \\ &\quad - \nabla \theta \cdot D^{\gamma-1} \nabla^\perp D^s \theta) \, dx. \end{aligned}$$

By Corollary 1.4 with $A^\gamma = D^{\gamma-1}\nabla^\perp$, $f = D^s\theta$, $g = \nabla\theta$, $p = p_1 = 2$, $p_2 = \infty$, we get

$$\|D^{\gamma-1}\nabla^\perp \cdot (D^s\theta\nabla\theta) - \nabla\theta \cdot D^{\gamma-1}\nabla^\perp D^s\theta\|_2 \lesssim \|D^s\theta\|_2 \|D^\gamma\nabla\theta\|_{\text{BMO}} \lesssim \|\theta\|_{H^s}^2.$$

- $1 \leq |\beta| \leq s - 2$. If $1 \leq |\beta| < s - 2$, then clearly by Sobolev embedding,

$$\|\partial^\beta g D^{s,\beta} f\|_2 \lesssim \|\partial^\beta \nabla\theta\|_\infty \|D^{s,\beta} D^{\gamma-1}\nabla^\perp\theta\|_2 \lesssim \|\theta\|_{H^s}^2.$$

Similarly, if $|\beta| = s - 2$ (in this case s will be an integer),

$$\|\partial^\beta g D^{s,\beta} f\|_2 \lesssim \|\partial^{s-1}\theta\|_{\infty-} \|D^{s,\beta} D^{\gamma-1}\nabla^\perp\theta\|_{2+} \lesssim \|\theta\|_{H^s}^2.$$

- $s - 2 < |\beta| < s - 1$. We have

$$\|\partial^\beta g D^{s,\beta} f\|_2 \lesssim \|\partial^\beta \nabla\theta\|_{(\frac{1}{2} - \frac{s-(|\beta|+1)}{2})^{-1}} \|D^{s,\beta} D^{\gamma-1}\nabla^\perp\theta\|_{(\frac{s-(|\beta|+1)}{2})^{-1}} \lesssim \|\theta\|_{H^s}^2.$$

Collecting the above estimates, we get for $s > \gamma + 2$,

$$\frac{d}{dt}(\|\theta\|_{H^s}^2) \lesssim \|\theta\|_{H^s}^3$$

which (together with standard mollification/regularisation arguments) easily yields the desired local wellposedness in H^s .

In Theorem 5.1, we state and prove a family of refined Kato–Ponce inequalities for the operator $D^s = (-\Delta)^{s/2}$. Those inequalities are proved with the help of Theorem 1.2. On the other hand, for the inhomogeneous operator J^s , we have the following generalised inequalities. Note that in the following inequality, some of the endpoint cases can be further improved along similar lines as in Theorem 5.1. For simplicity of presentation (and practical considerations), here we only state the simplest version.

Theorem 1.9. *Let $1 < p < \infty$. Let $1 < p_1, p_2, p_3, p_4 \leq \infty$ satisfy $1/p_1 + 1/p_2 = 1/p_3 + 1/p_4 = 1/p$. Then for any $f, g \in \mathcal{S}(\mathbb{R}^d)$, the following hold:*

- If $0 < s \leq 1$, then

$$\|J^s(fg) - fJ^s g\|_p \lesssim_{s,p_1,p_2,p_3,p_4,d} \|J^{s-1}\partial f\|_{p_1} \|g\|_{p_2}.$$

- If $s > 1$, then

$$\|J^s(fg) - fJ^s g\|_p \lesssim_{s,p_1,p_2,p_3,p_4,p,d} \|J^{s-1}\partial f\|_{p_1} \|g\|_{p_2} + \|\partial f\|_{p_3} \|J^{s-2}\partial g\|_{p_4}.$$

There are also many other reformulations and generalisations of the Kato–Ponce commutator inequalities (cf. [1] and the references therein). One popular variant is the following fractional Leibniz rule, which holds for any $s > 0$, and for $f, g \in \mathcal{S}(\mathbb{R}^d)$:

$$(1.15) \quad \|D^s(fg)\|_r \leq C_{s,d,p_1,p_2,q_1,q_2} \cdot (\|D^s f\|_{p_1} \|g\|_{q_1} + \|D^s g\|_{p_2} \|f\|_{q_2}),$$

where $1/r = 1/p_1 + 1/q_1 = 1/p_2 + 1/q_2$, $1 < r < \infty$, $1 < p_1, p_2, q_1, q_2 \leq \infty$, and $C_{s,d,p_1,p_2,q_1,q_2} > 0$ is a constant depending only on $(s, d, p_1, p_2, q_1, q_2)$. One should note that the same inequality also holds for the inhomogeneous operator J^s . Grafakos and Oh [13], and Muscalu and Schlag [19], have extended the inequality (1.15) to the wider range $1/2 < r < \infty$ under the assumption that $s > \max(0, d/r - d)$ or $s \in 2\mathbb{N}$. The end-point case $r = \infty$ was conjectured in Grafakos, Maldonado and Naibo [12] and solved in [2].

The rest of this paper is organized as follows. In Section 2 we collect some notation used in this paper and also some preliminary lemmas. In Section 3 we prove an important paraproduct estimate and some auxiliary lemmas. Section 4 is devoted to the proof of Theorem 1.2. In Section 5 we prove several refined inequalities for the operator D^s . In Section 6 we prove refined Kato–Ponce inequalities for the operator J^s . Section 7 contains several counterexamples for the operator J^s . Section 8 is devoted to the proof of Theorem 1.9. Section 9 contains further divergence-free counterexamples.

2. Notation and preliminaries

In this section we introduce some notation and collect some preliminaries used in this paper.

We adopt the following convention for the Fourier transform pair:

$$(2.1) \quad \begin{aligned} (\mathcal{F}f)(\xi) &= \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx, \\ f(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix \cdot \xi} d\xi. \end{aligned}$$

The inverse Fourier transform is sometimes denoted as \mathcal{F}^{-1} so that

$$f(x) = (\mathcal{F}^{-1}(\hat{f}))(x).$$

For any $x \in \mathbb{R}^d$, we denote $\langle x \rangle = (1 + |x|^2)^{1/2}$. Similarly, for any $s \in \mathbb{R}$ we define $\langle \nabla \rangle^s$ via its Fourier transform $\widehat{\langle \nabla \rangle^s}(\xi) = (1 + |\xi|^2)^{s/2}$. In this notation, $J^s = (I - \Delta)^{s/2} = \langle \nabla \rangle^s$.

For any real number $a \in \mathbb{R}$, we denote by $a+$ the quantity $a + \epsilon$ for sufficiently small $\epsilon > 0$. The numerical value of ϵ is unimportant and the needed smallness of ϵ is usually clear from the context. The notation $a-$ is similarly defined.

We denote by $\mathcal{S}(\mathbb{R}^d)$ the space of Schwartz functions and $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions. For any integer $k \geq 0$ and open set $U \subset \mathbb{R}^d$, we shall denote by $C_{\text{loc}}^k(U)$ the space of k -times continuously differentiable functions in U . For any function $f: \mathbb{R}^d \rightarrow \mathbb{R}$, we use $\|f\|_{L^p(\mathbb{R}^d)}$, $\|f\|_{L^p}$ or sometimes $\|f\|_p$ to denote the usual Lebesgue L^p norm for $0 < p \leq \infty$. For a sequence of real numbers $(a_j)_{j=-\infty}^{\infty}$, we denote

$$(a_j)_{l_j}^p = \|(a_j)_{j \in \mathbb{Z}}\|_{l_j^p} = \begin{cases} (\sum_{j \in \mathbb{Z}} |a_j|^p)^{1/p}, & \text{if } 0 < p < \infty, \\ \sup_j |a_j|, & \text{if } p = \infty. \end{cases}$$

We shall often use mixed-norm notation. For example, for a sequence of functions $f_j: \mathbb{R}^d \rightarrow \mathbb{R}$, we will denote (below $0 < q < \infty$)

$$\|(f_j)_{l_j^q}\|_p = \left\| \left(\sum_j |f_j(x)|^q \right)^{1/q} \right\|_{L_x^p(\mathbb{R}^d)},$$

with obvious modification for $q = \infty$.

For any two operators A, B , we shall denote by

$$[A, B] = AB - BA$$

the usual commutator.

For any two quantities X and Y , we denote $X \lesssim Y$ if $X \leq CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \geq CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. The dependence of the constant C on other parameters or constants are usually clear from the context and we will often suppress this dependence. We shall denote $X \lesssim_{Z_1, Z_2, \dots, Z_k} Y$ if $X \leq CY$ and the constant C depends on the quantities Z_1, \dots, Z_k .

For any two quantities X and Y , we shall denote $X \ll Y$ if $X \leq cY$ for some sufficiently small constant c . The smallness of the constant c is usually clear from the context. The notation $X \gg Y$ is similarly defined. Note that our use of \ll and \gg here is *different* from the usual Vinogradov notation in number theory or asymptotic analysis.

We will need to use the Littlewood–Paley (LP) frequency projection operators. To fix the notation, let ϕ_0 be a radial function in $C_c^\infty(\mathbb{R}^n)$ satisfying

$$0 \leq \phi_0 \leq 1, \quad \phi_0(\xi) = 1 \text{ for } |\xi| \leq 1, \quad \phi_0(\xi) = 0 \text{ for } |\xi| \geq 7/6.$$

Let $\phi(\xi) := \phi_0(\xi) - \phi_0(2\xi)$, which is supported in $1/2 \leq |\xi| \leq 7/6$. For any $f \in \mathcal{S}(\mathbb{R}^n)$, $j \in \mathbb{Z}$, define

$$\begin{aligned} \widehat{P_{\leq j} f}(\xi) &= \phi_0(2^{-j}\xi) \hat{f}(\xi), \\ \widehat{P_j f}(\xi) &= \phi(2^{-j}\xi) \hat{f}(\xi), \quad \xi \in \mathbb{R}^n. \end{aligned}$$

We will denote $P_{> j} = I - P_{\leq j}$ (I is the identity operator). Sometimes for simplicity of notation (and when there is no obvious confusion) we will write $f_j = P_j f$, $f_{\leq j} = P_{\leq j} f$ and $f_{a \leq \cdot \leq b} = \sum_{j=a}^b f_j$. By using the support property of ϕ , we have $P_j P_{j'} = 0$ whenever $|j - j'| > 1$. This property will be useful in product decompositions. For example, the Bony paraproduct for a pair of functions f, g takes the form

$$fg = \sum_{i \in \mathbb{Z}} f_i \tilde{g}_i + \sum_{i \in \mathbb{Z}} f_i g_{\leq i-2} + \sum_{i \in \mathbb{Z}} g_i f_{\leq i-2},$$

where $\tilde{g}_i = g_{i-1} + g_i + g_{i+1}$.

The fattened operators \tilde{P}_j are defined by

$$\tilde{P}_j = \sum_{l=-n_1}^{n_2} P_{j+l},$$

where $n_1 \geq 0$, $n_2 \geq 0$ are some finite integers whose values play no role in the argument.

Note that the Littlewood–Paley projection operators P_j defined above depend on the function ϕ . Sometimes it is desirable to use a different function $\tilde{\phi} \in \mathcal{S}(\mathbb{R}^d)$. In that case, to stress the dependence on $\tilde{\phi}$, we shall denote

$$\widehat{P_j^{\tilde{\phi}} f}(\xi) = \tilde{\phi}(\xi/2^j) \hat{f}(\xi).$$

We recall the Bernstein estimates/inequalities: for $1 \leq p \leq q \leq \infty$,

$$\begin{aligned} \|D^s P_j f\|_p &\sim 2^{js} \|f\|_p, \quad s \in \mathbb{R}; \\ \|P_{\leq j} f\|_q + \|P_j f\|_q &\lesssim 2^{jd(1/p-1/q)} \|f\|_p. \end{aligned}$$

In the above, $D^s = (-\Delta)^{s/2}$.

For $s \in \mathbb{R}$, $1 \leq p \leq \infty$, the homogeneous Besov $\dot{B}_{p,\infty}^s$ (semi-)norm is given by

$$\|f\|_{\dot{B}_{p,\infty}^s} = \sup_{j \in \mathbb{Z}} (2^{js} \|P_j f\|_p).$$

For any $f \in L_{\text{loc}}^1(\mathbb{R}^d)$, the BMO norm is given by

$$\|u\|_{\text{BMO}} = \sup_Q \frac{1}{|Q|} \int_Q |u(y) - u_Q| dy,$$

where u_Q is the average of u on Q , and the supreme is taken over all cubes Q in \mathbb{R}^d .

It is well known that the Besov $\dot{B}_{\infty,\infty}^0$ norm is weaker than the BMO norm. The following proposition records this fact. We omit the (standard) proof. For any $\lambda > 0$, $x_0 \in \mathbb{R}^d$, denote

$$f^{\lambda, x_0}(y) = f(x_0 + \lambda y), \quad y \in \mathbb{R}^d.$$

Proposition 2.1. *For any $f \in L_{\text{loc}}^1(\mathbb{R}^d)$, we have*

$$\|f\|_{\dot{B}_{\infty,\infty}^0} \lesssim_d \sup_{\lambda > 0, x_0 \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f^{\lambda, x_0}(y) - (f^{\lambda, x_0})_{Q_1}|}{(1 + |y|)^{d+1}} dy \lesssim_d \|f\|_{\text{BMO}},$$

where $Q_1 = [-1, 1]^d$.

We recall the definition of Hardy space $\mathcal{H}^1 = \{f \in L^1(\mathbb{R}^d) : \mathcal{R}_j f \in L^1, \forall 1 \leq j \leq d\}$ with the norm

$$\|f\|_{\mathcal{H}^1} = \|f\|_1 + \|\mathcal{R}f\|_1,$$

where $\mathcal{R}f = |\nabla|^{-1} \nabla f =: (\mathcal{R}_1, \dots, \mathcal{R}_d f)$. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ with $\int \varphi = 1$. Define $\varphi_t(x) = t^{-d} \varphi(x/t)$. An equivalent norm on \mathcal{H}^1 is given by (see [10])

$$\|f\|_{\mathcal{H}^1} = \left\| \sup_{t > 0} |\varphi_t \star f| \right\|_1.$$

Similarly, for $0 < p \leq 1$,

$$(2.2) \quad \|f\|_{\mathcal{H}^p} = \left\| \sup_{t>0} |\varphi_t \star f| \right\|_p.$$

Alternatively one can use the following characterization (cf. [11]):

$$\|f\|_{\mathcal{H}^p} \sim \|(P_j f)_{l_j^2}\|_p.$$

We will often use this latter characterisation without explicit mentioning.

We will sometimes use (cf. (3.5) in the proof of Lemma 3.1) the following useful fact: if $f \in L_{\text{loc}}^1(\mathbb{R}^d)$ satisfies $\|f\|_{\text{BMO}} < \infty$, $g \in \mathcal{H}^1(\mathbb{R}^d)$, and

$$\int_{\mathbb{R}^d} |f(x)g(x)| dx < \infty,$$

then

$$(2.3) \quad \left| \int_{\mathbb{R}^d} f(x)g(x) dx \right| \lesssim \|f\|_{\text{BMO}} \|g\|_{\mathcal{H}^1}.$$

Of course, without absolute convergence the \mathcal{H}^1 -BMO pairing is defined through a careful limiting process.

For any $f \in L_{\text{loc}}^1(\mathbb{R}^d)$, we shall denote by $\mathcal{M}f: \mathbb{R}^d \rightarrow [0, \infty]$ the usual Hardy–Littlewood maximal function defined as

$$(\mathcal{M}f)(x) = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f(x-y)| dy,$$

where $B_r = B(0, r)$ is the Euclidean open ball of radius r centered at the origin.

We will often use the following Fefferman–Stein inequality without explicit mentioning.

Lemma 2.2 ([9]). *Let $f = (f_j)_{j=1}^\infty$ be a sequence of locally integrable functions in \mathbb{R}^d . Let $1 < p < \infty$ and $1 < r \leq \infty$. Then*

$$\|(\mathcal{M}f_j)_{l_j^r}\|_p \lesssim_{r,p,d} \|(f_j)_{l_j^r}\|_p,$$

where $\mathcal{M}f$ is the usual Hardy–Littlewood maximal function.

Proof. See [9]. Note that the inequality therein was stated for $1 < r < \infty$. But for $r = \infty$ the inequality also holds trivially. \square

Lemma 2.3. *Suppose $u \in \mathcal{S}'(\mathbb{R}^d)$ with $\text{supp}(\hat{u}) \subset \{\xi : |\xi| < t\}$ for some $t > 0$. Then for any $0 < r < \infty$,*

$$\sup_{z \in \mathbb{R}^d} \frac{|u(x-z)|}{(1+t|z|)^{d/r}} \lesssim_{d,r} (\mathcal{M}(|u|^r)(x))^{1/r}, \quad \forall x \in \mathbb{R}^d.$$

In the above, $\mathcal{M}f$ denotes the usual Hardy–Littlewood maximal function.

Remark 2.4. For textbook proofs under slightly stronger conditions, see [20] or [11]. For example, in [11], the proof therein assumes the growth condition

$$(2.4) \quad \sup_{x \in \mathbb{R}^d} \frac{|u(x)|}{(1 + |x|)^{d/r}} < \infty.$$

Here we show that this condition can be removed. The removal of such conditions is particularly interesting for $0 < r < 1$.

Proof of Lemma 2.3. First note that $u \in C^\infty(\mathbb{R}^d)$ since \hat{u} is compactly supported. By Paley–Wiener (for distributions), the function u grows at most polynomially at the spatially infinity. More precisely, there exists an integer $N_0 \geq 0$ and a constant $A_0 > 0$, such that

$$(2.5) \quad |u(y)| \leq A_0 |y|^{N_0}, \quad \forall |y| \geq 2.$$

This estimate will be used below. Note that both constants (A_0 and N_0) may depend on u .

By scaling one can assume $t = 1$. Also by translation it suffices to prove the case $x = 0$. Since $u = P_{<2}u$, we have

$$u(z) = \int_{\mathbb{R}^d} \psi(z - y) u(y) dy,$$

where $\psi \in \mathcal{S}(\mathbb{R}^d)$ corresponds to $P_{<2}$. The convergence of the integral is not an issue thanks to the estimate (2.5).

Consider first the case $r \geq 1$. Clearly

$$\begin{aligned} |u(z)| &\lesssim_{d,r} \int_{|y-z| \leq 1+|z|} |u(y)| dy + \sum_{i=1}^{\infty} (2^i(1+|z|))^{-10d} \int_{|y-z| \sim 2^i(1+|z|)} |u(y)| dy \\ &\lesssim_{d,r} (\mathcal{M}(|u|^r)(0))^{1/r} \cdot (1+|z|)^{d/r}. \end{aligned}$$

Thus this case is OK.

Next consider the case $0 < r < 1$. We first assume the growth condition (2.4) and complete the estimate. We have

$$\begin{aligned} |u(z)| &\lesssim_{d,r} \int |\psi(z - y)| \cdot |u(y)|^r \cdot |u(y)|^{1-r} dy \\ &\lesssim_{d,r} \int |\psi(z - y)| |u(y)|^r (1 + |y|)^{d(1-r)/r} dy \left(\sup_{\tilde{y} \in \mathbb{R}^d} \frac{|u(\tilde{y})|}{(1 + |\tilde{y}|)^{d/r}} \right)^{1-r} \\ &\lesssim_{d,r} (1 + |z|)^{d/r} \cdot \mathcal{M}(|u|^r)(0) \cdot \left(\sup_{\tilde{y} \in \mathbb{R}^d} \frac{|u(\tilde{y})|}{(1 + |\tilde{y}|)^{d/r}} \right)^{1-r}. \end{aligned}$$

The desired inequality then follows.

Finally we show how to prove (2.4) for the case $0 < r < 1$. For any $|z| = R \geq 2$, by using (2.5), it is easy to check that

$$\begin{aligned} |u(z)| &\lesssim_{d,r,u} \int_{|y-z| > \frac{2}{3}R} \langle z-y \rangle^{-10d-N_0} |u(y)| dy + \int_{|y-z| \leq \frac{2}{3}R} |u(y)| dy \\ &\lesssim_{d,r,u} 1 + \mathcal{M}(|u|^r)(0) \cdot R^d \max_{|\tilde{y}| \leq 2R} |u(\tilde{y})|^{1-r}. \end{aligned}$$

Define, for $R \geq 2$,

$$U_R = 1 + \max_{|y| \leq R} |u(y)|.$$

Note that we may assume $\mathcal{M}(|u|^r)(0) < \infty$. Otherwise there is nothing to prove. It follows that, for some positive constant $B = B(u, d, r) \geq 2$,

$$U_R \leq B \cdot R^d \cdot U_{2R}^{1-r}, \quad \forall R \geq 2.$$

We inductively assume

$$U_R \leq A_k R^{N_k}, \quad \forall R \geq 2.$$

The base case $k = 0$ certainly holds in view of (2.5). Then

$$U_R \leq B \cdot A_k^{1-r} 2^{N_k(1-r)} \cdot R^{d+(1-r)N_k}.$$

Set $N_{k+1} = d + (1-r)N_k$. Clearly $N_k \rightarrow d/r$ as $k \rightarrow \infty$ and $N_k \leq M$ (for some $M > 0$) for all k . Set

$$A_{k+1} = B \cdot 2^{M(1-r)} A_k^{1-r}.$$

It is easy to check that A_k converges to some constant $A = B^{1/r} 2^{M(1-r)/r}$ as $k \rightarrow \infty$. Thus (2.4) is proved. \square

Lemma 2.5. *Suppose $(f_j)_{j \in \mathbb{Z}}$ is a sequence of functions satisfying*

$$\text{supp}(\widehat{f_j}) \subset \{\xi : |\xi| \leq B_1 2^j\}, \quad \forall j,$$

where $B_1 > 0$ is a constant. Then for any $0 < p, q < \infty$, we have

$$\|(P_j f_j)_{l_j^q}\|_p \lesssim \|(f_j)_{l_j^q}\|_p.$$

In the above P_j can be replaced by \tilde{P}_j or $\tilde{P}_{\leq j}$.

On the other hand, if $1 < p, q < \infty$, and $(g_j)_{j \in \mathbb{Z}}$ is a sequence of functions, then

$$(2.6) \quad \|(P_j g_j)_{l_j^q}\|_p \lesssim \|(g_j)_{l_j^q}\|_p.$$

Proof of Lemma 2.5. The inequality (2.6) follows easily from the simple fact that $|P_j g_j| \lesssim \mathcal{M}g_j$. Therefore we only need to prove the first inequality.

By Lemma 2.3,

$$|(P_j f_j)(x)| \lesssim \int 2^{jd} |\psi(2^j y)| \cdot (1 + 2^j |y|)^{d/r} \cdot \frac{|f_j(x-y)|}{(1 + 2^j |y|)^{d/r}} dy \lesssim (\mathcal{M}(|f_j|^r)(x))^{1/r}.$$

Now choose $0 < r < \min\{p, q\}$ and use Lemma 2.2. We get

$$\begin{aligned} \|(P_j f_j)_{l_j^q}\|_p &\lesssim \|(\mathcal{M}(|f_j|^r))_{l_j^{q/r}}^{1/r}\|_p = \|(\mathcal{M}(|f_j|^r))_{l_j^{q/r}}\|_{p/r}^{1/r} \\ &\lesssim \|(|f_j|^r)_{l_j^{q/r}}\|_{p/r}^{1/r} = \|(f_j)_{l_j^q}\|_p. \end{aligned} \quad \square$$

The following lemma is more or less trivial. But it is certainly relieving for some intermediate computations.

Lemma 2.6. *Let $L \geq 2$ be an integer. Suppose $f_j \in \mathcal{S}(\mathbb{R}^d)$, $j = -L, -L + 1, \dots, L - 1, L$, is a (finite) sequence of functions satisfying*

$$\text{supp}(\widehat{f_j}) \subset \{\xi : |\xi| < B_1 2^j\}, \quad \forall j,$$

for some constant $B_1 > 0$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be such that $\psi \equiv 0$ in a neighborhood of the origin. Then for all $1 < p < \infty$,

$$\left\| \sum_{j=-L}^L P_j^\psi f_j \right\|_p \lesssim_{p, \psi, B_1, d} \|(f_j)_{l_j^2}\|_p;$$

and for $0 < p \leq 1$,

$$\left\| \sum_{j=-L}^L P_j^\psi f_j \right\|_{\mathcal{H}^p} \lesssim_{p, \psi, B_1, d} \|(f_j)_{l_j^2}\|_p.$$

Proof of Lemma 2.6. Assume $\text{supp}(\psi) \subset \{\xi : 2^{-n_0} < |\xi| < 2^{n_0}\}$ for some integer $n_0 > 0$. Then clearly for all $0 < p < \infty$,

$$\left\| \left(P_k \left(\sum_{j=-L}^L P_j^\psi f_j \right) \right)_{l_k^2} \right\|_p \lesssim \sum_{|a| \leq n_0 + 10} \|(P_k P_{k+a}^\psi f_{k+a})_{l_k^2}\|_p \lesssim \|(f_j)_{l_j^2}\|_p,$$

where the last inequality follows from Lemma 2.5. □

Lemma 2.7. *Let $(a_j)_{j \in \mathbb{Z}}$ be a sequence of real numbers. Then for any $0 < \theta < 1$, $0 < p \leq \infty$, $s_1 \neq s_2$, $s = \theta s_1 + (1 - \theta)s_2$, we have*

$$\begin{aligned} (2^{js} a_j)_{l_j^p} &\lesssim_{\theta, p, s_1, s_2} |(2^{js_1} a_j)_{l_j^\infty}|^\theta \cdot |(2^{js_2} a_j)_{l_j^2}|^{1-\theta}; \\ (2^{js} a_j)_{l_j^p} &\lesssim_{\theta, p, s_1, s_2} |(2^{js_1} a_j)_{l_j^p}|^\theta \cdot |(2^{js_2} a_j)_{l_j^\infty}|^{1-\theta}. \end{aligned}$$

Remark 2.8. The condition $s_1 \neq s_2$ is crucial. For $s = s_1 = s_2$ the above inequalities are obviously false (unless $p = \infty$).

Proof of Lemma 2.7. The case $p = \infty$ is trivial. For $0 < p < \infty$ since $(2^{js} a_j)_{l_j^p}^p = (2^{jps} |a_j|^p)_{l_j^1}$, it suffices to prove the case $p = 1$, and we may assume $a_j \geq 0$. It is clear that the second inequality follows from the first inequality by using the the fact that $l_j^1 \hookrightarrow l_j^\infty$. Consider first the case $s_1 < s_2$. Let J_0 be chosen later. Then

$$\begin{aligned} \sum_j 2^{j(\theta s_1 + (1-\theta)s_2)} a_j &= \sum_{j \leq J_0} 2^{js_1} a_j \cdot 2^{j(1-\theta)(s_2-s_1)} + \sum_{j > J_0} 2^{js_2} a_j \cdot 2^{-j(1-\theta)(s_2-s_1)} \\ &\lesssim (2^{js_1} a_j)_{l_j^\infty} \cdot 2^{J_0(1-\theta)(s_2-s_1)} + (2^{js_2} a_j)_{l_j^\infty} \cdot 2^{-J_0(1-\theta)(s_2-s_1)}. \end{aligned}$$

Optimizing in J_0 then yields the result. The argument for $s_1 > s_2$ is similar. \square

Remark 2.9. Other proofs are available. For example (for the second inequality) if $s > 0$, $s_1 = s$, $s_2 = 0$, $\theta = 1/2$, $p = 1$, then

$$\begin{aligned} (2^{j\frac{1}{2}s} a_j)_{l_j^1}^2 &\lesssim \sum_{j_1 \leq j_2} 2^{j_1\frac{1}{2}s} 2^{j_2\frac{1}{2}s} |a_{j_1}| |a_{j_2}| \\ &\lesssim (a_j)_{l_j^\infty} \sum_{j_2} 2^{j_2 s} |a_{j_2}| \lesssim (a_j)_{l_j^\infty} (2^{js} a_j)_{l_j^1}. \end{aligned}$$

One should recognise this as the usual “squaring trick” with low to high re-ordering. See Remark 2.12 below.

Lemma 2.10. *Let $1 < r \leq \infty$, $0 < \theta < 1$ and recall $D = (-\Delta)^{1/2}$. Then for any $f \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\begin{aligned} \|D^{\theta s} f\|_r &\lesssim \|D^s f\|_p^\theta \cdot \|f\|_q^{1-\theta}, \quad \text{if } s \geq 0, 1 < p, q < \infty, \text{ and } \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}; \\ \|D^{\theta s} f\|_r &\lesssim \|D^s f\|_{\dot{B}_{\infty, \infty}^\theta}^\theta \cdot \|f\|_{r(1-\theta)}^{1-\theta}, \quad \text{if } s > 0 \\ &\quad (\text{this corresponds to } p = \infty, q = r(1-\theta)); \\ \|D^{\theta s} f\|_r &\lesssim \|D^s f\|_{r\theta}^\theta \cdot \|f\|_{\dot{B}_{\infty, \infty}^{1-\theta}}^{1-\theta}, \quad \text{if } s > 0 \text{ (correspondingly } p = r\theta, q = \infty). \end{aligned}$$

Remark. The second and third inequalities are false for $s = 0$.

Remark 2.11. By the same proof of lemma 2.10 below, one can show that more generally for $0 < \theta < 1$, $s = s_1\theta + s_2(1-\theta)$ with $s_1 \neq s_2$, $1/\theta < r \leq \infty$,

$$\|D^s f\|_r \lesssim \|D^{s_1} f\|_{r\theta}^\theta \|D^{s_2} f\|_{\dot{B}_{\infty, \infty}^{1-\theta}}^{1-\theta},$$

provided (of course) that these quantities are well-defined (especially when dealing with D^s operators with $s < 0$).

Proof of Lemma 2.10. The inequalities are trivial for $r = \infty$. Also the first inequality is trivial if $s = 0$. Thus we may assume $s > 0$ and $1 < r < \infty$. By Lemma 2.7,

we have

$$\begin{aligned} \|D^{\theta s} f\|_r &\sim \|(2^{j\theta s} f_j)_{l_j^2}\|_r \lesssim \|(2^{js} f_j)_{l_j^2}^\theta (f_j)_{l_j^\infty}^{1-\theta}\|_r \\ &\lesssim \|(2^{js} f_j)_{l_j^2}\|_p^\theta \cdot \|(f_j)_{l_j^\infty}\|_q^{1-\theta} \lesssim \|D^s f\|_p^\theta \cdot \|f\|_q^{1-\theta}, \quad \text{if } p, q < \infty. \end{aligned}$$

The argument for the second and third inequalities are similar. We omit details. \square

Remark 2.12. By using Remark 2.11 and taking $r = 2d/(d-2)$, $\theta = (d-2)/d$, $s = 0$, $s_1 = 1$, $s_2 = -(d-2)/2$, we obtain for $d \geq 3$ and $f \in \mathcal{S}(\mathbb{R}^d)$:

$$(2.7) \quad \|f\|_{2d/(d-2)} \lesssim \|\nabla f\|_2^{(d-2)/d} \|f\|_{\dot{B}_{\infty, \infty}^{-(d-2)/2}}^{2/d}.$$

By the obvious embedding $\dot{B}_{2d/(d-2), \infty}^0 \hookrightarrow \dot{B}_{\infty, \infty}^{-(d-2)/2}$, we obtain

$$\|f\|_{2d/(d-2)} \lesssim \|\nabla f\|_2^{(d-2)/d} \|f\|_{\dot{B}_{2d/(d-2), \infty}^0}^{2/d}.$$

This refined inequality plays an important role³ in the theory of linear profile decomposition for nonlinear Schrödinger equations (cf. Proposition 4.8 on page 36 of [16]). Note that the proof therein uses a squaring trick and a low to high ordering of the dyadic sum, in a way that is similar to the idea in Remark 2.9. Our treatment here seems much simpler.

We shall often use the following simple lemma, sometimes without explicit mentioning.

Lemma 2.13. *The following hold.*

- If $s > 0$, $1 < p < \infty$, then

$$\|(2^{-js} D^s P_{\leq j} f)_{l_j^2}\|_p + \|(2^{-js} D^s P_{\leq j} f)_{l_j^\infty}\|_p \lesssim_{s,p,d} \|f\|_p.$$

- If $s > 0$, $p = \infty$, then

$$\|(2^{-js} D^s P_{\leq j} f)_{l_j^\infty}\|_\infty \lesssim \|f\|_{\dot{B}_{\infty, \infty}^0}.$$

Proof of Lemma 2.13. We have

$$|2^{-js} (D^s P_{\leq j} f)(x)| = 2^{-js} \left| \sum_{k \leq j} (D^s \tilde{P}_k P_k f)(x) \right| \lesssim 2^{-js} \sum_{k \leq j} 2^{ks} \mathcal{M}(P_k f)(x).$$

Since $s > 0$, we have

$$\left(2^{-js} \sum_{k \leq j} 2^{ks} \mathcal{M}(P_k f) \right)_{l_j^2} \lesssim (\mathcal{M}(P_k f))_{l_k^2}.$$

³In fact the first inequality (2.7) already provides a quick route to the extraction of “bubbles” in linear profile decomposition.

Thus

$$\|(2^{-js} D^s P_{\leq j} f)_{l_j^2}\|_p \lesssim \|(\mathcal{M}(P_k f))_{l_k^2}\|_p \lesssim \|f\|_p.$$

Since the sequence l^2 norm controls l^∞ norm, the inequality for l_j^∞ follows. Finally,

$$\begin{aligned} \|(2^{-js} D^s P_{\leq j} f)_{l_j^\infty}\|_\infty &\lesssim (2^{-js} \|D^s P_{\leq j} f\|_\infty)_{l_j^\infty} \\ &\lesssim \left(2^{-js} \sum_{k \leq j} 2^{ks} \|f\|_{\dot{B}_{\infty, \infty}^0}\right)_{l_j^\infty} \lesssim \|f\|_{\dot{B}_{\infty, \infty}^0}. \quad \square \end{aligned}$$

The following lemma collects some useful properties of J^s and D^s operators. We will often use it without explicit mentioning in later computations.

Lemma 2.14 (Properties of J^s , D^s operators). *Let $s > 0$ and recall $D^s = (-\Delta)^{s/2}$, $J^s = (1 - \Delta)^{s/2}$. Let a be any given real number. Then the following inequalities hold for any $f \in \mathcal{S}(\mathbb{R}^d)$:*

$$(2.8) \quad \|P_{>a} D^s f\|_p \lesssim_{a,p,d,s} \|J^{s-1} \partial f\|_p, \quad \forall 1 < p < \infty,$$

$$(2.9) \quad \|P_j f\|_p \lesssim_{d,s} \begin{cases} 2^{-j} \|J^{s-1} \partial f\|_p, & \text{if } j \leq 0, 1 \leq p \leq \infty \\ 2^{-js} \|J^{s-1} \partial f\|_p, & \text{if } j > 0, 1 \leq p \leq \infty; \end{cases}$$

$$(2.10) \quad \|J^s f - f\|_p \lesssim_{p,d,s} \|J^{s-1} \partial f\|_p, \quad \forall 1 < p < \infty,$$

$$(2.11) \quad \|J^{-1} \partial f\|_{\text{BMO}} \lesssim_d \|f\|_{\text{BMO}},$$

$$(2.12) \quad \|J^s P_{>a} f\|_{\text{BMO}} \lesssim_{a,d,s} \|J^{s-1} \partial f\|_{\text{BMO}}.$$

Proof of Lemma 2.14. For (2.8), we write

$$P_{>a} D^s f = P_{>a} D^{s-2} (-\partial \cdot \partial) f = -P_{>a} D^{s-2} J^{-(s-1)} \partial \cdot (J^{s-1} \partial f).$$

It is easy to check that $-P_{>a} D^{s-2} J^{-(s-1)} \partial$ maps L^p to L^p for $1 < p < \infty$. Thus (2.8) holds.

For (2.9), consider first $j \leq 0$. Clearly

$$\begin{aligned} \|P_j f\|_p &= \|P_j D^{-2} \partial J^{1-s} \cdot J^{s-1} \partial f\|_p \lesssim_d 2^{-j} \|P_{<10} J^{1-s} \cdot J^{s-1} \partial f\|_p \\ &\lesssim_{s,d} 2^{-j} \|J^{s-1} \partial f\|_p. \end{aligned}$$

For $j > 0$, one can easily verify that the kernel $K_j = D^{-2} J^{1-s} \partial P_j \delta_0$ satisfies the point-wise bound

$$|K_j(x)| \lesssim_{d,m,s} 2^{j(-s+d)} (1 + 2^j |x|)^{-m}, \quad \forall m \geq 1.$$

Thus the inequality for $j > 0$ also holds.

For (2.10), we first bound the low frequency piece. By using Lemma 6.1, we have

$$\begin{aligned} \|(J^s - 1) P_{\leq 1} f\|_p &\lesssim_{s,p,d} \sum_{j \leq 1} 2^{2j} \|P_j f\|_p \\ &\lesssim_{s,p,d} \sum_{j \leq 1} 2^j \|J^{s-1} \partial f\|_p \lesssim \|J^{s-1} \partial f\|_p, \end{aligned}$$

where in the second inequality we have used (2.9). For the high frequency piece, we first note that

$$\|P_{>1}f\|_p \lesssim_{s,p,d} \sum_{j>1} 2^{-js} \|J^{s-1}\partial f\|_p \lesssim \|J^{s-1}\partial f\|_p.$$

On the other hand, by writing $P_{>1}J^s f = -P_{>1}D^{-2}J\partial \cdot J^{s-1}\partial f$, it is clear that

$$\|P_{>1}J^s f\|_p \lesssim_{s,p,d} \|J^{s-1}\partial f\|_p, \quad \forall 1 < p < \infty.$$

The inequality (2.11) follows easily from the fact that $J^{-1}\partial$ is a standard singular integral operator.

The last inequality (2.12) is similarly proved by writing

$$J^s P_{>a}f = -P_{>a}D^{-2}J\partial \cdot J^{s-1}\partial f. \quad \square$$

3. Paraproduct estimates

Lemma 3.1. *Let $\phi \in C_c^\infty(\mathbb{R}^d)$, $\psi \in C_c^\infty(\mathbb{R}^d)$, and $\psi \equiv 0$ in a neighborhood of the origin. Define*

$$\begin{aligned} \widehat{P_j^\phi f}(\xi) &= \phi(\xi/2^j) \widehat{f}(\xi), \\ \widehat{P_j^\psi f}(\xi) &= \psi(\xi/2^j) \widehat{f}(\xi), \quad \xi \in \mathbb{R}^d. \end{aligned}$$

Then for any $1 < p < \infty$, $f \in L^p(\mathbb{R}^d)$, $g \in L_{\text{loc}}^1(\mathbb{R}^d)$ with $\|g\|_{\text{BMO}} < \infty$, the series

$$\sum_{j \in \mathbb{Z}} P_j^\phi f \cdot P_j^\psi g$$

converges in L^p , and satisfies

$$(3.1) \quad \left\| \sum_{j \in \mathbb{Z}} P_j^\phi f \cdot P_j^\psi g \right\|_p \lesssim_{p,d,\phi,\psi} \|f\|_p \|g\|_{\text{BMO}}.$$

In particular, for the usual Littlewood–Paley projector P_j , we have

$$\left\| \sum_j P_{\leq j} f \cdot P_j g \right\|_p \lesssim_{p,d} \|f\|_p \|g\|_{\text{BMO}}.$$

If $\|D^{-1}f\|_p < \infty$ and $\|Dg\|_{\text{BMO}} < \infty$, then

$$(3.2) \quad \left\| \sum_j P_{\leq j} f \cdot P_j g \right\|_p \lesssim_{p,d} \|D^{-1}f\|_p \|Dg\|_{\text{BMO}}.$$

More generally, if $s > 0$, $f \in L^p$, $g \in L_{\text{loc}}^1(\mathbb{R}^d)$ with $\|g\|_{\text{BMO}} < \infty$, then

$$(3.3) \quad \left\| \sum_j D^s P_{\leq j} f \cdot D^{-s} P_j g \right\|_p \lesssim_{p,d,s} \|f\|_p \|g\|_{\text{BMO}}.$$

Remark 3.2. The estimate (3.1) actually holds under the weaker assumption that $\psi(0) = 0$. But the argument is slightly more involved.

Proof of Lemma 3.1. For simplicity of notation we shall write $\lesssim_{p,d,\phi,\psi}$ as \lesssim .

Step 1: We first show for any integer $L_1 \geq 2$, $L_2 \geq 2$,

$$(3.4) \quad \left\| \underbrace{\sum_{-L_1 \leq j \leq L_2} P_j^\phi f \cdot P_j^\psi g}_{=:S} \right\|_p \lesssim \|f\|_p \|g\|_{\text{BMO}}.$$

Note that the summand is well-defined in L^p since $\|P_j^\psi g\|_\infty \lesssim \|g\|_{\text{BMO}}$. We shall write $\sum_{-L_1 \leq j \leq L_2}$ simply as \sum_j and keep in mind that the summation in j is finite. By a density argument we only need to prove (3.4) for $f \in C_c^\infty(\mathbb{R}^d)$.

It suffices to prove that, for any $h \in C_c^\infty(\mathbb{R}^d)$,

$$|\langle S, h \rangle| \lesssim \|f\|_p \|g\|_{\text{BMO}} \|h\|_{p'},$$

where $p' = p/(p-1)$, and $\langle \cdot, \cdot \rangle$ denotes the usual L^2 pairing.

Now observe

$$(3.5) \quad |\langle S, h \rangle| = \left| \left\langle g, \sum_j P_j^\psi (h P_j^\phi f) \right\rangle \right| \lesssim \|g\|_{\text{BMO}} \left\| \sum_j P_j^\psi (h P_j^\phi f) \right\|_{\mathcal{H}^1},$$

where we used (2.3).

Thus we only need to show

$$\left\| \sum_j P_j^\psi (h P_j^\phi f) \right\|_{\mathcal{H}^1} \lesssim \|h\|_{p'} \|f\|_p.$$

Without loss of generality, we assume $P_j^\phi = P_{\leq j}$, $P_j^\psi = P_j$ the usual Littlewood–Paley projectors. The argument can be easily modified for the general case.

By frequency localization,

$$\begin{aligned} \sum_j P_j (P_{\leq j} f h) &= \sum_j P_j (P_{\leq j-3} f P_{j-2 < \cdot < j+2} h) + \sum_j P_j (P_{j-2 \leq \cdot \leq j} f P_{\leq j+3} h) \\ &=: \sum_j P_j (f_{\leq j-3} \tilde{h}_j) + \sum_j P_j (\tilde{f}_j h_{\leq j+3}). \end{aligned}$$

Clearly, by Lemma 2.6,

$$\left\| \sum_j P_j (f_{\leq j-3} \tilde{h}_j) \right\|_{\mathcal{H}^1} \lesssim \|(f_{\leq j-3} \tilde{h}_j)_{l_j^2}\|_1 \lesssim \|(f_{\leq j-3})_{l_j^\infty} (\tilde{h}_j)_{l_j^2}\|_1 \lesssim \|f\|_p \|h\|_{p'}.$$

The other piece is estimated similarly. Thus (3.4) holds.

Step 2: Convergence of the series in L^p . First observe that for any $M_1 > L_1 \geq 2$,

$$\left\| \sum_{-M_1 < j < -L_1} P_j^\phi f P_j^\psi g \right\|_p \lesssim \|\tilde{P}_{< -L_1+10} f\|_p \|g\|_{\text{BMO}} \rightarrow 0, \quad \text{as } L_1 \rightarrow \infty,$$

where $\tilde{P}_{<-L_1+10}$ denotes a fattened frequency projection. We only need to show for any $M_2 > L_2 \geq 2$,

$$\left\| \sum_{L_2 < j < M_2} P_j^\phi f P_j^\psi g \right\|_p \rightarrow 0, \quad \text{as } L_2 \rightarrow \infty.$$

By a density argument we may assume $f \in C_c^\infty(\mathbb{R}^d)$. It is easy to check that

$$\|\phi(0)f - P_j^\phi f\|_p \lesssim 2^{-j} \|\nabla f\|_p.$$

Thus

$$\left\| \sum_{L_2 < j < M_2} (\phi(0)f - P_j^\phi f) P_j^\psi g \right\|_p \lesssim \sum_{L_2 < j < M_2} 2^{-j} \|\nabla f\|_p \|g\|_{\text{BMO}} \rightarrow 0, \quad \text{as } L_2 \rightarrow \infty.$$

It remains for us to show

$$(3.6) \quad \left\| f \sum_{L_2 < j < M_2} P_j^\psi g \right\|_p \rightarrow 0, \quad \text{as } L_2 \rightarrow \infty.$$

Assume $\text{supp}(f) \subset B(0, R_0)$ for some $R_0 \geq 2$. Let $\chi \in C_c^\infty$ be such that $\chi(x) = 1$ for $|x| \leq 2R_0$ and $\chi(x) = 0$ for $|x| > 3R_0$. Denote the average of g on $B(0, 1)$ as g_{B_1} . Then it is not difficult to check that

$$\begin{aligned} & \sum_{L_2 < j < M_2} \|f P_j^\psi((1 - \chi)(g - g_{B_1}))\|_p \\ & \lesssim \sum_{L_2 < j < M_2} \|f\|_p (2^j R_0)^{-1} \|(1 + |y|)^{-(1+d)} |g(y) - g_{B_1}|\|_{L_y^1(\mathbb{R}^d)} \\ & \lesssim \sum_{L_2 < j < M_2} \|f\|_p 2^{-j} \|g\|_{\text{BMO}} \rightarrow 0, \quad \text{as } L_2 \rightarrow \infty. \end{aligned}$$

On the other hand, note that $\chi \cdot (g - g_{B_1}) \in L^q$ for any $1 \leq q < \infty$. Assuming $\text{supp}(\psi) \subset \{\xi : 2^{-n_0} < |\xi| < 2^{n_0}\}$ for some integer n_0 , then

$$\begin{aligned} & \left\| f \sum_{L_2 < j < M_2} P_j^\psi(\chi(g - g_{B_1})) \right\|_p \\ & \lesssim \|f\|_{2p} \|P_{>L_2-n_0-10}(\chi(g - g_{B_1}))\|_{2p} \rightarrow 0, \quad \text{as } L_2 \rightarrow \infty. \end{aligned}$$

Thus (3.6) is proved and the series converges in L^p .

Step 3: Proof of (3.2). In this case, one just observes that $f = -\nabla \cdot (-\Delta)^{-1} \nabla f$, and

$$\sum_j P_{\leq j} f P_j g = - \sum_{l=1}^d \sum_j (2^{-j} \partial_l P_{\leq j} \cdot (-\Delta)^{-1} \partial_l f) (2^j D^{-1} P_j D g).$$

Note that for each l one can write $2^{-j} \partial_l P_{\leq j} = P_j^{\phi_l}$, and $2^j D^{-1} P_j = P_j^\psi$, for some functions $\phi_l \in \mathcal{S}(\mathbb{R}^d)$, $\psi \in \mathcal{S}(\mathbb{R}^d)$ with ψ vanishing near the origin. The desired inequality then easily follows from (3.1).

Step 4: Proof of (3.3). This is similar to the argument in Step 1. By duality and density, it suffices to prove that, for any $f, h \in C_c^\infty(\mathbb{R}^d)$,

$$\|(D^{-s}P_j(hD^sP_{\leq j}f))\|_{l_j^2} \lesssim \|f\|_p \|h\|_{p'}.$$

One can then split h as $h_{<j-2}$ and $\tilde{h}_j = h_{[j-2, j+2]}$ and proceed to estimate

$$\|(D^{-s}P_j(hD^sP_{\leq j}f))\|_{l_j^2} \lesssim \|(h_{<j-2}\tilde{f}_j)\|_{l_j^2} + \|(\tilde{h}_j2^{-js}D^s f_{\leq j})\|_{l_j^2} \lesssim \|f\|_p \|h\|_{p'}. \quad \square$$

Remark 3.3. The BMO norm on the right-hand side of (3.1) cannot be replaced by a weaker norm such as $\|\cdot\|_{\dot{B}_{\infty, \infty}^0}$. For a counterexample one can take $p = 2$, $d = 1$, $g = f$, and we shall *disprove*

$$\left\| \sum_j (P_j f)^2 \right\|_2 \lesssim \|f\|_2 \cdot \|f\|_{\dot{B}_{\infty, \infty}^0}.$$

To this end, take $\widehat{\phi}_0 \in C_c^\infty(\mathbb{R})$ such that $0 \leq \widehat{\phi}_0(\xi) \leq 1$ for all ξ , $\widehat{\phi}_0(\xi) = 1$ for $|\xi| < 1/10$, and $\widehat{\phi}_0(\xi) = 0$ for $|\xi| > 1/5$. Then for some $\rho_0 > 0$, we have

$$|\phi_0(x)| \gtrsim 1, \quad \text{for all } |x| < \rho_0.$$

Now take

$$f = \sum_{j \geq 10} a_j \lambda_j^{1/2} \phi_0(\lambda_j x) e^{i2^j x} =: \sum_{j \geq 10} a_j f_j,$$

where $a_j = j^{-(1/2+\delta)}$, $\lambda_j = j^\epsilon$, with $1/10 > \epsilon \geq 4\delta > 0$. It is easy to check that

$$\begin{aligned} \sum_j a_j^2 < \infty &\Rightarrow \|f\|_2 < \infty, \\ \sup_j |a_j| \lambda_j^{1/2} < \infty &\Rightarrow \|f\|_{\dot{B}_{\infty, \infty}^0} < \infty. \end{aligned}$$

On the other hand, for $\rho_0/\lambda_{l+1} < |x| < \rho_0/\lambda_l$, we have

$$\sum_j |P_j f(x)|^2 = \sum_j |a_j|^2 \cdot \lambda_j \cdot |\phi_0(\lambda_j x)|^2 \gtrsim \sum_{j < l} |a_j|^2 \cdot \lambda_j.$$

Thus

$$\begin{aligned} \int \left(\sum_j |(P_j f)(x)|^2 \right) dx &\gtrsim \sum_l \left(\sum_{j < l} a_j^2 \lambda_j \right)^2 \cdot \left(\frac{\rho_0}{\lambda_l} - \frac{\rho_0}{\lambda_{l+1}} \right) \\ &\gtrsim \sum_l \left(\sum_{j < l} j^{-1-2\delta} j^\epsilon \right)^2 \cdot \frac{1}{l^{1+\epsilon}} \gtrsim \sum_l l^{2(\epsilon-2\delta)} \cdot \frac{1}{l^{1+\epsilon}} = \infty. \end{aligned}$$

Finally we should point it out that by considering real and imaginary parts, one can also make the above counterexample real-valued.

Remark 3.4. For the periodic domain $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, it is much easier to construct counterexamples to the estimate

$$\left\| \sum_j (P_j f)^2 \right\|_{L^2(\mathbb{T})} \lesssim \|f\|_{L^2(\mathbb{T})} \|f\|_{\dot{B}_{\infty, \infty}^0(\mathbb{T})}.$$

Indeed if the above estimate were true, then by Hölder, one gets for any periodic function f with mean zero,

$$\|f\|_{L^4(\mathbb{T})} \lesssim \|f\|_{\dot{B}_{\infty, \infty}^0(\mathbb{T})}.$$

Now take $\lambda_j = 4^j$, $c_j = 1/\sqrt{j}$ and consider f in the form of a lacunary series,

$$f = \sum_{j \geq 1} c_j e^{i\lambda_j x}.$$

Clearly

$$\|f\|_{\dot{B}_{\infty, \infty}^0(\mathbb{T})} \lesssim \sup_j |c_j| < \infty \quad \text{and} \quad \|f\|_{L^2(\mathbb{T})} \sim \left(\sum_j |c_j|^2 \right)^{1/2} = \infty,$$

which is an obvious contradiction.

Remark 3.5. The first part of the statements of Lemma 3.1 can also be deduced from the following proposition due to Coifman–Meyer (see [5], Chapter V. Proposition 2): let $\sigma = \sigma(\xi, \eta) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d \setminus (0, 0))$ satisfy

$$(3.7) \quad \bullet \quad |\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \lesssim_{\alpha, \beta} (|\xi| + |\eta|)^{-(|\alpha| + |\beta|)}, \quad \forall \alpha, \beta, \forall (\xi, \eta) \neq (0, 0),$$

$$(3.8) \quad \bullet \quad \sigma(\xi, 0) = 0.$$

Define

$$\sigma(D)(f, g)(x) = \int e^{ix \cdot (\xi + \eta)} \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) d\xi d\eta.$$

Then for any $1 < p < \infty$,

$$(3.9) \quad \|\sigma(D)(f, g)\|_p \lesssim_{\sigma, p, d} \|f\|_p \|g\|_{\text{BMO}}.$$

In our setting

$$\sigma(\xi, \eta) = \sum_j \phi(2^{-j}\xi) \psi(2^{-j}\eta)$$

and it is easy to check that it satisfies the conditions (3.7)–(3.8). See also Theorem 3.11 and Theorem 3.14 for more refined results.

Remark 3.6. Taking $f = g$ and using the Littlewood–Paley projection P_j in Lemma 3.1, we get

$$\left\| \sum_j f_j^2 \right\|_p \lesssim \|f\|_p \|f\|_{\text{BMO}}.$$

Fix any $0 < p_0 < p$. Then

$$\|f\|_{2p}^2 \lesssim \|f\|_p \|f\|_{\text{BMO}} \lesssim \|f\|_{p_0}^\theta \|f\|_{2p}^{1-\theta} \|f\|_{\text{BMO}},$$

where $\theta \in (0, 1)$ satisfies $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{2p}$. This in turn yields

$$\|f\|_{2p} \lesssim \|f\|_{p_0}^{p_0/(2p)} \|f\|_{\text{BMO}}^{1-p_0/(2p)}.$$

By another interpolation if necessary, one can then get for any $q > p_0$,

$$\|f\|_q \lesssim \|f\|_{p_0}^{p_0/q} \|f\|_{\text{BMO}}^{1-p_0/p},$$

which is the usual BMO refinement of Hölder interpolation inequality (albeit with no explicit constants).

Similarly, by writing

$$fg = \sum_j f_{\leq j+2} g_j + \sum_j f_{> j+2} g_j = \sum_j f_{\leq j+2} g_j + \sum_j f_j g_{< j-2}$$

and applying Lemma 3.1, we get for any $1 < p < \infty$,

$$\|fg\|_p \lesssim \|f\|_p \|g\|_{\text{BMO}} + \|g\|_p \|f\|_{\text{BMO}}.$$

These (and more) bilinear BMO type inequalities were derived by Kozono and Taniuchi [17] and have important applications in Navier–Stokes and Euler equations.

The idea of duality used in Lemma 3.1 is quite useful. For example the following well-known commutator estimate can be proved along similar lines as in the proof of Lemma 3.1.

Proposition 3.7. *Let the dimension $d \geq 2$ and let $\mathcal{R}_{ij} = \Delta^{-1} \partial_i \partial_j$, $1 \leq i, j \leq d$ be the usual Riesz transform on \mathbb{R}^d . Then for any $1 < p < \infty$, $a \in L_{\text{loc}}^1(\mathbb{R}^d)$ with $\|a\|_{\text{BMO}} < \infty$, we have*

$$\|[\mathcal{R}_{ij}, a]f\|_p \lesssim_{p,d} \|a\|_{\text{BMO}} \|f\|_p, \quad \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Proof. Without loss of generality, we prove the inequality for $\mathcal{R}_{11} = \Delta^{-1} \partial_{11}$. By duality, it suffices to prove for any $g \in C_c^\infty(\mathbb{R}^d)$,

$$\|f \mathcal{R}_{11} g - g \mathcal{R}_{11} f\|_{\mathcal{H}^1} \lesssim \|f\|_p \|g\|_{p'}.$$

Write

$$\begin{aligned} f \mathcal{R}_{11} g &= \sum_j f_{<j-2} \mathcal{R}_{11} g_j + \sum_j f_j \mathcal{R}_{11} g_{<j-2} + \sum_j f_j \mathcal{R}_{11} \tilde{g}_j, \\ (\mathcal{R}_{11} f) g &= \sum_j (\mathcal{R}_{11} f)_{<j-2} g_j + \sum_j (\mathcal{R}_{11} f)_j g_{<j-2} + \sum_j (\mathcal{R}_{11} f)_j \tilde{g}_j. \end{aligned}$$

It is easy to check that

$$\begin{aligned} \left\| \sum_j f_{<j-2} \mathcal{R}_{11} g_j \right\|_{\mathcal{H}^1} &\lesssim \left\| (f_{<j-2} \mathcal{R}_{11} g_j)_{l_j^2} \right\|_1 \lesssim \|f\|_p \|g\|_{p'}, \\ \left\| \sum_j (f_j \mathcal{R}_{11} g_{<j-2} + (\mathcal{R}_{11} f)_{<j-2} g_j + (\mathcal{R}_{11} f)_j g_{<j-2}) \right\|_{\mathcal{H}^1} &\lesssim \|f\|_p \|g\|_{p'}. \end{aligned}$$

For the diagonal piece, denote $A = \Delta^{-1} f_j$, $B = \Delta^{-1} \tilde{g}_j$, and observe

$$f_j \mathcal{R}_{11} \tilde{g}_j - (\mathcal{R}_{11} f_j) \tilde{g}_j = \sum_{k=1}^d (\partial_{kk} A \partial_{11} B - \partial_{11} A \partial_{kk} B).$$

In terms of the frequency variables $(\eta, \xi - \eta)$ (i.e., $\hat{A}(\eta)$, $\hat{B}(\xi - \eta)$), note that

$$\eta_k^2 (\xi_1 - \eta_1)^2 - \eta_1^2 (\xi_k - \eta_k)^2 = \eta_k^2 \xi_1^2 - 2\eta_k^2 \eta_1 \xi_1 - \eta_1^2 \xi_k^2 + 2\eta_1^2 \eta_k \xi_k.$$

Therefore we can write

$$\sum_{k=1}^d (\partial_{kk} A \partial_{11} B - \partial_{11} A \partial_{kk} B) = O(\partial^2(\partial^2 A \cdot B)) + O(\partial(\partial^3 A \cdot B)).$$

Clearly then

$$\begin{aligned} \left\| \sum_j (f_j \mathcal{R}_{11} \tilde{g}_j - \mathcal{R}_{11} f_j \tilde{g}_j) \right\|_{\mathcal{H}^1} &\lesssim \sum_k 2^{2k} \sum_{j \geq k-4} \|\tilde{P}_j f \cdot 2^{-2j} \tilde{P}_j g\|_1 + \sum_k 2^k \sum_{j \geq k-4} 2^{-j} \|\tilde{P}_j f \tilde{P}_j g\|_1 \\ &\lesssim \|(\tilde{P}_j f \cdot \tilde{P}_j g)_{l_j^1}\|_1 \lesssim \|f\|_p \|g\|_{p'}. \end{aligned}$$

The desired inequality then follows. \square

Proposition 3.8. *Denote by H the usual Hilbert transform on \mathbb{R} . Then for any $1 < p < \infty$, and for any integers $l, m \geq 0$, we have*

$$\|\partial_x^l [H, a] \partial_x^m f\|_p \lesssim_{l,m,p} \|\partial_x^{l+m} a\|_{\text{BMO}} \|f\|_p.$$

Remark 3.9. In [7], Dawson, McGahagan and Ponce proved

$$\|\partial_x^l [H, a] \partial_x^m f\|_p \lesssim_{l,m,p} \|\partial_x^{l+m} a\|_{\infty} \|f\|_p.$$

Here Proposition 3.8 gives a slight improvement replacing the L^∞ -norm by BMO-norm.

Proof of Proposition 3.8. Write

$$(\partial_x^l [H, a] \partial_x^m f)(x) = \frac{1}{(2\pi)^2} \int \sigma(\xi, \eta) \hat{a}(\xi) \hat{f}(\eta) e^{i(\xi+\eta) \cdot x} d\xi d\eta,$$

where

$$\sigma(\xi, \eta) = i^{l+m} (\xi + \eta)^l \eta^m \cdot (-i) \cdot (\operatorname{sgn}(\xi + \eta) - \operatorname{sgn}(\eta)).$$

By a slight abuse of notation, we shall denote

$$\sigma(a, f) = \partial_x^l [H, a] \partial_x^m f.$$

Now note that neglecting the measure zero sets such as $\xi + \eta = 0$ or $\eta = 0$, the factor $\operatorname{sgn}(\xi + \eta) - \operatorname{sgn}(\eta)$ in $\sigma(\xi, \eta)$ does not vanish only when $\xi + \eta > 0, \eta < 0$ or $\xi + \eta < 0, \eta > 0$. In either cases easy to check that $|\eta| < |\xi|$. Then clearly

$$\sigma(a, f) = \sum_j \sigma(P_j a, P_{<j-2} f) + \sum_j \sigma(P_j a, \tilde{P}_j f),$$

where $\tilde{P}_j = P_{[j-2, j+2]}$. For the first piece, write

$$\begin{aligned} \sum_j \sigma(P_j a, P_{<j-2} f) &= H \left(\sum_j \partial_x^l (P_j a \partial_x^m P_{<j-2} f) \right) - \sum_j \partial_x^l (P_j a \partial_x^m H P_{<j-2} f) \\ &= H \left(\sum_j \partial_x^l ((D^{-(l+m)} P_j D^{l+m} a) \partial_x^m P_{<j-2} f) \right) \\ &\quad - \sum_j \partial_x^l ((D^{-(l+m)} P_j D^{l+m} a) \partial_x^m P_{<j-2} H f) \\ &= H(\sigma_1(D^{l+m} a, f)) + \sigma_2(D^{l+m} a, H f), \end{aligned}$$

where the symbols σ_1, σ_2 satisfy the conditions (3.7)–(3.8) (with ξ and η swapped therein). Thus

$$\left\| \sum_j \sigma(P_j a, P_{<j-2} f) \right\|_p \lesssim \|D^{l+m} a\|_{\text{BMO}} \|f\|_p \lesssim \|\partial_x^{l+m} a\|_{\text{BMO}} \|f\|_p.$$

For the second piece, write

$$\begin{aligned} \sum_j \sigma(P_j a, \tilde{P}_j f) &= H \left(\sum_j \partial_x^l (P_j a \partial_x^m \tilde{P}_j f) \right) - \sum_j \partial_x^l (P_j a \partial_x^m \tilde{P}_j H f) \\ &= H \left(\sum_j \partial_x^l (D^{-(l+m)} P_j D^{l+m} a \partial_x^m \tilde{P}_j f) \right) \\ &\quad - \sum_{\tilde{l}=0}^l \binom{l}{\tilde{l}} \sum_j \partial_x^{\tilde{l}} (D^{-(l+m)} P_j D^{l+m} a) \partial_x^{m+l-\tilde{l}} \tilde{P}_j H f. \end{aligned}$$

It is easy to check that the associated symbols again satisfy (3.7)–(3.8), and we have

$$\left\| \sum_j \sigma(P_j a, \tilde{P}_j f) \right\|_p \lesssim \|\partial_x^{l+m} a\|_{\text{BMO}} \|f\|_p. \quad \square$$

In [7], Dawson, McGahagan and Ponce also proved the following inequality: let $0 \leq \alpha < 1$, $0 < \beta < 1$, $0 < \alpha + \beta \leq 1$, $1 < p, q < \infty$, $\delta > 1/q$, then

$$\|D^\alpha[D^\beta, a]D^{1-(\alpha+\beta)}f\|_{L^p(\mathbb{R})} \lesssim_{\alpha, \beta, p, q, \delta} \|J^\delta \partial_x a\|_{L^q(\mathbb{R})} \|f\|_{L^p(\mathbb{R})},$$

where $D = (-\partial_{xx})^{1/2}$ and $J^\delta = (1 - \partial_{xx})^{\delta/2}$. Note that

$$D^\alpha[D^\beta, a]D^{1-(\alpha+\beta)}f = D^{\alpha+\beta}(aD^{1-(\alpha+\beta)}f) - D^\alpha(aD^{1-\alpha}f).$$

The next proposition gives a sharp version of the above estimate. Moreover it holds on any \mathbb{R}^d , $d \geq 1$.

Proposition 3.10. *For any $0 \leq \alpha < 1$, $0 < \beta \leq 1 - \alpha$, $1 < p < \infty$, we have*

$$\begin{aligned} \|D^{\alpha+\beta}(aD^{1-(\alpha+\beta)}f) - D^\alpha(aD^{1-\alpha}f) + \beta \nabla a \cdot D^{-1} \nabla f\|_{L^p(\mathbb{R}^d)} \\ \lesssim_{\alpha, \beta, p, d} \|Da\|_{\text{BMO}} \|f\|_{L^p(\mathbb{R}^d)}. \end{aligned}$$

Consequently,

$$\|D^{\alpha+\beta}(aD^{1-(\alpha+\beta)}f) - D^\alpha(aD^{1-\alpha}f)\|_{L^p(\mathbb{R}^d)} \lesssim_{\alpha, \beta, p, d} \|\nabla a\|_{L^\infty(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)}.$$

Remark. Clearly for dimension $d = 1$, by using Sobolev embedding $W^{\delta, q} \hookrightarrow L^\infty(\mathbb{R})$ for $\delta > 1/q$, we recover the Dawson–McGahagan–Ponce estimate.

Proof of Proposition 3.10. First we denote

$$\sigma(a, f) = D^{\alpha+\beta}(aD^{1-(\alpha+\beta)}f) - D^\alpha(aD^{1-\alpha}f)$$

with the symbol (by a slight abuse of notation)

$$\sigma(\xi, \eta) = |\xi + \eta|^{\alpha+\beta} |\eta|^{1-(\alpha+\beta)} - |\xi + \eta|^\alpha |\eta|^{1-\alpha}.$$

We then have

$$\sigma(a, f) = \sum_j \sigma(a_j, f_{<j-2}) + \sum_j \sigma(a_{<j-2}, f_j) + \sum_j \sigma(a_j, \tilde{f}_j).$$

For the high-low piece, we observe

$$\sum_j D^\alpha(a_j D^{1-\alpha} f_{<j-2}) = \sum_j \tilde{P}_j(\tilde{P}_j(Da) 2^{-j(1-\alpha)} D^{1-\alpha} f_{<j-2}),$$

where by an abuse of notation we use \tilde{P}_j to denote generic smooth frequency projection operators with frequency $|\xi| \sim 2^j$. Noting that $0 \leq \alpha < 1$, we have

$$\left\| \sum_j \tilde{P}_j(\tilde{P}_j(Da) 2^{-j(1-\alpha)} D^{1-\alpha} f_{<j-2}) \right\|_p \lesssim \|Da\|_{\dot{B}_{\infty, \infty}^0} \|f\|_p \lesssim \|Da\|_{\text{BMO}} \|f\|_p.$$

Similar estimate holds for the piece corresponding to the operator $D^{\alpha+\beta}$ if $\alpha+\beta < 1$. For $\alpha+\beta = 1$, one can use the duality argument as in the proof of Lemma 3.1, it is easy to check that

$$\left\| \sum_j \tilde{P}_j(\tilde{P}_j(Da)f_{<j-2}) \right\|_p \lesssim \|Da\|_{\text{BMO}} \|f\|_p.$$

Thus

$$\left\| \sum_j \sigma(a_j, f_{<j-2}) \right\|_p \lesssim \|Da\|_{\text{BMO}} \|f\|_p.$$

The argument for the diagonal piece is similar, and we have

$$\left\| \sum_j \sigma(a_j, \tilde{f}_j) \right\|_p \lesssim \|Da\|_{\text{BMO}} \|f\|_p.$$

Now we focus on the low-high piece where a correction term is needed for the final estimate. Note that on this piece $|\xi| \ll |\eta|$, and we shall write

$$\begin{aligned} \sigma(\xi, \eta) &= |\xi + \eta|^{\alpha+\beta} |\eta|^{1-(\alpha+\beta)} - |\xi + \eta|^\alpha |\eta|^{1-\alpha} \\ &= (|\xi + \eta|^{\alpha+\beta} - |\eta|^{\alpha+\beta}) |\eta|^{1-(\alpha+\beta)} - (|\xi + \eta|^\alpha - |\eta|^\alpha) |\eta|^{1-\alpha} \\ &=: \sigma_2(\xi, \eta) - \sigma_3(\xi, \eta). \end{aligned}$$

We now consider the piece corresponding to σ_3 (the estimate for σ_2 will be similar). Observe

$$|\eta + \xi|^\alpha - |\eta|^\alpha = \alpha \int_0^1 |\eta + \theta\xi|^{\alpha-2} (\eta + \theta\xi) d\theta \cdot \xi = \alpha |\eta|^{\alpha-2} \eta \cdot \xi + \sum_{i,j=1}^d \tilde{\sigma}_{ij}(\xi, \eta) \xi_i \xi_j,$$

where $\tilde{\sigma}_{ij}(\xi, \eta)$ is of order $|\eta|^{\alpha-2}$ when $|\xi| \ll |\eta|$. It is also easy to check that the m^{th} derivatives of $\tilde{\sigma}_{ij}$ decays as $O(|\eta|^{\alpha-2-m})$ when $|\xi| \ll |\eta|$. To simplify notation we shall write $\sum_{i,j=1}^d \tilde{\sigma}_{ij} \xi_i \xi_j$ simply as $\tilde{\sigma}(\xi, \eta) \xi^2$. Now we can write

$$\sum_j \sigma_3(a_{<j-2}, f_j) = - \sum_j \alpha \nabla a_{<j-2} \cdot D^{-1} \nabla f_j + \sum_j b_j,$$

where

$$b_j(x) = \frac{1}{(2\pi)^{2d}} \int \tilde{\sigma}(\xi, \eta) \xi^2 |\eta|^{1-\alpha} \widehat{a_{<j-2}}(\xi) \widehat{f_j}(\eta) e^{i(\xi+\eta) \cdot x} d\xi d\eta.$$

It is not difficult to check that

$$\left\| \sum_j b_j \right\|_p \lesssim \|(2^{-j} \mathcal{M}(\partial^2 a_{<j-2}) \mathcal{M}(f_j))_{l_j^2}\|_p \lesssim \|\partial a\|_{\dot{B}_{\infty, \infty}^0} \|f\|_p.$$

On the other hand, observe

$$\left\| \sum_j \nabla a_{\geq j-2} \cdot D^{-1} \nabla f_j \right\|_p \lesssim \|\nabla a\|_{\text{BMO}} \|f\|_p.$$

The desired result then easily follows. \square

Theorem 3.11. Let $n_0 = 2d + 2$ and $\sigma = \sigma(\xi, \eta) \in C_{\text{loc}}^{n_0}(\mathbb{R}^d \times \mathbb{R}^d \setminus (0, 0))$ satisfy

- $\sigma(\xi, 0) = 0$, for any ξ ;
- $|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \lesssim_{\alpha, \beta, d} (|\xi| + |\eta|)^{-(|\alpha| + |\beta|)}$, for any $(\xi, \eta) \neq (0, 0)$, and for any $|\alpha| + |\beta| \leq n_0$.

For $f, g \in \mathcal{S}(\mathbb{R}^d)$, define

$$\sigma(D)(f, g)(x) = \int \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{ix \cdot (\xi + \eta)} d\xi d\eta.$$

Then for any $1 < p < \infty$, we have

$$\|\sigma(D)(f, g)\|_p \lesssim_{p, d} \|\sigma\|_\star \|f\|_p \|g\|_{\text{BMO}},$$

where

$$\|\sigma\|_\star = \sup_{\substack{(\xi, \eta) \neq (0, 0) \\ |\alpha| + |\beta| \leq n_0}} \left| (|\xi| + |\eta|)^{|\alpha| + |\beta|} (\partial_\xi^\alpha \partial_\eta^\beta \sigma)(\xi, \eta) \right|.$$

Proof of Theorem 3.11. In this proof we shall ignore the dependence on (p, d) and write $\lesssim_{p, d}$ simply as \lesssim . It suffices to show that, for any $h \in C_c^\infty(\mathbb{R}^d)$,

$$\langle \sigma(D)(f, g), h \rangle \lesssim \|\sigma\|_\star \|f\|_p \|g\|_{\text{BMO}} \|h\|_{p'},$$

where $p' = p/(p-1)$.

We then only need to show that, for any $f \in \mathcal{S}(\mathbb{R}^d)$, $h \in C_c^\infty(\mathbb{R}^d)$,

$$\|\tilde{\sigma}(D)(f, h)\|_{\mathcal{H}^1} \lesssim \|\sigma\|_\star \|f\|_p \|h\|_{p'},$$

where

$$\tilde{\sigma}(D)(f, h)(x) = \int \sigma(\xi, -\xi - \eta) \hat{f}(\xi) \hat{h}(\eta) e^{i(\xi + \eta) \cdot x} d\xi d\eta.$$

Now write

$$\tilde{\sigma}(D)(f, h) = \sum_j \tilde{\sigma}(D)(f_{\leq j-2}, h_j) + \sum_j \tilde{\sigma}(D)(f_j, h_{\leq j-2}) + \sum_j \tilde{\sigma}(D)(f_j, \tilde{h}_j),$$

where $\tilde{h}_j = h_{j-1} + h_j + h_{j+1}$. We refer to these three summands as low-high, high-low and diagonal pieces respectively.

Low-high piece. Note that $|\xi| \ll |\eta|$ and $|\xi + \eta| \sim |\eta| \sim 2^j$. Thus

$$\left\| \sum_j \tilde{\sigma}(D)(f_{\leq j-2}, h_j) \right\|_{\mathcal{H}^1} \lesssim \|(\tilde{\sigma}(D)(f_{\leq j-2}, h_j))_{l_j^2}\|_{L_x^1}.$$

Now we need a simple lemma.

Lemma 3.12. *Suppose $\chi_1, \chi_2 \in C_c^\infty(\mathbb{R}^d)$ are such that χ_1 or χ_2 is supported on an annulus. Then*

$$\begin{aligned} & \left| \int \sigma(\xi, -\xi - \eta) \chi_1\left(\frac{\xi}{2^j}\right) \chi_2\left(\frac{\eta}{2^j}\right) e^{i\xi \cdot (x-y)} e^{i\eta \cdot (x-z)} d\xi d\eta \right| \\ & \lesssim \sup_{\substack{|\alpha|+|\beta| \leq 2d+1 \\ (\xi, \eta) \neq (0,0)}} (|\xi| + |\eta|)^{|\alpha|+|\beta|} |\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \cdot \frac{2^{2jd}}{(1+2^j|x-y|+2^j|x-z|)^{2d+1}}. \end{aligned}$$

Proof of Lemma 3.12. Without loss of generality, we can assume $|x-y| \geq |x-z|$. If $|x-y| \leq 2^{-j}$, then the bound is trivial. Now assume $|x-y| > 2^{-j}$, then just integrate by parts in the variable ξ up to $(2d+1)$ -times and note that $|\xi| + |\xi + \eta| \sim |\xi| + |\eta|$. More precisely if one denotes $\tilde{\xi} = 2^{-j}\xi$ and $\tilde{\eta} = 2^{-j}\eta$, then by the support assumption on χ_1 and χ_2 , one has $|\tilde{\xi}| + |\tilde{\xi} + \tilde{\eta}| \sim 1$. It follows that for $|\tilde{\xi}| \lesssim 1$, $|\tilde{\eta}| \sim 1$ or $|\tilde{\xi}| \sim 1$, $|\tilde{\eta}| \lesssim 1$, we have

$$\begin{aligned} & \max_{|\alpha|+|\beta|=2d+1} |\partial_\xi^\alpha \partial_\eta^\beta (\sigma(2^j \tilde{\xi}, -2^j(\tilde{\xi} + \tilde{\eta})))| \\ & \lesssim 2^{j(2d+1)} \max_{|\alpha|+|\beta|=2d+1} (|\tilde{\xi}| + |\tilde{\xi} + \tilde{\eta}|) |(\partial_\xi^\alpha \partial_\eta^\beta \sigma)(2^j \tilde{\xi}, -2^j(\tilde{\xi} + \tilde{\eta}))| \lesssim \|\sigma\|_\star. \quad \square \end{aligned}$$

By using Lemma 3.12, it is easy to check that

$$\begin{aligned} & |\tilde{\sigma}(D)(f_{\leq j-2}, h_j)(x)| \\ & \lesssim \|\sigma\|_\star \cdot 2^{jd} \int \frac{|f_{\leq j-2}(y)|}{(1+2^j|x-y|)^{d+1/2}} dy \cdot 2^{jd} \int \frac{|h_j(z)|}{(1+2^j|x-z|)^{d+1/2}} dz \\ & \lesssim \|\sigma\|_\star \cdot \mathcal{M}f_{\leq j-2}(x) \cdot \mathcal{M}h_j(x). \end{aligned}$$

Therefore,

$$\begin{aligned} \|\tilde{\sigma}(D)(f_{\leq j-2}, h_j)\|_{l_j^2} & \lesssim \|\sigma\|_\star \cdot \|(\mathcal{M}f_{\leq j-2})\|_{l_j^\infty} \cdot \|(\mathcal{M}h_j)\|_{l_j^2} \\ & \lesssim \|\sigma\|_\star \cdot \|f\|_p \cdot \|h\|_{p'}. \end{aligned}$$

This finishes the estimate of the low-high piece. The estimate for the high-low piece is similar.

Diagonal-piece. In this case we need to consider the integral

$$\tilde{\sigma}(D)(f_j, \tilde{h}_j)(x) = \int \sigma(\xi, -\xi - \eta) \chi_1\left(\frac{\xi}{2^j}\right) \chi_2\left(\frac{\eta}{2^j}\right) \hat{f}_j(\xi) \widehat{h}_j(\eta) e^{i(\xi+\eta) \cdot x} d\xi d\eta,$$

where χ_1, χ_2 are smooth cut-off functions with support in the annulus $\{z \in \mathbb{R}^d : 2^{-m_0} \leq |z| \leq 2^{m_0}\}$ for some integer $m_0 > 0$.

We now consider two subcases.

Subcase 1: $|\xi + \eta| \geq 2^{j-m_0-10}$. Note that in this case $|\xi + \eta| \sim 2^j$. By using frequency localization and Lemma 3.12, we have

$$\begin{aligned} \left\| \sum_j P_{>j-m_0-10}(\tilde{\sigma}(D)(f_j, \tilde{h}_j)) \right\|_{\mathcal{H}^1} & \sim \left\| \sum_j P_{j-m_0-10 < \cdot < j+m_0+100}(\tilde{\sigma}(D)(f_j, \tilde{h}_j)) \right\|_{\mathcal{H}^1} \\ & \lesssim \|(\mathcal{M}f_j \cdot \mathcal{M}\tilde{h}_j)\|_{l_j^2} \lesssim \|f\|_p \|h\|_{p'}. \end{aligned}$$

Subcase 2: $|\xi + \eta| < 2^{j-m_0-10}$. In yet other words, $|\xi + \eta| \ll \min\{|\xi|, |\eta|\}$. Then since $\sigma(\xi, 0) = 0$ by assumption, we just need to consider

$$\int \chi_{|\xi+\eta| < 2^{j-m_0-10}} (\sigma(\xi, -\xi-\eta) - \sigma(\xi, 0)) \chi_1\left(\frac{\xi}{2^j}\right) \chi_2\left(\frac{\eta}{2^j}\right) \hat{f}_j(\xi) \hat{h}_j(\eta) e^{i(\xi+\eta)\cdot x} d\xi d\eta,$$

where χ is a smooth cut-off function. Now we need another simple lemma.

Lemma 3.13. *For all $0 \leq \theta \leq 1$, we have*

$$\begin{aligned} & \left| \int \chi_{|\xi+\eta| < 2^{j-m_0-10}} (\partial_\eta \sigma)(\xi, -\theta(\xi + \eta)) \chi_1\left(\frac{\xi}{2^j}\right) \chi_2\left(\frac{\eta}{2^j}\right) e^{i\xi\cdot(x-y)} e^{i\eta\cdot(x-z)} d\xi d\eta \right| \\ & \lesssim \sup_{\substack{|\alpha|+|\beta| \leq 2d+2 \\ (\xi, \eta) \neq (0,0)}} (|\xi| + |\eta|)^{|\alpha|+|\beta|} |\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \\ & \quad \cdot 2^{2jd-j} \cdot (1 + 2^j|x-y| + 2^j|x-z|)^{-(2d+1)}. \end{aligned}$$

The proof of Lemma 3.13 is similar to Lemma 3.12 and therefore we omit it. By Lemma 3.13, we then have

$$\begin{aligned} \left\| \sum_j \tilde{\sigma}(D)(f_j, \tilde{h}_j) \right\|_{\dot{B}_{1,1}^0} & \lesssim \sum_k 2^k \sum_{j \geq k+m_0+10} 2^{-j} \|\mathcal{M}f_j \cdot \mathcal{M}\tilde{h}_j\|_1 \\ & \lesssim \sum_j \|\mathcal{M}f_j \cdot \mathcal{M}\tilde{h}_j\|_1 \\ & \lesssim \|(\mathcal{M}f_j)_{l_j^2}\|_p \cdot \|(\mathcal{M}\tilde{h}_j)_{l_j^2}\|_{p'} \lesssim \|f\|_p \|h\|_{p'}. \end{aligned}$$

This concludes the proof of Theorem 3.11. \square

Remark. A close inspection shows that the diagonal piece in fact belongs to $\dot{B}_{1,1}^0$ which embeds into \mathcal{H}^1 .

It is possible to refine Theorem 3.11 further. The following only requires $2d+1$ derivatives on the symbol σ .

Theorem 3.14. *Let $n_0 = 2d+1$ and $\sigma = \sigma(\xi, \eta) \in C_{\text{loc}}^{n_0}(\mathbb{R}^d \times \mathbb{R}^d \setminus (0,0))$ satisfy*

- $\sigma(\xi, 0) = 0$, for any ξ ;
- $|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \lesssim_{\alpha, \beta, d} (|\xi| + |\eta|)^{-(|\alpha|+|\beta|)}$, for any $(\xi, \eta) \neq (0,0)$, and for any $|\alpha| + |\beta| \leq n_0$.

For $f, g \in \mathcal{S}(\mathbb{R}^d)$ define

$$\sigma(D)(f, g)(x) = \int \sigma(\xi, \eta) \hat{f}(\xi) \hat{g}(\eta) e^{ix\cdot(\xi+\eta)} d\xi d\eta.$$

Then for any $1 < p < \infty$, we have

$$\|\sigma(D)(f, g)\|_p \lesssim_{p,d} \|\sigma\|_\star \|f\|_p \|g\|_{\text{BMO}},$$

where

$$\|\sigma\|_\star = \sup_{\substack{(\xi, \eta) \neq (0,0) \\ |\alpha|+|\beta| \leq n_0}} (|\xi| + |\eta|)^{|\alpha|+|\beta|} |\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)|.$$

Proof of Theorem 3.14. We shall use the same notation as in the proof of Theorem 3.11 and sketch the needed modifications. It suffices to show

$$\sum_k \sum_{j \geq k+m_0+10} \|P_k(\tilde{\sigma}(D)(f_j, \tilde{h}_j))\|_{L_x^1} \lesssim \|f\|_p \|h\|_{p'}.$$

Clearly,

$$\begin{aligned} & \|P_k(\tilde{\sigma}(D)(f_j, \tilde{h}_j))\|_{L_x^1} \\ & \lesssim \left\| \int \chi\left(\frac{\xi+\eta}{2^k}\right) \chi(2^{-j}\xi) \chi(2^{-j}\eta) \sigma(\xi, -\xi-\eta) e^{i(\xi \cdot (x-y) + \eta \cdot (x-z))} \right. \\ (3.10) \quad & \left. \cdot d\xi d\eta f_j(y) \tilde{h}_j(z) dy dz \right\|_{L_x^1}, \end{aligned}$$

where we have slightly abused the notation and denote all smooth cutoff functions by the same symbol χ (whose support is on the annulus $\{z : |z| \sim 1\}$). Now set $\tilde{\xi} = \xi + \eta$, $\tilde{\eta} = \eta$. Then

$$\begin{aligned} & \left| \int \chi\left(\frac{\xi+\eta}{2^k}\right) \chi(2^{-j}\xi) \chi(2^{-j}\eta) \sigma(\xi, -\xi-\eta) e^{i(\xi \cdot (x-y) + \eta \cdot (x-z))} d\xi d\eta \right| \\ (3.11) \quad & = \left| \int \chi(2^{-k}\tilde{\xi}) \chi(2^{-j}(\tilde{\xi}-\tilde{\eta})) \chi(2^{-j}\tilde{\eta}) \sigma(\tilde{\xi}-\tilde{\eta}, -\tilde{\xi}) e^{i(\tilde{\xi} \cdot (x-y) + \tilde{\eta} \cdot (y-z))} d\tilde{\xi} d\tilde{\eta} \right|. \end{aligned}$$

Case 1: $|y-z| \geq \frac{1}{10}|x-y|$ or $|y-z| \geq \frac{1}{10}|x-z|$. In this case easy to check that $|z-x| + |y-x| \lesssim |y-z|$. Integrating by parts in $\tilde{\eta}$ up to $2d+1$ times, we obtain

$$\begin{aligned} (3.11) & \lesssim (1+2^j|y-z|)^{-(2d+1)} \cdot 2^{jd} \cdot 2^{kd} \\ & \lesssim (1+2^j|x-y|)^{-(d+1/2)} (1+2^j|x-z|)^{-(d+1/2)} \cdot 2^{jd} \cdot 2^{(k-j)d}. \end{aligned}$$

It follows easily that in this case,

$$(3.10) \lesssim 2^{(k-j)d} \|\mathcal{M}f_j \mathcal{M}\tilde{h}_j\|_{L_x^1}.$$

Summing in k and j then easily yields the desired inequality.

Case 2: $|y-z| < \frac{1}{10} \min\{|x-y|, |x-z|\}$. In this case $|x-y| \sim |x-z|$. We estimate (3.11) in several different ways. First integrating by parts in $\tilde{\xi}$ up to $d+1$ times and then in $\tilde{\eta}$ up to d times, we get

$$(3.11) \lesssim (1+2^k|x-y|)^{-(d+1)} (1+2^j|y-z|)^{-d} 2^{jd} 2^{kd}.$$

Then integrating by parts in $\tilde{\xi}$ up to d times and then in $\tilde{\eta}$ up to $d+1$ times, we get

$$(3.11) \lesssim (1+2^k|x-y|)^{-d} (1+2^j|y-z|)^{-(d+1)} 2^{jd} 2^{kd}.$$

Interpolating these two estimates gives us

$$(3.12) \quad (3.11) \lesssim (1+2^k|x-y|)^{-(d+1/2)} (1+2^j|y-z|)^{-(d+1/2)} 2^{jd} 2^{kd}.$$

Since $\sigma(\xi, 0) = 0$ for any ξ , we have

$$\sigma(\tilde{\xi} - \tilde{\eta}, -\tilde{\xi}) = -\tilde{\xi} \cdot \int_0^1 (\partial_\eta \sigma)(\tilde{\xi} - \tilde{\eta}, -\theta \tilde{\xi}) d\theta.$$

It is then not difficult to check that

$$(3.11) \lesssim 2^{k-j} (1 + 2^k |x - y|)^{-d} (1 + 2^j |y - z|)^{-d} 2^{jd} 2^{kd}.$$

Interpolating this estimate with (3.12) yields

$$(3.11) \lesssim 2^{\frac{1}{2}(k-j)} (1 + 2^k |x - y|)^{-(d+1/4)} (1 + 2^j |y - z|)^{-(d+1/4)} 2^{jd} 2^{kd}.$$

We then get

$$\begin{aligned} (3.10) &\lesssim 2^{kd} 2^{\frac{1}{2}(k-j)} \int (1 + 2^k |x - y|)^{-(d+1/4)} |f_j(y)| (\mathcal{M}\tilde{h}_j)(y) dy dx \\ &\lesssim 2^{\frac{1}{2}(k-j)} \int |f_j(y)| (\mathcal{M}\tilde{h}_j)(y) dy. \end{aligned}$$

Summing in k and j then easily implies the desired result. \square

The next two simple lemmas will be needed in Section 5. It is interesting that one can obtain some BMO (or Besov) refinements of some terms in Kato–Ponce inequalities.

Lemma 3.15. *Let $s > 1$ be an integer and let $1 < p < \infty$. Let \mathcal{R} be the usual Riesz-type operator. Then for any $f, g \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\|\partial^s f \cdot \mathcal{R}g\|_p \lesssim_{s,p,d,r_1,r_2} \|D^s f\|_{\text{BMO}} \|g\|_p + \|\partial f\|_{r_1} \|D^{s-1}g\|_{r_2},$$

where $1/p = 1/r_1 + 1/r_2$, and $1 < r_1, r_2 < \infty$. The notation ∂^s denotes any differentiation operator $\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$ with $|\alpha| = s$. If $r_1 = \infty$, $r_2 = p$, then

$$\|\partial^s f \cdot \mathcal{R}g\|_p \lesssim_{s,p,d} \|D^s f\|_{\text{BMO}} \|g\|_p + \|\partial f\|_{\dot{B}_{\infty,\infty}^0} \|D^{s-1}g\|_p.$$

And if $r_1 = p$, $r_2 = \infty$, then

$$\|\partial^s f \cdot \mathcal{R}g\|_p \lesssim_{s,p,d} \|D^s f\|_{\text{BMO}} \|g\|_p + \|\partial f\|_p \|D^{s-1}g\|_{\dot{B}_{\infty,\infty}^0}.$$

Remark 3.16. The same estimates also hold for $\|\partial^s f \cdot g\|_p$.

Proof of Lemma 3.15. Write

$$\partial^s f \cdot \mathcal{R}g = \sum_j \partial^s f_{<j-2} \mathcal{R}g_j + \sum_j \partial^s f_{\geq j-2} \mathcal{R}g_j = \sum_j \partial^s f_{<j-2} \mathcal{R}g_j + \sum_j \partial^s f_j \mathcal{R}g_{\leq j+2}.$$

First note that by Lemma 3.1, we have

$$\left\| \sum_j \partial^s f_j \cdot \mathcal{R}g_{\leq j+2} \right\|_p \lesssim \|\partial^s f\|_{\text{BMO}} \|\mathcal{R}g\|_p \lesssim \|D^s f\|_{\text{BMO}} \|g\|_p.$$

Thus we only need to estimate the piece $\sum_j \partial^s f_{<j-2} \mathcal{R}g_j$.

Consider first the case $r_2 < \infty$. Then,

$$\begin{aligned} \left\| \sum_j \partial^s f_{<j-2} \mathcal{R}g_j \right\|_p &\lesssim \|(\partial^s f_{<j-2} \cdot \mathcal{R}g_j)_{l_j^2}\|_p \\ &\lesssim \|(2^{-j(s-1)} \partial^s f_{<j-2})_{l_j^\infty} (2^{j(s-1)} \mathcal{R}g_j)_{l_j^2}\|_p \\ &\lesssim \begin{cases} \|\partial f\|_{r_1} \|D^{s-1}g\|_{r_2}, & \text{if } r_1 < \infty, \\ \|\partial f\|_{\dot{B}_{\infty,\infty}^0} \|D^{s-1}g\|_p, & \text{if } r_1 = \infty. \end{cases} \end{aligned}$$

If $r_2 = \infty$, then $r_1 = p$ and

$$\begin{aligned} \left\| \sum_j \partial^s f_{<j-2} \mathcal{R}g_j \right\|_p &\lesssim \|(2^{-j(s-1)} \partial^s f_{<j-2})_{l_j^2} \cdot (2^{j(s-1)} \mathcal{R}g_j)_{l_j^\infty}\|_p \\ &\lesssim \|(2^{-j(s-1)} \partial^s f_{<j-2})_{l_j^2}\|_p \cdot \|D^{s-1}g\|_{\dot{B}_{\infty,\infty}^0} \lesssim \|\partial f\|_p \|D^{s-1}g\|_{\dot{B}_{\infty,\infty}^0}. \quad \square \end{aligned}$$

Lemma 3.17. *Let $s > 1$ and $1 < p < \infty$. Let $1 < p_1, p_2 \leq \infty$ satisfy $1/p_1 + 1/p_2 = 1/p$. Then for any $f, g \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$(3.13) \quad \|\partial f \cdot D^{s-2} \partial g\|_p \lesssim \|\partial f\|_{p_1} \|D^{s-1}g\|_{p_2}, \quad \text{if } 1 < p_1 \leq \infty, 1 < p_2 < \infty,$$

$$(3.14) \quad \|\partial f \cdot D^{s-2} \partial g\|_p \lesssim \|D^s f\|_{p_1} \|g\|_{p_2} + \|\partial f\|_p \|D^{s-1}g\|_{\text{BMO}},$$

$$\text{if } 1 < p_1 < \infty, 1 < p_2 < \infty,$$

$$(3.15) \quad \|\partial f \cdot D^{s-2} \partial g\|_p \lesssim \|D^s f\|_p \|g\|_{\dot{B}_{\infty,\infty}^0} + \|\partial f\|_p \|D^{s-1}g\|_{\text{BMO}},$$

$$(3.16) \quad \|\partial f \cdot D^{s-2} \partial g\|_p \lesssim \|D^s f\|_{\dot{B}_{\infty,\infty}^0} \|g\|_p + \|\partial f\|_p \|D^{s-1}g\|_{\text{BMO}}.$$

Proof of Lemma 3.17. The first inequality is trivial. For (3.14), by using frequency localization and Lemma 3.1, we have

$$\begin{aligned} \|\partial f \cdot D^{s-2} \partial g\|_p &\lesssim \left\| \sum_j \partial f_j \cdot D^{s-2} \partial g_{<j-2} \right\|_p + \left\| \sum_j \partial f_{\leq j+2} \cdot D^{s-2} \partial g_j \right\|_p \\ &\lesssim \|(2^{j(s-1)} \partial f_j)_{l_j^2} (2^{-j(s-1)} D^{s-2} \partial g_{<j-2})_{l_j^\infty}\|_p + \|\partial f\|_p \|D^{s-1}g\|_{\text{BMO}} \\ &\lesssim \|D^s f\|_{p_1} \|g\|_{p_2} + \|\partial f\|_p \|D^{s-1}g\|_{\text{BMO}}. \end{aligned}$$

For (3.15), just observe

$$\|(2^{-j(s-1)} D^{s-2} \partial g_{<j-2})_{l_j^\infty}\|_\infty \lesssim \|g\|_{\dot{B}_{\infty,\infty}^0}.$$

For (3.16), we have

$$\begin{aligned} \left\| \sum_j \partial f_j \cdot D^{s-2} \partial g_{<j-2} \right\|_p &\lesssim \|(2^{j(s-1)} \partial f_j)_{l_j^\infty}\|_\infty \|(2^{-j(s-1)} D^{s-2} \partial g_{<j-2})_{l_j^2}\|_p \\ &\lesssim \|D^s f\|_{\dot{B}_{\infty,\infty}^0} \|g\|_p. \quad \square \end{aligned}$$

4. Proof of Theorem 1.2

In the first subsection, we shall give the proof of Theorem 1.2 for the case $1 < p < \infty$. In the second subsection, we sketch the needed modification for the case $1/2 < p \leq 1$.

4.1. The case $1 < p < \infty$

To prove Theorem 1.2 for the case $1 < p < \infty$, we first prove the following.

Proposition 4.1. *Let $s > 0$ and $1 < p < \infty$. Let $s_1, s_2 \geq 0$ and $s_1 + s_2 = s$. Then for any $f, g \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$(4.1) \quad \left\| D^s(fg) - \sum_j \left(\sum_{|\alpha| \leq s_1} \frac{1}{\alpha!} \partial^\alpha f_{\leq j-2} D^{s,\alpha} g_j + \sum_{|\beta| \leq s_2} \frac{1}{\beta!} \partial^\beta g_{\leq j-2} D^{s,\beta} f_j \right) \right\|_p \lesssim \begin{cases} \|D^{s_1} f\|_{p_1} \cdot \|D^{s_2} g\|_{p_2}, & \text{if } 1 < p_1, p_2 < \infty, 1/p = 1/p_1 + 1/p_2; \\ \|D^{s_1} f\|_p \|D^{s_2} g\|_{\dot{B}_{\infty,\infty}^0}, & \text{if } p_1 = p, p_2 = \infty; \\ \|D^{s_1} f\|_{\dot{B}_{\infty,\infty}^0} \cdot \|D^{s_2} g\|_p, & \text{if } p_1 = \infty, p_2 = p. \end{cases}$$

Proof of Proposition 4.1. We write

$$fg = \sum_{j \in \mathbb{Z}} f_j \tilde{g}_j + \sum_{j \in \mathbb{Z}} f_{\leq j-2} g_j + \sum_{j \in \mathbb{Z}} g_{\leq j-2} f_j,$$

where $\tilde{g}_j = g_{j-1} + g_j + g_{j+1}$. We shall analyze each term separately.

1) *The diagonal piece.* Denote $h = \sum_j f_j \tilde{g}_j$. Then

$$\begin{aligned} \|D^s h\|_p &\lesssim \|(2^{ks} P_k h)_{l_k^2}\|_p = \left\| \left(2^{ks} P_k \left(\sum_{j \geq k-10} f_j \tilde{g}_j \right) \right)_{l_k^2} \right\|_p \\ &\lesssim \sum_{l \geq -10} \|(2^{ks} f_{k+l} \tilde{g}_{k+l})_{l_k^2}\|_p \lesssim \sum_{l \geq -10} 2^{-ls} \|(2^{ks} f_k \tilde{g}_k)_{l_k^2}\|_p \lesssim \|(2^{ks} f_k \tilde{g}_k)_{l_k^2}\|_p. \end{aligned}$$

Now discuss three cases.

If $p_1 < \infty$ and $p_2 < \infty$, then

$$\|(2^{ks} f_k \tilde{g}_k)_{l_k^2}\|_p \lesssim \|(2^{ks_1} f_k)_{l_k^2}\|_{p_1} \cdot \|(2^{ks_2} \tilde{g}_k)_{l_k^\infty}\|_{p_2} \lesssim \|D^{s_1} f\|_{p_1} \cdot \|D^{s_2} g\|_{p_2}.$$

If $p_1 = p$ and $p_2 = \infty$, then

$$\|(2^{ks} f_k \tilde{g}_k)_{l_k^2}\|_p \lesssim \|D^{s_1} f\|_p \cdot \|D^{s_2} g\|_{\dot{B}_{\infty,\infty}^0}.$$

Similarly if $p_1 = \infty$ and $p_2 = p$, then

$$\|(2^{ks} f_k \tilde{g}_k)_{l_k^2}\|_p \lesssim \|D^{s_1} f\|_{\dot{B}_{\infty,\infty}^0} \|D^{s_2} g\|_p.$$

Thus the diagonal piece is OK.

2) *The low-high piece.* For each j , write

$$(4.2) \quad D^s(f_{\leq j-2}g_j) = [D^s, f_{\leq j-2}]g_j + f_{\leq j-2}D^s g_j.$$

We shall simplify further the commutator piece $[D^s, f_{\leq j-2}]g_j$. To write out its explicit form, we need to use a fattened frequency cut-off \tilde{P}_j such that $\tilde{P}_j P_j = P_j$. More precisely, let $\tilde{\phi} \in C_c^\infty(\mathbb{R}^d)$ be such that

$$\tilde{\phi}(\xi) = 1, \quad \forall \xi \in \text{supp}(\phi),$$

where we recall $\widehat{P_j \delta_0}(\xi) = \phi(2^{-j}\xi)$. Define $\widehat{\tilde{P}_j \delta_0}(\xi) = \tilde{\phi}(2^{-j}\xi)$. Clearly

$$(D^s \tilde{P}_j \delta_0)(y) = 2^{j(d+s)} \psi_1(2^j y)$$

where $\widehat{\psi_1}(\xi) = |\xi|^s \tilde{\phi}(\xi)$. Then

$$(4.3) \quad \begin{aligned} & ([D^s, f_{\leq j-2}]g_j)(x) \\ &= 2^{j(d+s)} \int_{\mathbb{R}^d} \psi_1(2^j y) (f_{\leq j-2}(x-y) - f_{\leq j-2}(x)) g_j(x-y) dy. \end{aligned}$$

Denote

$$h(\theta) = f_{\leq j-2}(x - \theta y).$$

Set $m_0 = [s_1]$. We then Taylor expand $h(\theta)$ up to m_0^{th} term:

$$(4.4) \quad h(1) - h(0) = h'(0) + \dots + \frac{h^{(m_0)}(0)}{m_0!} + \underbrace{\int_0^1 \frac{(1-\theta)^{m_0}}{m_0!} h^{(m_0+1)}(\theta) d\theta}_{\text{error}}.$$

Contribution of the “error” term. We first show that the contribution of the “error term” in (4.4) to (4.3) can be bounded by $\|D^{s_1} f\|_{p_1} \|D^{s_2} f\|_{p_2}$ (after summation in j).

It is easy to check that

$$h^{(m_0+1)}(\theta) = \sum_{|\alpha|=m_0+1} C_\alpha (\partial^\alpha f_{\leq j-2})(x - \theta y) \cdot y^\alpha,$$

where C_α are constant coefficients whose value do not matter in our estimates.

By Lemma 2.3, we have

$$|(\partial^{m_0+1} f_{\leq j-2})(x - \theta y)| \lesssim (1 + 2^j |y|)^d \mathcal{M}(|\partial^{m_0+1} f_{\leq j-2}|)(x).$$

Then

$$\begin{aligned} & 2^{j(d+s)} \int_{\mathbb{R}^d} |\psi_1(2^j y)| \cdot |y|^{m_0+1} \cdot |\partial^{m_0+1} f_{\leq j-2}(x - \theta y)| \cdot |g_j(x-y)| dy \\ & \lesssim 2^{-j(m_0+1-s_1)} \mathcal{M}(|\partial^{m_0+1} f_{\leq j-2}|)(x) \cdot (\mathcal{M}g_j)(x) \cdot 2^{js_2}. \end{aligned}$$

Now discuss two cases.

Case 1: $p_1 \leq \infty, p_2 < \infty$. Then

$$\begin{aligned}
& \left\| (2^{j(d+s)} \int \psi_1(2^j y) \cdot \int_0^1 \frac{(1-\theta)^{m_0}}{m_0!} h^{(m_0+1)}(\theta) d\theta \cdot g_j(x-y) dy)_{l_j^2} \right\|_p \\
& \lesssim \|(2^{-j(m_0+1-s_1)} \mathcal{M}(|\partial^{m_0+1} f_{\leq j-2}|))_{l_j^\infty} \cdot (\mathcal{M}g_j \cdot 2^{js_2})_{l_j^2}\|_p \\
& \lesssim \|\mathcal{M}(\sup_j 2^{-j(m_0+1-s_1)} |\partial^{m_0+1} f_{\leq j-2}|)\|_{p_1} \cdot \|(\mathcal{M}(2^{js_2} g_j))_{l_j^2}\|_{p_2} \\
& \lesssim \begin{cases} \|D^{s_1} f\|_{p_1} \cdot \|D^{s_2} g\|_{p_2}, & \text{if } p_1 < \infty, p_2 < \infty, \\ \|D^{s_1} f\|_{\dot{B}_{\infty, \infty}^0} \cdot \|D^{s_2} g\|_p, & \text{if } p_1 = \infty, p_2 < \infty. \end{cases}
\end{aligned}$$

Case 2: $p_1 = p, p_2 = \infty$. Then

$$\begin{aligned}
& \left\| (2^{j(d+s)} \int \psi_1(2^j y) \cdot \int_0^1 \frac{(1-\theta)^{m_0}}{m_0!} h^{(m_0+1)}(\theta) d\theta \cdot g_j(x-y) dy)_{l_j^2} \right\|_p \\
& \lesssim \|(2^{-j(m_0+1-s_1)} \mathcal{M}(|\partial^{m_0+1} f_{\leq j-2}|))_{l_j^2} \cdot (\mathcal{M}g_j \cdot 2^{js_2})_{l_j^\infty}\|_p \\
& \lesssim \|(\mathcal{M}(2^{-j(m_0+1-s_1)} |\partial^{m_0+1} f_{\leq j-2}|))_{l_j^2}\|_p \cdot \|D^{s_2} g\|_{\dot{B}_{\infty, \infty}^0} \\
& \lesssim \|D^{s_1} f\|_p \cdot \|D^{s_2} g\|_{\dot{B}_{\infty, \infty}^0}.
\end{aligned}$$

Thus the contribution of the ‘‘error’’ term to (4.3) is OK for us.

The form of the other terms in (4.4). It is easy to check that

$$h'(0) + \cdots + \frac{h^{(m_0)}(0)}{m_0!} = \sum_{0 < |\alpha| \leq m_0} \frac{1}{\alpha!} (\partial^\alpha f_{\leq j-2})(x) \cdot (-1)^{|\alpha|} y^\alpha.$$

Therefore in (4.3), we have

$$\begin{aligned}
& 2^{j(d+s)} \int_{\mathbb{R}^d} \psi_1(2^j y) \cdot \left(\sum_{l=1}^{m_0} \frac{h^{(l)}(0)}{l!} \right) g_j(x-y) dy \\
& = \sum_{0 < |\alpha| \leq m_0} 2^{j(d+s)} \int_{\mathbb{R}^d} \psi_1(2^j y) \cdot \frac{1}{\alpha!} (\partial^\alpha f_{\leq j-2})(x) \cdot (-1)^{|\alpha|} \cdot y^\alpha \cdot g_j(x-y) dy \\
& = \sum_{0 < |\alpha| \leq m_0} \frac{1}{\alpha!} \cdot (-1)^{|\alpha|} \cdot 2^{js} \cdot \mathcal{F}^{-1}(i^\alpha \partial_\xi^\alpha (\widehat{\psi_1}(\xi/2^j)) \widehat{g_j}(\xi))(x) \cdot (\partial^\alpha f_{\leq j-2})(x) \\
& = \sum_{0 < |\alpha| \leq m_0} \frac{1}{\alpha!} \mathcal{F}^{-1}(i^{-|\alpha|} \partial_\xi^\alpha (|\xi|^s) \widehat{g_j}(\xi))(x) \cdot (\partial^\alpha f_{\leq j-2})(x) \\
& = \sum_{0 < |\alpha| \leq m_0} \frac{1}{\alpha!} D^{s, \alpha} g_j \cdot \partial^\alpha f_{\leq j-2}.
\end{aligned}$$

Note that the second term in (4.2) corresponds to $\alpha = 0$. Thus the low-high piece can be written as

$$\left(\sum_j \sum_{0 \leq |\alpha| \leq [s_1]} \frac{1}{\alpha!} \partial^\alpha f_{\leq j-2} D^{s, \alpha} g_j \right) + E_1,$$

where

$$(4.5) \quad \|E_1\|_p \lesssim \begin{cases} \|D^{s_1} f\|_{p_1} \cdot \|D^{s_2} g\|_{p_2}, & \text{if } 1 < p_1, p_2 < \infty; \\ \|D^{s_1} f\|_p \|D^{s_2} g\|_{\dot{B}_{\infty, \infty}^0}, & \text{if } p_1 = p, p_2 = \infty; \\ \|D^{s_1} f\|_{\dot{B}_{\infty, \infty}^0} \cdot \|D^{s_2} g\|_p, & \text{if } p_1 = \infty, p_2 = p. \end{cases}$$

3) *The high-low piece.* This is similar to the low-high piece. One can get the results by symmetry.

Collecting all the estimates, we obtain

$$(4.6) \quad D^s(fg) = \sum_j \left(\sum_{|\alpha| \leq s_1} \frac{1}{\alpha!} \partial^\alpha f_{\leq j-2} D^{s, \alpha} g_j + \sum_{|\beta| \leq s_2} \frac{1}{\beta!} \partial^\beta g_{\leq j-2} D^{s, \beta} f_j \right) + \text{error},$$

where “error” term satisfies the same bound as in (4.5). \square

We are now ready to complete the proof of Theorem 1.2 for the case $1 < p < \infty$.

Proof of Theorem 1.2, case $1 < p < \infty$. By Proposition 4.1, we just need to simplify the expression in (4.6). For this we have to discuss several cases.

Case 1: $1 < p_1 < \infty$ and $1 < p_2 < \infty$. Note that

$$\begin{aligned} \sum_j f_{\leq j-2} D^s g_j &= f D^s g - \sum_j f_{> j-2} D^s g_j = f D^s g - \sum_j f_j D^s g_{< j+2} \\ &= f D^s g - \sum_j f_j D^s g_{j-2 \leq \cdot < j+2} - \sum_j f_j D^s g_{< j-2}. \end{aligned}$$

Clearly,

$$\begin{aligned} \left\| \sum_j f_j D^s g_{j-2 \leq \cdot < j+2} \right\|_p &\lesssim \left\| (2^{js_1} f_j)_{l_j^2} (2^{-js_1} D^s g_{j-2 \leq \cdot < j+2})_{l_j^2} \right\|_p \\ &\lesssim \|D^{s_1} f\|_{p_1} \cdot \|D^{s_2} g\|_{p_2}; \end{aligned}$$

and by frequency localization,

$$\left\| \sum_j f_j D^s g_{< j-2} \right\|_p \lesssim \left\| (2^{js_1} f_j)_{l_j^2} \cdot (2^{-js_1} D^s g_{< j-2})_{l_j^\infty} \right\|_p \lesssim \|D^{s_1} f\|_{p_1} \cdot \|D^{s_2} g\|_{p_2}.$$

Similarly, for each $0 < |\alpha| \leq [s_1]$, $0 \leq |\beta| \leq [s_2]$, it holds that

$$\begin{aligned} \left\| \sum_j \partial^\alpha f_{> j-2} D^{s, \alpha} g_j \right\|_p &= \left\| \sum_j \partial^\alpha f_j D^{s, \alpha} g_{< j+2} \right\|_p \lesssim \|D^{s_1} f\|_{p_1} \|D^{s_2} g\|_{p_2}; \\ \left\| \sum_j \partial^\beta g_{> j-2} D^{s, \beta} f_j \right\|_p &= \left\| \sum_j \partial^\beta g_j D^{s, \beta} f_{< j+2} \right\|_p \lesssim \|D^{s_1} f\|_{p_1} \cdot \|D^{s_2} g\|_{p_2}. \end{aligned}$$

Thus for $1 < p_1, p_2 < \infty$, $s_1 \geq 0$, $s_2 \geq 0$, $s_1 + s_2 = s$,

$$\left\| D^s(fg) - \sum_{|\alpha| \leq s_1} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{|\beta| \leq s_2} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f \right\|_p \lesssim \|D^{s_1} f\|_{p_1} \|D^{s_2} g\|_{p_2}.$$

Case 2: $p_1 = p$, $p_2 = \infty$. If $|\alpha| = s_1$ (in this case s_1 has to be an integer), then

$$\left\| \sum_j \partial^\alpha f_{\leq j-2} D^{s,\alpha} g_j \right\|_p \lesssim \|D^{s_1} f\|_p \|D^{s_2} g\|_{\text{BMO}}.$$

If $|\alpha| < s_1$, then rewrite

$$\begin{aligned} \sum_j \partial^\alpha f_{> j-2} D^{s,\alpha} g_j &= \sum_j \partial^\alpha f_j D^{s,\alpha} g_{< j+2} \\ &= \sum_j \partial^\alpha f_j D^{s,\alpha} g_{< j-2} + \sum_j \partial^\alpha f_j D^{s,\alpha} g_{j-2 \leq \cdot < j+2}. \end{aligned}$$

Clearly then

$$\begin{aligned} &\left\| \sum_j \partial^\alpha f_{> j-2} D^{s,\alpha} g_j \right\|_p \\ &\lesssim \left\| (\partial^\alpha f_j \cdot 2^{j(s_1-|\alpha|)})_{l_j^\infty} \cdot (2^{-j(s_1-|\alpha|)} D^{s,\alpha} g_{< j-2})_{l_j^\infty} \right\|_p + \|D^{s_1} f\|_p \cdot \|D^{s_2} g\|_{\text{BMO}} \\ &\lesssim \|D^{s_1} f\|_p \|D^{s_2} g\|_{\dot{B}_{\infty,\infty}^0} + \|D^{s_1} f\|_p \|D^{s_2} g\|_{\text{BMO}} \lesssim \|D^{s_1} f\|_p \|D^{s_2} g\|_{\text{BMO}}. \end{aligned}$$

On the other hand, for each $|\beta| = s_2$, we have

$$\begin{aligned} \left\| \sum_j \partial^\beta g_{> j-2} D^{s,\beta} f_j \right\|_p &= \left\| \sum_j \partial^\beta g_j D^{s,\beta} f_{< j+2} \right\|_p \\ &\lesssim \|\partial^\beta g\|_{\text{BMO}} \|D^{s,\beta} f\|_p \lesssim \|D^{s_1} f\|_p \cdot \|D^{s_2} g\|_{\text{BMO}}. \end{aligned}$$

Similarly, for each $0 \leq |\beta| < s_2$, we have

$$\begin{aligned} &\left\| \sum_j \partial^\beta g_{> j-2} D^{s,\beta} f_j \right\|_p \\ &= \left\| \sum_j \partial^\beta g_j D^{s,\beta} f_{< j+2} \right\|_p \\ &\lesssim \left\| \sum_j \partial^\beta g_j D^{s,\beta} f_{< j-2} \right\|_p + \left\| \sum_j \partial^\beta g_j D^{s,\beta} f_{j-2 \leq \cdot < j+2} \right\|_p \\ &\lesssim \|(\partial^\beta g_j D^{s,\beta} f_{< j-2})_{l_j^2}\|_p + \|D^{s_1} f\|_p \cdot \|D^{s_2} g\|_{\text{BMO}} \\ &\lesssim \|(2^{(s_2-|\beta|)j} \partial^\beta g_j)_{l_j^\infty} (2^{-(s_2-|\beta|)j} D^{s,\beta} f_{< j-2})_{l_j^2}\|_p + \|D^{s_1} f\|_p \cdot \|D^{s_2} g\|_{\text{BMO}} \\ &\lesssim \|D^{s_2} g\|_{\dot{B}_{\infty,\infty}^0} \|D^{s_1} f\|_p + \|D^{s_1} f\|_p \cdot \|D^{s_2} g\|_{\text{BMO}} \\ &\lesssim \|D^{s_1} f\|_p \cdot \|D^{s_2} g\|_{\text{BMO}}. \end{aligned}$$

Collecting the estimates, we have the following:

If s_1 is not an integer, then

$$\left\| D^s(fg) - \sum_{|\alpha| \leq [s_1]} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{|\beta| \leq [s_2]} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f \right\|_p \lesssim \|D^{s_1} f\|_p \|D^{s_2} g\|_{\text{BMO}}.$$

If $s_1 \geq 0$ is an integer, then

$$\left\| D^s(fg) - \sum_{|\alpha| < s_1} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{|\beta| \leq [s_2]} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f \right\|_p \lesssim \|D^{s_1} f\|_p \|D^{s_2} g\|_{\text{BMO}}.$$

Then clearly for all $s_1 \geq 0$, we have

$$\left\| D^s(fg) - \sum_{|\alpha| < s_1} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{|\beta| \leq s_2} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f \right\|_p \lesssim \|D^{s_1} f\|_p \|D^{s_2} g\|_{\text{BMO}}.$$

Case 3): $p_1 = \infty$, $p_2 = p$. This is similar to Case 2 (with f and g swapped). Clearly,

$$\left\| D^s(fg) - \sum_{|\alpha| \leq s_1} \frac{1}{\alpha!} \partial^\alpha f D^{s,\alpha} g - \sum_{|\beta| < s_2} \frac{1}{\beta!} \partial^\beta g D^{s,\beta} f \right\|_p \lesssim \|D^{s_1} f\|_{\text{BMO}} \|D^{s_2} g\|_p. \quad \square$$

4.2. Proof of Theorem 1.2, case $1/2 < p \leq 1$

Here we shall use the condition $s > d/p - d$ or $s \in 2\mathbb{N}$. A close inspection of the proof for the case $1 < p < \infty$ shows that we only need to modify the estimate for the diagonal piece $\| \sum_j D^s(f_j \tilde{g}_j) \|_p$. For the low-high and high-low pieces, the estimate works for the whole range $1/2 < p < \infty$, $s > 0$ (note that when $1/2 < p \leq 1$ we require $1 < p_1, p_2 < \infty$). The constraint $s > d/p - d$ or $s \in 2\mathbb{N}$ for $1/2 < p \leq 1$ is only needed for the diagonal piece. To deal with this situation, we shall use the approach in Grafakos–Oh [13] and write

$$(D^s(f_j \tilde{g}_j))(x) = \frac{1}{(2\pi)^{2d}} \int 2^{js} \cdot |2^{-j}(\xi + \eta)|^s \chi(2^{-j}\xi) \chi(2^{-j}\eta) \widehat{f}_j(\xi) \widehat{g}_j(\eta) e^{i(\xi+\eta) \cdot x} d\xi d\eta,$$

where χ is a smooth cut-off function with support in $\{\xi : 2^{-m_0} < |\xi| < 2^{m_0}\}$ for some integer $m_0 \geq 1$. Let $\chi_1 \in C_c^\infty(\mathbb{R}^d)$ be such that $\chi_1(z) = 1$ for $|z| \leq 2^{m_0+1}$ and $\chi_1(z) = 0$ for $|z| \geq 2^{m_0+1.5}$. By using Fourier series, it is easy to check that for $|z| \leq 2^{m_0+2}$,

$$|z|^s \chi_1(z) = \sum_{m \in \mathbb{Z}^d} C_m^s e^{2\pi i z \cdot m/L},$$

where $L = 2^{m_0+2}$ and $C_m^s = O((1 + |m|)^{-d-s})$. If $s \in 2\mathbb{N}$, then $C_m^s = O((1 + |m|)^{-C})$ for any $C > 0$. We then have

$$\sum_j D^s(f_j \tilde{g}_j) = \sum_{m \in \mathbb{Z}^d} C_m^s \sum_j 2^{js} P_j^m f_j P_j^m \tilde{g}_j,$$

where $\widehat{P_j^m f}(\xi) = \phi_1(2^{-j}\xi)\hat{f}(\xi)$, and $\phi_1(\xi) = \chi(\xi)e^{2\pi im \cdot \xi/L}$. It follows that

$$\begin{aligned} \left\| \sum_j D^s(f_j \tilde{g}_j) \right\|_p^p &\lesssim \sum_{m \in \mathbb{Z}^d} |C_m^s|^p \left\| \sum_j 2^{js} P_j^m f_j P_j^m \tilde{g}_j \right\|_p^p \\ &\lesssim \sum_{m \in \mathbb{Z}^d} |C_m^s|^p (\| (2^{js_1} P_j^m f_j)_{l_j^2} \|_{p_1} \| (2^{js_2} P_j^m \tilde{g}_j)_{l_j^2} \|_{p_2})^p \\ &\lesssim (\|D^{s_1} f\|_{p_1} \|D^{s_2} g\|_{p_2})^p, \end{aligned}$$

where in the last inequality above, we have used $p(d+s) > d$ (for $s \in 2\mathbb{N}$ we just use the fast decay of C_m^s) and the fact that the operator norm of P_j^m on $L^r(\mathbb{R}^d, l^2)$ ($1 < r < \infty$) is bounded by $C_{r,d} \cdot \log(10 + |m|)$ ($C_{r,d}$ depends only on r and d).

5. Refined Kato–Ponce inequalities

Theorem 5.1. *Let $s > 0$, $1 < p < \infty$, $1 < p_1, p_2, p_3, p_4 \leq \infty$ and $1/p_1 + 1/p_2 = 1/p_3 + 1/p_4 = 1/p$. Then the following hold for any $f, g \in \mathcal{S}(\mathbb{R}^d)$:*

- If $0 < s < 1$, then

$$\begin{aligned} \|D^s(fg) - fD^s g - gD^s f\|_p &\lesssim \begin{cases} \|D^s f\|_{p_1} \|g\|_{p_2}, & \text{if } 1 < p_1, p_2 < \infty; \\ \|D^s f\|_p \|g\|_{\text{BMO}}, & \text{if } p_1 = p, p_2 = \infty. \end{cases} \\ \|D^s(fg) - fD^s g\|_p &\lesssim \|D^s f\|_{\text{BMO}} \|g\|_p, \quad \text{if } p_1 = \infty, p_2 = p. \end{aligned}$$

- If $s = 1$, then

$$\begin{aligned} \|D(fg) - fDg - gDf + \partial f \cdot D^{-1} \partial g\|_p &\lesssim \|Df\|_{p_1} \|g\|_{p_2}, \quad \text{if } 1 < p_1, p_2 < \infty; \\ \|D(fg) - fDg - gDf\|_p &\lesssim \|Df\|_p \|g\|_{\text{BMO}}, \quad \text{if } p_1 = p, p_2 = \infty; \\ \|D(fg) - fDg + \partial f \cdot D^{-1} \partial g\|_p &\lesssim \|Df\|_{\text{BMO}} \|g\|_p, \quad \text{if } p_1 = \infty, p_2 = p. \end{aligned}$$

- If $1 < s < 2$, then

$$\begin{aligned} \|D^s(fg) - fD^s g - gD^s f + s \partial f \cdot D^{s-2} \partial g\|_p &\lesssim \|D^s f\|_{p_1} \|g\|_{p_2}, \\ &\quad \text{if } 1 < p_1, p_2 < \infty; \\ \|D^s(fg) - fD^s g - gD^s f + s \partial f \cdot D^{s-2} \partial g\|_p &\lesssim \|D^s f\|_p \|g\|_{\text{BMO}}, \\ &\quad \text{if } p_1 = p, p_2 = \infty; \\ \|D^s(fg) - fD^s g + s \partial f \cdot D^{s-2} \partial g\|_p &\lesssim \|D^s f\|_{\text{BMO}} \|g\|_p, \\ &\quad \text{if } p_1 = \infty, p_2 = p. \end{aligned}$$

- If $s \geq 2$ and $1 < p_1 < \infty$, then

$$\|D^s(fg) - fD^s g - gD^s f + s \partial f \cdot D^{s-2} \partial g\|_p \lesssim A + B,$$

where

$$A = \begin{cases} \|D^s f\|_{p_1} \|g\|_{p_2}, & \text{if } 1 < p_1, p_2 < \infty; \\ \|D^s f\|_p \|g\|_{\text{BMO}}, & \text{if } p_1 = p, p_2 = \infty; \end{cases}$$

$$B = \begin{cases} \|\partial f\|_{p_3} \cdot \|D^{s-1} g\|_{p_4}, & \text{if } 1 < p_3, p_4 < \infty; \\ \|\partial f\|_{\dot{B}_{\infty, \infty}^0} \|D^{s-1} g\|_p, & \text{if } p_3 = \infty, p_4 = p; \\ \|\partial f\|_p \|D^{s-1} g\|_{\dot{B}_{\infty, \infty}^0}, & \text{if } p_3 = p, p_4 = \infty. \end{cases}$$

- If $s \geq 2$ and $p_1 = \infty, p_2 = p$, then

$$\|D^s(fg) - fD^s g + s\partial f \cdot D^{s-2}\partial g\|_p \lesssim \|D^s f\|_{\text{BMO}} \|g\|_p + B,$$

where B is defined the same as above.

Proof of Theorem 5.1. We have to discuss several cases. The following discussions are a little bit tedious and duly long.

- The case $0 < s < 1$.

Taking $s_1 = s, s_2 = 0$ in (1.4), we get

$$\|D^s(fg) - fD^s g - gD^s f\|_p \lesssim \|D^s f\|_{p_1} \|g\|_{p_2}, \quad \text{if } 1 < p_1, p_2 < \infty.$$

Similarly, from (1.5)–(1.6) we get

$$\begin{aligned} \|D^s(fg) - fD^s g - gD^s f\|_p &\lesssim \|D^s f\|_p \|g\|_{\text{BMO}}, & \text{if } p_1 = p, p_2 = \infty, \\ \|D^s(fg) - fD^s g\|_p &\lesssim \|D^s f\|_{\text{BMO}} \|g\|_p, & \text{if } p_1 = \infty, p_2 = p. \end{aligned}$$

- The case $1 \leq s < 2$.

Subcase $1 < p_1, p_2 < \infty$. Again taking $s_1 = s, s_2 = 0$ in the inequalities (1.4), we get

$$\left\| D^s(fg) - fD^s g - \sum_{|\alpha|=1} \partial^\alpha f D^{s, \alpha} g - gD^s f \right\|_p \lesssim \|D^s f\|_{p_1} \|g\|_{p_2}, \quad \text{if } 1 < p_1, p_2 < \infty.$$

Note that (recall $\partial = (\partial_1, \dots, \partial_d)$)

$$\sum_{|\alpha|=1} \partial^\alpha f D^{s, \alpha} g = -s\partial f \cdot D^{s-2}\partial g.$$

Thus for $1 \leq s < 2$ and $1 < p_1, p_2 < \infty$, we get

$$\|D^s(fg) - fD^s g - gD^s f + s\partial f \cdot D^{s-2}\partial g\|_p \lesssim \|D^s f\|_{p_1} \|g\|_{p_2}.$$

Subcase $p_1 = p, p_2 = \infty$. Consider first $s = 1$. By using (1.5), we have

$$\|D^s(fg) - fD^s g - gD^s f\|_p \lesssim \|D^s f\|_p \|g\|_{\text{BMO}}.$$

Next if $1 < s < 2$, then by (1.5) (note that the terms $|\alpha| = 1$ are now included),

$$\|D^s(fg) - fD^s g - gD^s f + s\partial f \cdot D^{s-2}\partial g\|_p \lesssim \|D^s f\|_p \|g\|_{\text{BMO}}.$$

Subcase $p_1 = \infty, p_2 = p$. By using (1.6), obviously we have

$$\|D^s(fg) - fD^s g + s\partial f \cdot D^{s-2}\partial g\|_p \lesssim \|D^s f\|_{\text{BMO}} \|g\|_p.$$

- The case $s = 2$.

In this case since $D^2 = -\Delta$, we can directly use the formula

$$\Delta(fg) - f\Delta g - g\Delta f = 2\partial f \cdot \partial g.$$

Thus

$$D^s(fg) - fD^s g - gD^s f + s\partial f \cdot D^{s-2}\partial g = 0$$

and no estimate is needed.

- The case $s > 2$.

Subcase 1: $1 < p_1, p_2 < \infty$. By (1.4), we have

$$\begin{aligned} \|D^s(fg) - fD^s g - gD^s f + s\partial f \cdot D^{s-2}\partial g\|_p \\ \lesssim \|D^s f\|_{p_1} \|g\|_{p_2} + \sum_{2 \leq |\alpha| \leq s} \|\partial^\alpha f \cdot D^{s,\alpha} g\|_p. \end{aligned}$$

Then for each $2 \leq |\alpha| < s$, by Lemma 2.10,

$$\begin{aligned} \|D^{|\alpha|} f\|_{\left(\frac{1}{p_1} \cdot \frac{|\alpha|-1}{s-1} + \frac{1}{p_3} \cdot \frac{s-|\alpha|}{s-1}\right)^{-1}} &\lesssim \begin{cases} \|D^s f\|_{p_1}^{\frac{|\alpha|-1}{s-1}} \|Df\|_{p_3}^{\frac{s-|\alpha|}{s-1}}, & \text{if } 1 < p_3 < \infty; \\ \|D^s f\|_{p_1}^{\frac{|\alpha|-1}{s-1}} \|Df\|_{\dot{B}_{\infty,\infty}^0}^{\frac{s-|\alpha|}{s-1}}, & \text{if } p_3 = \infty; \end{cases} \\ \|D^{s-|\alpha|} g\|_{\left(\frac{1}{p_2} \cdot \frac{|\alpha|-1}{s-1} + \frac{1}{p_4} \cdot \frac{s-|\alpha|}{s-1}\right)^{-1}} &\lesssim \begin{cases} \|g\|_{p_2}^{\frac{|\alpha|-1}{s-1}} \|D^{s-1} g\|_{p_4}^{\frac{s-|\alpha|}{s-1}}, & \text{if } p_4 < \infty; \\ \|g\|_{p_2}^{\frac{|\alpha|-1}{s-1}} \|D^{s-1} g\|_{\dot{B}_{\infty,\infty}^0}^{\frac{s-|\alpha|}{s-1}}, & \text{if } p_4 = \infty. \end{cases} \end{aligned}$$

Note that $p \leq \left(\frac{1}{p_1} \cdot \frac{|\alpha|-1}{s-1} + \frac{1}{p_3} \cdot \frac{s-|\alpha|}{s-1}\right)^{-1} < \infty$ and $p \leq \left(\frac{1}{p_2} \cdot \frac{|\alpha|-1}{s-1} + \frac{1}{p_4} \cdot \frac{s-|\alpha|}{s-1}\right)^{-1} < \infty$. Clearly,

$$\begin{aligned} \|\partial^\alpha f\|_{\left(\frac{1}{p_1} \cdot \frac{|\alpha|-1}{s-1} + \frac{1}{p_3} \cdot \frac{s-|\alpha|}{s-1}\right)^{-1}} &\lesssim \|D^{|\alpha|} f\|_{\left(\frac{1}{p_1} \cdot \frac{|\alpha|-1}{s-1} + \frac{1}{p_3} \cdot \frac{s-|\alpha|}{s-1}\right)^{-1}}; \\ \|D^{s,\alpha} g\|_{\left(\frac{1}{p_2} \cdot \frac{|\alpha|-1}{s-1} + \frac{1}{p_4} \cdot \frac{s-|\alpha|}{s-1}\right)^{-1}} &\lesssim \|D^{s-|\alpha|} g\|_{\left(\frac{1}{p_2} \cdot \frac{|\alpha|-1}{s-1} + \frac{1}{p_4} \cdot \frac{s-|\alpha|}{s-1}\right)^{-1}}. \end{aligned}$$

If $|\alpha| = s$, then obviously

$$\sum_{|\alpha|=s} \|\partial^\alpha f \cdot D^{s,\alpha} g\|_p \lesssim \|D^s f\|_{p_1} \|g\|_{p_2}.$$

Thus

$$\begin{aligned} & \sum_{2 \leq |\alpha| \leq s} \|\partial^\alpha f \cdot D^{s,\alpha} g\|_p \\ & \lesssim \|D^s f\|_{p_1} \|g\|_{p_2} + \begin{cases} \|\partial f\|_{p_3} \|D^{s-1} g\|_{p_4}, & \text{if } 1 < p_3, p_4 < \infty; \\ \|\partial f\|_{\dot{B}_{\infty,\infty}^0} \|D^{s-1} g\|_p, & \text{if } p_3 = \infty, p_4 = p; \\ \|\partial f\|_p \|D^{s-1} g\|_{\dot{B}_{\infty,\infty}^0}, & \text{if } p_3 = p, p_4 = \infty. \end{cases} \end{aligned}$$

Subcase 2: $p_1 = \infty$ and $p_2 = p$. By (1.6),

$$\begin{aligned} & \|D^s(fg) - fD^s g + s\partial f \cdot D^{s-2} \partial g\|_p \\ & \lesssim \|D^s f\|_{\text{BMO}} \|g\|_p + \sum_{2 \leq |\alpha| \leq s} \|\partial^\alpha f \cdot D^{s,\alpha} g\|_p. \end{aligned}$$

For each $2 \leq |\alpha| < s$, by Lemma 2.10,

$$\begin{aligned} \|D^{|\alpha|} f\|_{\left(\frac{1}{p_3}, \frac{s-|\alpha|}{s-1}\right)^{-1}} & \lesssim \|D^s f\|_{\dot{B}_{\infty,\infty}^0}^{\frac{|\alpha|-1}{s-1}} \|Df\|_{p_3}^{\frac{s-|\alpha|}{s-1}}, & \text{if } 1 < p_3 < \infty; \\ \|D^{|\alpha|} f\|_{\dot{B}_{\infty,1}^0} & \lesssim \|D^s f\|_{\dot{B}_{\infty,\infty}^0}^{\frac{|\alpha|-1}{s-1}} \|Df\|_{\dot{B}_{\infty,\infty}^0}^{\frac{s-|\alpha|}{s-1}}, & \text{if } p_3 = \infty; \\ \|D^{s-|\alpha|} g\|_{\left(\frac{1}{p}, \frac{|\alpha|-1}{s-1} + \frac{1}{p_4}, \frac{s-|\alpha|}{s-1}\right)^{-1}} & \lesssim \|g\|_p^{\frac{|\alpha|-1}{s-1}} \|D^{s-1} g\|_{p_4}^{\frac{s-|\alpha|}{s-1}}, & \text{if } 1 < p_4 < \infty; \\ \|D^{s-|\alpha|} g\|_{\left(\frac{1}{p}, \frac{|\alpha|-1}{s-1}\right)^{-1}} & \lesssim \|g\|_p^{\frac{|\alpha|-1}{s-1}} \|D^{s-1} g\|_{\dot{B}_{\infty,\infty}^0}^{\frac{s-|\alpha|}{s-1}}, & \text{if } p_4 = \infty. \end{aligned}$$

The second inequality above can be easily proved by splitting into low and high frequencies.

For $|\alpha| = s$ (in this case s is an integer), by Lemma 3.15, we have

$$\begin{aligned} & \sum_{|\alpha|=s} \|\partial^\alpha f \cdot D^{s,\alpha} g\|_p \\ & \lesssim \|D^s f\|_{\text{BMO}} \|g\|_p + \begin{cases} \|\partial f\|_{p_3} \|D^{s-1} g\|_{p_4}, & \text{if } 1 < p_3, p_4 < \infty; \\ \|\partial f\|_{\dot{B}_{\infty,\infty}^0} \|D^{s-1} g\|_p, & \text{if } p_3 = \infty, p_4 = p; \\ \|\partial f\|_p \|D^{s-1} g\|_{\dot{B}_{\infty,\infty}^0}, & \text{if } p_3 = p, p_4 = \infty. \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} & \sum_{2 \leq |\alpha| \leq s} \|\partial^\alpha f \cdot D^{s,\alpha} g\|_p \\ & \lesssim \|D^s f\|_{\text{BMO}} \|g\|_p + \begin{cases} \|\partial f\|_{p_3} \|D^{s-1} g\|_{p_4}, & \text{if } 1 < p_3, p_4 < \infty; \\ \|\partial f\|_{\dot{B}_{\infty,\infty}^0} \|D^{s-1} g\|_p, & \text{if } p_3 = \infty, p_4 = p; \\ \|\partial f\|_p \|D^{s-1} g\|_{\dot{B}_{\infty,\infty}^0}, & \text{if } p_3 = p, p_4 = \infty. \end{cases} \end{aligned}$$

Subcase 3: $p_1 = p, p_2 = \infty$. By (1.5), we have

$$\begin{aligned} & \|D^s(fg) - fD^s g - gD^s f + s\partial f \cdot D^{s-2} \partial g\|_p \\ & \lesssim \|D^s f\|_p \|g\|_{\text{BMO}} + \sum_{2 \leq |\alpha| < s} \|\partial^\alpha f \cdot D^{s,\alpha} g\|_p. \end{aligned}$$

For each $2 \leq |\alpha| < s$, by Lemma 2.10,

$$\begin{aligned} \|D^{|\alpha|}f\|_{\left(\frac{1}{p} \cdot \frac{|\alpha|-1}{s-1} + \frac{1}{p_3} \cdot \frac{s-|\alpha|}{s-1}\right)^{-1}} &\lesssim \begin{cases} \|D^s f\|_p^{\frac{|\alpha|-1}{s-1}} \|Df\|_{p_3}^{\frac{s-|\alpha|}{s-1}}, & \text{if } 1 < p_3 < \infty; \\ \|D^s f\|_p^{\frac{|\alpha|-1}{s-1}} \|Df\|_{\dot{B}_{\infty,\infty}^0}^{\frac{s-|\alpha|}{s-1}}, & \text{if } p_3 = \infty; \end{cases} \\ \|D^{s-|\alpha|}g\|_{\left(\frac{1}{p_4} \cdot \frac{s-|\alpha|}{s-1}\right)^{-1}} &\lesssim \|g\|_{\dot{B}_{\infty,\infty}^0}^{\frac{|\alpha|-1}{s-1}} \|D^{s-1}g\|_{p_4}^{\frac{s-|\alpha|}{s-1}}, \quad \text{if } 1 < p_4 < \infty; \\ \|D^{s-|\alpha|}g\|_{\dot{B}_{\infty,1}^0} &\lesssim \|g\|_{\dot{B}_{\infty,\infty}^0}^{\frac{|\alpha|-1}{s-1}} \|D^{s-1}g\|_{\dot{B}_{\infty,\infty}^0}^{\frac{s-|\alpha|}{s-1}}, \quad \text{if } p_4 = \infty. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{2 \leq |\alpha| < s} \|\partial^\alpha f \cdot D^{s,\alpha}g\|_p \\ &\lesssim \|D^s f\|_p \|g\|_{\text{BMO}} + \begin{cases} \|\partial f\|_{p_3} \|D^{s-1}g\|_{p_4}, & \text{if } 1 < p_3, p_4 < \infty; \\ \|\partial f\|_{\dot{B}_{\infty,\infty}^0} \|D^{s-1}g\|_p, & \text{if } p_3 = \infty, p_4 = p; \\ \|\partial f\|_p \|D^{s-1}g\|_{\dot{B}_{\infty,\infty}^0}, & \text{if } p_3 = p, p_4 = \infty. \end{cases} \end{aligned}$$

□

Corollary 5.2. *Let $s > 0$ and $1 < p < \infty$. Then for any $f, g \in \mathcal{S}(\mathbb{R}^d)$,*

$$(5.1) \quad \|D^s(fg) - fD^s g\|_p \lesssim \|D^s f\|_{p_1} \|g\|_{p_2} + \|\partial f\|_{p_3} \|D^{s-1}g\|_{p_4},$$

where $1 < p_1, p_2, p_3, p_4 \leq \infty$, and $1/p_1 + 1/p_2 = 1/p_3 + 1/p_4 = 1/p$.

For $0 < s < 1$ and $1 < p < \infty$, the following inequality holds:

$$\|D^s(fg) - fD^s g\|_p \lesssim \|D^s f\|_{p_1} \|g\|_{p_2},$$

where $1/p_1 + 1/p_2 = 1/p$, and $1 < p_1, p_2 \leq \infty$.

For $s = 1$ and $1 < p < \infty$,

$$(5.2) \quad \|D(fg) - fDg\|_p \lesssim \|\partial f\|_{p_1} \|g\|_{p_2},$$

where $1/p_1 + 1/p_2 = 1/p$, and $1 < p_1, p_2 \leq \infty$.

For $p_1 = \infty$, $p_2 = p$ and $1 < p < \infty$, we have the following BMO-refinements:

- If $0 < s < 1$, then

$$(5.3) \quad \|D^s(fg) - fD^s g\|_p \lesssim \|D^s f\|_{\text{BMO}} \|g\|_p.$$

- If $s \geq 1$, then

$$(5.4) \quad \|D^s(fg) - fD^s g\|_p \lesssim \|\partial f\|_{p_3} \|D^{s-1}g\|_{p_4} + \|D^s f\|_{\text{BMO}} \|g\|_p,$$

where $1/p_3 + 1/p_4 = 1/p$, $1 < p_3 \leq \infty$, $1 < p_4 < \infty$.

- Also for $s \geq 1$,

$$(5.5) \quad \|D^s(fg) - fD^s g\|_p \lesssim \|\partial f\|_p \|D^{s-1}g\|_{\text{BMO}} + \|D^s f\|_{\text{BMO}} \|g\|_p.$$

Proof of Corollary 5.2. We shall follow the same order as in the statement of Theorem 5.1 and discuss the regimes $0 < s < 1$, $s = 1$ and $s > 1$, respectively.

- The case $0 < s < 1$. Clearly by Theorem 5.1,

$$\|D^s(fg) - fD^s g\|_p \lesssim \begin{cases} \|D^s f\|_{p_1} \|g\|_{p_2}, & \text{if } 1 < p_1 < \infty, 1 < p_2 \leq \infty, \\ \|D^s f\|_{\text{BMO}} \|g\|_p, & \text{if } p_1 = \infty, p_2 = p. \end{cases}$$

Thus (5.2) and (5.3) hold.

- The case $s = 1$. If $1 < p_1 < \infty, 1 < p_2 < \infty$, then

$$\|D(fg) - fDg\|_p \lesssim \|Df\|_{p_1} \|g\|_{p_2} + \|\partial f \cdot D^{-1}\partial g\|_p \lesssim \|\partial f\|_{p_1} \|g\|_{p_2}.$$

If $p_1 = p, p_2 = \infty$, then recall

$$\|D(fg) - fDg - gDf\|_p \lesssim \|Df\|_p \|g\|_{\text{BMO}}.$$

Thus

$$\|D(fg) - fDg\|_p \lesssim \|\partial f\|_p \|g\|_{\infty}.$$

So (5.2) holds except the case $p_1 = \infty$.

If $p_1 = \infty, p_2 = p$, then

$$\|D(fg) - fDg\|_p \lesssim \|Df\|_{\text{BMO}} \|g\|_p + \|\partial f \cdot D^{-1}\partial g\|_p$$

Clearly if $1 < p_3 \leq \infty, 1 < p_4 < \infty$ with $1/p_3 + 1/p_4 = 1/p$, then

$$\|\partial f \cdot D^{-1}\partial g\|_p \lesssim \|\partial f\|_{p_3} \|g\|_{p_4}.$$

Thus (5.4) holds for $s = 1$, and also (5.2) holds.

On the other hand, if $p_3 = p, p_4 = \infty$, then

$$\begin{aligned} \|\partial f \cdot D^{-1}\partial g\|_p &\lesssim \left\| \sum_j \partial f_{\leq j+2} \cdot D^{-1}\partial g_j \right\|_p + \left\| \sum_j \partial f_j \cdot D^{-1}\partial g_{< j-2} \right\|_p \\ &\lesssim \|\partial f\|_p \|g\|_{\text{BMO}} + \|Df\|_{\text{BMO}} \|g\|_p. \end{aligned}$$

Clearly (5.5) holds for $s = 1$.

• The case $s > 1$. Consider first $1 < p_1, p_2 < \infty$. By Theorem 5.1 and Lemma 3.17, we have

$$\begin{aligned} \|D^s(fg) - fD^s g\|_p &\lesssim \|D^s f\|_{p_1} \|g\|_{p_2} + \|\partial f \cdot D^{s-2}\partial g\|_p + \|\partial f\|_{p_3} \|D^{s-1}g\|_{p_4} \\ &\lesssim \|D^s f\|_{p_1} \|g\|_{p_2} + \|\partial f\|_{p_3} \|D^{s-1}g\|_{p_4}, \end{aligned}$$

for any $1 < p_3, p_4 \leq \infty$. Thus (5.1) holds. Similarly, one can easily check that (5.4) and (5.5) hold. \square

Corollary 5.3. *Let $1 < p < \infty$. Let $1 < \widehat{p}_1, p_2, p_3, p_4 \leq \infty$ satisfy $1/p_1 + 1/p_2 = 1/p_3 + 1/p_4 = 1/p$. Then for any $f, g \in \mathcal{S}(\mathbb{R}^d)$, the following hold:*

- If $0 < s \leq 1$, then

$$\|D^s(fg) - fD^s g\|_p \lesssim \|D^{s-1}\partial f\|_{p_1} \|g\|_{p_2}.$$

- If $s > 1$, then

$$\|D^s(fg) - fD^s g\|_p \lesssim \|D^{s-1}\partial f\|_{p_1} \|g\|_{p_2} + \|\partial f\|_{p_3} \|D^{s-1}g\|_{p_4}.$$

Proof. This directly follows from Corollary 5.2 and the identity $D^s f = -D^{-1}\partial \cdot D^{s-1}\partial f$. The only difference is for the case $p_1 = \infty, p_2 = p$. In that case since we have BMO-refinements the inequality is obvious. \square

For $s > 0$, let A^s be a differential operator such that its symbol $\widehat{A^s}(\xi)$ is a homogeneous function of degree s and $\widehat{A^s}(\xi) \in C^\infty(\mathbb{S}^{d-1})$. Then the following corollary can be proved in much the same way as for the operator D^s .

Corollary 5.4. *Let $s > 0$ and $1 < p < \infty$. Then the following hold for any $f, g \in \mathcal{S}(\mathbb{R}^d)$:*

- If $0 < s \leq 1$, then

$$\|A^s(fg) - fA^s g - gA^s f\|_p \lesssim_{A^s, s, p, d} \|D^s f\|_p \|g\|_{\text{BMO}}.$$

- If $s > 1$, then

$$(5.6) \quad \begin{aligned} & \|A^s(fg) - fA^s g - gA^s f - \partial f \cdot A^{s, \partial} g\|_p \\ & \lesssim_{A^s, s, p, d} \|D^s f\|_p \|g\|_{\text{BMO}} + \|\partial f\|_{\dot{B}_{\infty, \infty}^0} \|D^{s-1}g\|_p, \end{aligned}$$

where

$$\widehat{A^{s, \partial}}(\xi) = \frac{1}{i} \partial_\xi(\widehat{A^s}(\xi)).$$

In fact a stronger inequality holds for $s > 1$:

$$(5.7) \quad \begin{aligned} & \|A^s(fg) - fA^s g - gA^s f - \partial f \cdot A^{s, \partial} g\|_p \\ & \lesssim_{A^s, s, p, d} \|D^s f\|_p \|g\|_{\dot{B}_{\infty, \infty}^0} + \|\partial f\|_{\dot{B}_{\infty, \infty}^0} \|D^{s-1}g\|_p, \end{aligned}$$

- For all $s > 0$,

$$(5.8) \quad \|A^s(fg) - fA^s g\|_p \lesssim_{A^s, s, p, d} \|D^s f\|_p \|g\|_\infty + \|\partial f\|_\infty \|D^{s-1}g\|_p.$$

Proof of Corollary 5.4. All the statements except (5.7) follow by mimicking the proof for the operator D^s . The argument proceeds by using Corollary 1.4 together with further interpolation inequalities. We omit the details.

On the other hand, it is possible to give a more “direct” proof of the above inequalities. We illustrate this by sketching a proof for (5.8) which is most useful in practice. To this end, we first decompose

$$fg = \sum_j f_{<j-2} g_j + \sum_j f_j g_{<j-2} + \sum_j f_j \tilde{g}_j.$$

Then

$$\begin{aligned} \|[A^s, f_{<j-2}]g_j\|_p &\lesssim \|(2^{js} \mathcal{M}(\partial f_{<j-2}) 2^{-j} \mathcal{M}g_j)_{l_j^2}\|_p \lesssim \|D^{s-1}g\|_p \|\partial f\|_\infty; \\ \|[A^s, g_{<j-2}]f_j\|_p &\lesssim \|(2^{js} \mathcal{M}(\partial g_{<j-2}) 2^{-j} \mathcal{M}f_j)_{l_j^2}\|_p \lesssim \|D^s f\|_p \|g\|_{\dot{B}_{\infty,\infty}^0}, \\ \|A^s(f_j \tilde{g}_j)\|_p &\lesssim \|(2^{ks} P_k(\sum_{j>k-10} f_j \tilde{g}_j)_{l_k^2})\|_p \lesssim \|(2^{js} f_j \tilde{g}_j)_{l_j^2}\|_p \lesssim \|D^s f\|_p \|g\|_{\dot{B}_{\infty,\infty}^0}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_j f_{<j-2} A^s g_j &= f A^s g - \sum_j f_j A^s g_{\leq j+2}, \\ \sum_j g_{<j-2} A^s f_j &= g A^s f - \sum_j g_j A^s f_{\leq j+2}. \end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned} \left\| \sum_j f_j A^s g_{\leq j+2} \right\|_p &\lesssim \|Df\|_{\text{BMO}} \|A^s D^{-1}g\|_p \lesssim \|\partial f\|_\infty \|D^{s-1}g\|_p; \\ \left\| \sum_j g_j A^s f_{\leq j+2} \right\|_p &\lesssim \|g\|_{\text{BMO}} \|A^s f\|_p \lesssim \|g\|_\infty \|D^s f\|_p. \end{aligned}$$

Thus the inequality (5.8) follows. A close inspection of the above shows that we actually proved

$$\|A^s(fg) - fA^s g - gA^s f\|_p \lesssim_{s,p,d} \|D^s f\|_p \|g\|_{\text{BMO}} + \|\partial f\|_\infty \|D^{s-1}g\|_p.$$

We now show how to prove (5.7). For this we just need to modify the estimate of the pieces $\sum_j f_j A^s g_{\leq j+2}$, $\sum_j g_j A^s f_{\leq j+2}$ and $\sum_j [A^s, f_{<j-2}]g_j$ in the preceding argument. We first explain how to estimate the first two pieces. In the following computation we shall avoid using Lemma 3.1 since this is where BMO-norm comes in. Now split

$$\sum_j f_j A^s g_{\leq j+2} = \sum_j f_j A^s g_{<j-2} + \sum_j f_j A^s g_{j-2 \leq \cdot \leq j+2}.$$

By frequency localization,

$$\begin{aligned} \left\| \sum_j f_j A^s g_{<j-2} \right\|_p &\lesssim \|(f_j A^s g_{<j-2})_{l_j^2}\|_p \\ &\lesssim \|(2^{js} f_j)_{l_j^2}\|_p \|(2^{-js} A^s g_{<j-2})_{l_j^\infty}\|_\infty \lesssim \|D^s f\|_p \|g\|_{\dot{B}_{\infty,\infty}^0}. \end{aligned}$$

Denote $\tilde{g}_j = g_{j-2 \leq \cdot \leq j+2}$. Then

$$(5.9) \quad \left\| \sum_j f_j A^s \tilde{g}_j \right\|_p \lesssim \|(2^{j \frac{s+1}{2}} f_j)_{l_j^2} (2^{-j \frac{s+1}{2}} A^s \tilde{g}_j)_{l_j^2}\|_p \lesssim \|D^{\frac{s+1}{2}} f\|_{2p} \|D^{\frac{s-1}{2}} g\|_{2p}.$$

For $s > 0$ and $s \neq 1$, we have⁴

$$\|D^{\frac{s+1}{2}} f\|_{2p} \lesssim \|D^s f\|_p^{1/2} \|\partial f\|_{\dot{B}_{\infty, \infty}^0}^{1/2} \quad \text{and} \quad \|D^{\frac{s-1}{2}} g\|_{2p} \lesssim \|D^{s-1} g\|_p^{1/2} \|g\|_{\dot{B}_{\infty, \infty}^0}^{1/2}.$$

Thus for $s > 0$ and $s \neq 1$,

$$\left\| \sum_j f_j A^s g_{\leq j+2} \right\|_p \lesssim \|D^s f\|_p \|g\|_{\dot{B}_{\infty, \infty}^0} + \|\partial f\|_{\dot{B}_{\infty, \infty}^0} \|D^{s-1} g\|_p.$$

On the other hand for the piece $\sum_j g_j A^s f_{\leq j+2}$ we only need to handle the part $\sum_j g_j A^s f_{< j-2}$ since the other part is treated the same way as in (5.9). Now for $s > 1$, by using Lemma 2.13,

$$\begin{aligned} \left\| \sum_j g_j A^s f_{< j-2} \right\|_p &\lesssim \|(g_j A^s f_{< j-2})_{l_j^2}\|_p \\ &\lesssim \|(2^{j(s-1)} g_j)_{l_j^2}\|_p \|(2^{-j(s-1)} A^s f_{< j-2})_{l_j^\infty}\|_\infty \lesssim \|D^{s-1} g\|_p \|\partial f\|_{\dot{B}_{\infty, \infty}^0}. \end{aligned}$$

Next for the commutator piece $[A^s, f_{< j-2}]g_j$, we denote

$$K_j = \sum_{a=-10}^{10} A^s P_{j+a} \delta_0,$$

and write

$$(5.10) \quad \begin{aligned} &([A^s, f_{< j-2}]g_j)(x) \\ &= \int K_j(y) (f_{< j-2}(x-y) - f_{< j-2}(x) - (\partial f_{< j-2})(x) \cdot (-y)) g_j(x-y) dy \end{aligned}$$

$$(5.11) \quad + \partial f_{< j-2}(x) \int K_j(y) (-y) g_j(x-y) dy.$$

For (5.10), it is easy to check that

$$\left\| \sum_j (5.10) \right\|_p \lesssim \|(\mathcal{M}(\partial^2 f_{< j-2}) \cdot 2^{j(s-2)} \mathcal{M} g_j)_{l_j^2}\|_p \lesssim \|D^{s-1} g\|_p \|\partial f\|_{\dot{B}_{\infty, \infty}^0}.$$

On the other hand,

$$\sum_j (5.11) = \partial f \cdot A^{s, \partial} g - \sum_j \partial f_j \cdot A^{s, \partial} g_{\leq j+2}.$$

⁴These interpolation inequalities can be proved in the same way as in Lemma 2.10.

Clearly,

$$\left\| \sum_j \partial f_j A^{s,\partial} g_{<j-2} \right\|_p \lesssim \|(2^{j(s-1)} \partial f_j 2^{-j(s-1)} A^{s,\partial} g_{<j-2})_{l_j^2}\|_p \lesssim \|D^s f\|_p \|g\|_{\dot{B}_{\infty,\infty}^0}.$$

The piece $\sum_j \partial f_j \cdot A^{s,\partial} g_{j-2 \leq \cdot \leq j+2}$ can be estimated in the same way as in (5.9):

$$\left\| \sum_j \partial f_j \cdot A^{s,\partial} g_{j-2 \leq \cdot \leq j+2} \right\|_p \lesssim \|D^s f\|_p \|g\|_{\dot{B}_{\infty,\infty}^0} + \|\partial f\|_{\dot{B}_{\infty,\infty}^0} \|D^{s-1} g\|_p.$$

Thus (5.7) holds. \square

Remark 5.5. As was already mentioned in the introduction, Corollary 5.4 can be used to prove (1.14) and its L^p -versions. Indeed set $A_j^{s+1} = D^s \partial_j$, $f = u_j$, $g = B$, then

$$\|D^s \partial_j(u_j B) - u_j D^s \partial_j B\|_p \lesssim_{s,p,d} \|D^{s+1} u\|_p \|B\|_{\infty} + \|\partial u\|_{\infty} \|D^s B\|_p,$$

or upon summing in j (and using $\nabla \cdot u = 0$),

$$\|D^s((u \cdot \nabla)B) - (u \cdot \nabla)(D^s B)\|_p \lesssim_{s,p,d} \|D^{s+1} u\|_p \|B\|_{\infty} + \|\partial u\|_{\infty} \|D^s B\|_p.$$

Now if $s > d/p$, we can use Sobolev embedding to get

$$(5.12) \quad \|D^s((u \cdot \nabla)B) - (u \cdot \nabla)D^s B\|_p \lesssim_{s,p,d} \|J^s \nabla u\|_p \|J^s B\|_p.$$

Remark 5.6. One can construct divergence-free counterexamples to the estimate (5.12) for the borderline case $s = d/p$. The key is to use the estimate (5.6) (for the operator $A_j^{s+1} = D^s \partial_j$). It is easy to check that (by using $\nabla \cdot u = 0$)

$$\begin{aligned} & \|D^s \partial_j(u_j B) - u_j D^s \partial_j B + \partial u_j \cdot s D^{s-2} \partial \partial_j B\|_p \\ & \lesssim_{s,p,d} \|D^{s+1} u\|_p \|B\|_{\text{BMO}} + \|\partial u\|_{\dot{B}_{\infty,\infty}^0} \|D^s B\|_p. \end{aligned}$$

Upon summing in j and using $s = d/p$, we get

$$\|D^s((u \cdot \nabla)B) - (u \cdot D^s \nabla)B + s \sum_{j=1}^d (\partial u_j \cdot D^{s-2} \partial) \partial_j B\|_p \lesssim_{s,p,d} \|D^{s+1} u\|_p \|D^s B\|_p.$$

Now to finish the construction it suffices for us to exhibit a pair of divergence-free u , B with the property

$$\|J^{1+d/p} u\|_p + \|J^{d/p} B\|_p \leq 1,$$

such that

$$\left\| \sum_{j=1}^d (\partial u_j \cdot D^{s-2} \partial) \partial_j B \right\|_p \gg 1.$$

For a construction of such examples, see Remark 1.17 in [2]. Alternatively one can use the idea in Section 9 of this paper.

6. Refined Kato–Ponce inequalities for the operator J^s

Lemma 6.1. *Let $s > 0$ and $\tilde{J}^s = J^s - I$ (i.e., $\widehat{\tilde{J}^s}(\xi) = (1 + |\xi|^2)^{s/2} - 1$). For any $\phi \in C_c^\infty(\mathbb{R}^d)$ with $\text{supp}(\phi) \subset \{\xi : 0 < c_1 < |\xi| < c_2 < \infty\}$, define P_j^ϕ as*

$$\widehat{P_j^\phi} f(\xi) = \phi\left(\frac{\xi}{2^j}\right) \hat{f}(\xi), \quad j \in \mathbb{Z}.$$

Define $K_j = \tilde{J}^s P_j^\phi \delta_0$ where δ_0 is the usual Dirac delta function. Then for any integer $m \geq 1$, and any $x \in \mathbb{R}^d$,

$$\begin{aligned} |K_j(x)| &\lesssim_{m,\phi,d,s} 2^{j(2+d)} (1 + 2^j|x|)^{-m}, & \text{if } j \leq 0, \\ |K_j(x)| &\lesssim_{m,\phi,d,s} 2^{j(s+d)} (1 + 2^j|x|)^{-m}, & \text{if } j > 0. \end{aligned}$$

Proof. Consider first $j \leq 0$. Clearly

$$\begin{aligned} K_j(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} ((1 + |\xi|^2)^{s/2} - 1) \phi\left(\frac{\xi}{2^j}\right) e^{i\xi \cdot x} d\xi \\ &= \frac{1}{(2\pi)^d} \cdot 2^{jd} \int_{\mathbb{R}^d} ((1 + |2^j \tilde{\xi}|^2)^{s/2} - 1) \phi(\tilde{\xi}) e^{i\tilde{\xi} \cdot 2^j x} d\tilde{\xi}. \end{aligned}$$

If $|2^j x| \leq 1$, then since $|(1 + |2^j \tilde{\xi}|^2)^{s/2} - 1| \lesssim 2^{2j}$ for $|\tilde{\xi}| \sim 1$, $j \leq 0$, we get

$$|K_j(x)| \lesssim 2^{jd} \cdot 2^{2j} = 2^{j(d+2)}.$$

If $|2^j x| > 1$, then we can integrate by parts m -times and get

$$|K_j(x)| \lesssim 2^{jd} (2^j|x|)^{-m} \cdot 2^{2j}.$$

The case for $j > 0$ is similar. We omit details. \square

Theorem 6.2. *Let $s > 0$ and $1 < p < \infty$. Then the following hold for any $f, g \in \mathcal{S}(\mathbb{R}^d)$:*

- If $0 < s \leq 1$, then

$$\|J^s(fg) - fJ^s g - g(J^s f - f)\|_p \lesssim_{s,d,p} \|J^{s-1} \partial f\|_p \|g\|_{\text{BMO}}.$$

- If $s > 1$, then

$$\begin{aligned} \|J^s(fg) - fJ^s g - g(J^s f - f) + s \partial f \cdot J^{s-2} \partial g\|_p \\ \lesssim_{s,p,d} \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}_{\infty,\infty}^0} + \|\partial f\|_{\dot{B}_{\infty,\infty}^0} \|J^{s-2} \partial g\|_p. \end{aligned}$$

Proof. Define $\tilde{J}^s = J^s - I$. On the Fourier side, we have $\widehat{\tilde{J}^s}(\xi) = (1 + |\xi|^2)^{s/2} - 1$. Note that for $|\xi| \ll 1$, $\widehat{\tilde{J}^s}(\xi) \sim |\xi|^2$. This property will be useful for controlling low frequencies in later computations.

Now

$$(6.1) \quad \begin{aligned} J^s(fg) - fJ^s g &= \tilde{J}^s(fg) - f\tilde{J}^s g \\ &= \sum_j (\tilde{J}^s(f_{<j-2}g_j) - f_{<j-2}\tilde{J}^s g_j) \end{aligned}$$

$$(6.2) \quad + \sum_j (\tilde{J}^s(f_j g_{<j-2}) - f_j \tilde{J}^s g_{<j-2})$$

$$(6.3) \quad + \sum_j (\tilde{J}^s(f_j \tilde{g}_j) - f_j \tilde{J}^s \tilde{g}_j),$$

where $\tilde{g}_j = \sum_{a=-2}^2 g_{j+a}$.

Estimate of (6.1). Let $\tilde{P}_j = \sum_{l=-10}^{10} P_{j+l}$ and $K_j = \tilde{J}^s \tilde{P}_j \delta_0$. Then

$$(6.4) \quad \begin{aligned} &\tilde{J}^s(f_{<j-2}g_j) - f_{<j-2}\tilde{J}^s g_j \\ &= \int_{\mathbb{R}^d} K_j(y)(f_{<j-2}(x-y) - f_{<j-2}(x))g_j(x-y) dy \\ &= \int_{\mathbb{R}^d} K_j(y)(f_{<j-2}(x-y) - f_{<j-2}(x) + \partial f_{<j-2}(x) \cdot y)g_j(x-y) dy \end{aligned}$$

$$(6.5) \quad - \int_{\mathbb{R}^d} K_j(y)\partial f_{<j-2}(x) \cdot y g_j(x-y) dy.$$

Estimate of (6.4). By the fundamental theorem of calculus and Lemma 2.3, we have

$$\begin{aligned} &|f_{<j-2}(x-y) - f_{<j-2}(x) - \partial f_{<j-2}(x) \cdot (-y)| \\ &\lesssim \int_0^1 |(\partial^2 f_{<j-2})(x - \theta y)| d\theta \cdot |y|^2 \\ &\lesssim \mathcal{M}(\partial^2 f_{<j-2})(x) \cdot (1 + 2^j |y|)^d \cdot |y|^2. \end{aligned}$$

Therefore by Lemma 6.1,

$$\begin{aligned} &\left\| \sum_{j \leq 0} (6.4) \right\|_p \lesssim \sum_{j \leq 0} \|\partial^2 f_{<j-2}\|_p \cdot \|g_j\|_\infty \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}_{\infty, \infty}^0}. \\ &\left\| \sum_{j > 0} (6.4) \right\|_p \lesssim \|(2^{js} \mathcal{M}(\partial^2 f_{<j-2})) \cdot 2^{-2j} \cdot \mathcal{M}g_j\|_{l_j^2(j>0)} \|p\|. \end{aligned}$$

If $0 < s \leq 1$, then

$$\begin{aligned} &\|(2^{-2j} 2^{js} \mathcal{M}(\partial^2 f_{<j-2}))\|_{l_j^2(j>0)} \|p\| \lesssim \|(2^{j(s-2)} |\partial^2 f_{<j-2}|)\|_{l_j^2(j>0)} \|p\| \\ &\lesssim \|\partial^2 f_{<-4}\|_p + \|(2^{j(s-2)} |\partial^2 f_{-4 \leq \cdot < j-2}|)\|_{l_j^2(j>0)} \|p\| \\ &\lesssim \|J^{s-1} \partial f\|_p + \|D^s f_{\geq -4}\|_p \lesssim \|J^{s-1} \partial f\|_p. \end{aligned}$$

If $s > 1$, then

$$\begin{aligned} \| (2^{-2j} \mathcal{M}(\partial^2 f_{<j-2}) \cdot 2^{js} \cdot \mathcal{M}g_j)_{l^2_{j>0}} \|_p &\lesssim \| (\|\partial f\|_{\dot{B}^0_{\infty,\infty}} \cdot 2^{j(s-1)} \mathcal{M}g_j)_{l^2_{j>0}} \|_p \\ &\lesssim \|\partial f\|_{\dot{B}^0_{\infty,\infty}} \|J^{s-2} \partial g\|_p. \end{aligned}$$

Thus

$$\left\| \sum_j (6.4) \right\|_p \lesssim \begin{cases} \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}^0_{\infty,\infty}}, & \text{if } 0 < s \leq 1, \\ \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}^0_{\infty,\infty}} + \|\partial f\|_{\dot{B}^0_{\infty,\infty}} \cdot \|J^{s-2} \partial g\|_p, & \text{if } s > 1. \end{cases}$$

Estimate of (6.5). Note that

$$\begin{aligned} \int_{\mathbb{R}^d} K_j(y) y g_j(x-y) dy &= \mathcal{F}^{-1} \left(\frac{1}{-i} \partial_\xi (\widehat{K}_j(\xi)) \widehat{g}_j(\xi) \right) \\ &= \mathcal{F}^{-1} \left(\frac{1}{-i} \partial_\xi \left((1 + |\xi|^2)^{s/2} - 1 \right) \tilde{\psi} \left(\frac{\xi}{2^j} \right) \psi \left(\frac{\xi}{2^j} \right) \widehat{g}_j(\xi) \right) \\ &= \mathcal{F}^{-1} \left(\frac{1}{-i} \partial_\xi \left((1 + |\xi|^2)^{s/2} \right) \psi \left(\frac{\xi}{2^j} \right) \widehat{g}_j(\xi) \right) = s J^{s-2} \partial g_j. \end{aligned}$$

So

$$\sum_j (6.5) = -s \sum_j \partial f_{<j-2} \cdot J^{s-2} \partial g_j.$$

If $0 < s < 1$, then

$$\begin{aligned} \left\| \sum_j \partial f_{<j-2} \cdot J^{s-2} \partial g_j \right\|_p &\lesssim \sum_{j \leq 0} \|\partial f_{<0}\|_p \|g_j\|_\infty \cdot 2^j + \|(|\partial f_{<j-2}| \cdot |J^{s-2} \partial g_j|)_{l^2_{j>0}}\|_p \\ &\lesssim \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}^0_{\infty,\infty}} + \|(|\partial f_{<j-2}| \cdot 2^{j(s-1)})_{l^2_{j>0}}\|_p \cdot \|g\|_{\dot{B}^0_{\infty,\infty}} \\ &\lesssim \|J^{s-1} \partial f\|_p \cdot \|g\|_{\dot{B}^0_{\infty,\infty}}. \end{aligned}$$

If $s = 1$, then by Lemma 3.1,

$$\left\| \sum_j \partial f_{<j-2} \cdot J^{-1} \partial g_j \right\|_p \lesssim \|\partial f\|_p \|g\|_{\text{BMO}}.$$

If $s > 1$, then we write

$$\begin{aligned} \sum_j \partial f_{<j-2} \cdot J^{s-2} \partial g_j &= \partial f \cdot J^{s-2} \partial g - \sum_j \partial f_{\geq j-2} \cdot J^{s-2} \partial g_j \\ &= \partial f \cdot J^{s-2} \partial g - \sum_j \partial f_j \cdot J^{s-2} \partial g_{\leq j+2}. \end{aligned}$$

Note that by using Lemma 3.1, we have

$$\left\| \sum_j \partial f_j \cdot J^{s-2} \partial g_{\leq j+2} \right\|_p \lesssim \|\partial f\|_{\text{BMO}} \cdot \|J^{s-2} \partial g\|_p.$$

This bound can be improved as we show below.

First we deal with the low frequency piece:

$$\left\| \sum_{j \leq 10} \partial f_j \cdot J^{s-2} \partial P_{\leq j+2} g \right\|_p \lesssim \sum_{j \leq 10} \|\partial f\|_p \|g\|_{\dot{B}_{\infty, \infty}^0} \cdot 2^j \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}_{\infty, \infty}^0}.$$

For non-low frequencies, we first decompose $g = g_{\leq 1} + g_{>1}$ and bound the piece containing $g_{\leq 1}$ as

$$\left\| \sum_{j > 10} \partial f_j \cdot J^{s-2} \partial P_{\leq j+2}(g_{\leq 1}) \right\|_p \lesssim \sum_{j > 10} \|\partial f_j\|_p \|g\|_{\dot{B}_{\infty, \infty}^0} \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}_{\infty, \infty}^0}.$$

For the piece containing $g_{>1}$, we further write

$$\begin{aligned} & \sum_{j > 10} \partial f_j \cdot J^{s-2} \partial P_{\leq j+2}(g_{>1}) \\ &= \sum_{j > 10} \partial f_j \cdot J^{s-2} \partial P_{\leq j-2}(g_{>1}) + \sum_{j > 10} \partial f_j \cdot J^{s-2} \partial P_{j-2 < \cdot \leq j+2}(g_{>1}) \\ &= \sum_{j > 10} \partial f_j \cdot J^{s-2} \partial P_{>1} P_{\leq j-2} g + \sum_{j > 10} \partial f_j \cdot J^{s-2} \partial \tilde{g}_j. \end{aligned}$$

Firstly thanks to frequency localization,

$$\begin{aligned} & \left\| \sum_{j > 10} \partial f_j \cdot J^{s-2} \partial P_{>1} P_{\leq j-2} g \right\|_p \lesssim \|(\partial f_j \cdot J^{s-2} \partial P_{>1} P_{\leq j-2} g)_{l_j^2(j > 10)}\|_p \\ & \lesssim \|(2^{j(s-1)} \partial f_j)_{l_j^2(j > 10)}\|_p \|(2^{-j(s-1)} J^{s-2} \partial P_{>1} P_{\leq j-2} g)_{l_j^\infty(j > 10)}\|_\infty \\ & \lesssim \|J^{s-1} \partial f\|_p \cdot \|g\|_{\dot{B}_{\infty, \infty}^0}. \end{aligned}$$

Then recalling $s > 1$,

$$\begin{aligned} & \left\| \sum_{j > 10} \partial f_j \cdot J^{s-2} \partial P_{j-2 < \cdot \leq j+2}(g_{>1}) \right\|_p \\ & \lesssim \|(2^{j \frac{s-1}{2}} \partial f_j)_{l_j^2(j > 10)}\|_{2p} \|(2^{-j \frac{s-1}{2}} J^{s-2} \partial P_{j-2 < \cdot \leq j+2}(g_{>1}))_{l_j^2(j > 10)}\|_{2p} \\ & \lesssim \|J^{s-1} \partial f\|_p^{1/2} \|\partial f\|_{\dot{B}_{\infty, \infty}^0}^{1/2} \cdot \|g\|_{\dot{B}_{\infty, \infty}^0}^{1/2} \cdot \|J^{s-2} \partial g\|_p^{1/2} \\ & \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}_{\infty, \infty}^0} + \|\partial f\|_{\dot{B}_{\infty, \infty}^0} \|J^{s-2} \partial g\|_p, \end{aligned}$$

where in the second inequality above we have used (a version of) Lemma 2.10.

Thus for $0 < s < 1$,

$$\left\| \sum_j (6.5) \right\|_p \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}_{\infty, \infty}^0};$$

for $s = 1$,

$$\left\| \sum_j (6.5) \right\|_p \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\text{BMO}};$$

for $s > 1$,

$$\left\| \left(\sum_j (6.5) \right) - s \partial f \cdot J^{s-2} \partial g \right\|_p \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}_{\infty, \infty}^0} + \|\partial f\|_{\dot{B}_{\infty, \infty}^0} \|J^{s-2} \partial g\|_p.$$

Estimate of (6.2). We first estimate the piece $\sum_j f_j \tilde{J}^s g_{<j-2}$.

Clearly by frequency localization,

$$\left\| \sum_j f_j \tilde{J}^s g_{<j-2} \right\|_p \lesssim \|(f_j \tilde{J}^s g_{<j-2})\|_{l_j^2} \|g\|_p.$$

For $j \leq 0$, by Lemma 6.1,

$$\|\tilde{J}^s g_{<j-2}\|_{\infty} \lesssim \sum_{k < j} 2^{2k} \|g_k\|_{\infty} \lesssim 2^{2j} \|g\|_{\dot{B}_{\infty, \infty}^0}.$$

For $j > 0$,

$$\|\tilde{J}^s g_{<j-2}\|_{\infty} \lesssim \|\tilde{J}^s g_{<0}\|_{\infty} + \sum_{0 \leq k < j-2} 2^{ks} \|g_k\|_{\infty} \lesssim \|g\|_{\dot{B}_{\infty, \infty}^0} \cdot 2^{js}.$$

Thus

$$\begin{aligned} \|(f_j \tilde{J}^s g_{<j-2})\|_{l_j^2} &\lesssim (\|(f_j \cdot 2^{2j})\|_{l_j^2(j \leq 0)} + \|(f_j \cdot 2^{js})\|_{l_j^2(j > 0)}) \cdot \|g\|_{\dot{B}_{\infty, \infty}^0} \\ &\lesssim \|J^{s-1} \partial f\|_p \cdot \|g\|_{\dot{B}_{\infty, \infty}^0}. \end{aligned}$$

Next we estimate the piece $\sum_j \tilde{J}^s (f_j g_{<j-2})$. Write

$$\begin{aligned} \sum_j \tilde{J}^s (f_j g_{<j-2}) &= \sum_j [\tilde{J}^s, g_{<j-2}] f_j + \sum_j g_{<j-2} \tilde{J}^s f_j \\ &= \sum_j [\tilde{J}^s, g_{<j-2}] f_j + g \tilde{J}^s f - \sum_j g_{\geq j-2} \tilde{J}^s f_j \\ &= \sum_j [\tilde{J}^s, g_{<j-2}] f_j - \sum_j g_j \tilde{J}^s f_{\leq j+2} + g \tilde{J}^s f. \end{aligned}$$

By Lemma 6.1,

$$\begin{aligned} |([\tilde{J}^s, g_{<j-2}] f_j)(x)| &\lesssim \int_{\mathbb{R}^d} |K_j(y)| \cdot |g_{<j-2}(x-y) - g_{<j-2}(x)| \cdot |f_j(x-y)| dy \\ &\lesssim \|\partial g_{<j-2}\|_{\infty} \cdot \int_{\mathbb{R}^d} |K_j(y)| \cdot |y| \cdot |f_j(x-y)| dy \\ &\lesssim \|\partial g_{<j-2}\|_{\infty} \cdot \begin{cases} 2^j \mathcal{M} f_j(x), & \text{if } j \leq 0, \\ 2^{j(s-1)} \mathcal{M} f_j(x), & \text{if } j > 0. \end{cases} \end{aligned}$$

Thus by frequency localization,

$$\begin{aligned} \left\| \sum_j [\tilde{J}^s, g_{<j-2}] f_j \right\|_p &\lesssim \|([\tilde{J}^s, g_{<j-2}] f_j)_{l_j^2}\|_p \\ &\lesssim \|g\|_{\dot{B}_{\infty,\infty}^0} \cdot (\|(2^{2j} \mathcal{M} f_j)_{l_j^2(j \leq 0)}\|_p + \|(2^{j s} \mathcal{M} f_j)_{l_j^2(j > 0)}\|_p) \\ &\lesssim \|g\|_{\dot{B}_{\infty,\infty}^0} \cdot \|J^{s-1} \partial f\|_p. \end{aligned}$$

By Lemma 3.1, it is easy to see that for $s > 0$:

$$\left\| \sum_j g_j \tilde{J}^s f_{\leq j+2} \right\|_p \lesssim \|g\|_{\text{BMO}} \|\tilde{J}^s f\|_p \lesssim \|g\|_{\text{BMO}} \|J^{s-1} \partial f\|_p.$$

However we shall improve this bound below in the case $s > 1$. Write $f_{\leq j+2} = f_{<j-2} + \tilde{f}_j$ with $\tilde{f}_j := f_{j-2 \leq \cdot \leq j+2}$. Then

$$\begin{aligned} \left\| \sum_j g_j \tilde{J}^s f_{<j-2} \right\|_p &\lesssim \|(g_j \tilde{J}^s f_{<j-2})_{l_j^2}\|_p \\ &\lesssim \|(g_j 2^j)_{l_j^2(j \leq 0)}\|_p \cdot \|\partial f\|_{\dot{B}_{\infty,\infty}^0} + \|(g_j \cdot 2^{j(s-1)})_{l_j^2(j > 0)}\|_p \cdot \|\partial f\|_{\dot{B}_{\infty,\infty}^0} \\ &\lesssim \|J^{s-2} \partial g\|_p \|\partial f\|_{\dot{B}_{\infty,\infty}^0}. \end{aligned}$$

For \tilde{f}_j , we bound the low frequency piece as

$$\left\| \sum_{j \leq 10} g_j \tilde{J}^s \tilde{f}_j \right\|_p \lesssim \sum_{j \leq 10} \|g_j\|_{\infty} \cdot \|\tilde{f}_j\|_p \cdot 2^{2j} \lesssim \|g\|_{\dot{B}_{\infty,\infty}^0} \cdot \|J^{s-1} \partial f\|_p.$$

For the non-low frequency piece, we have

$$\begin{aligned} \left\| \sum_{j > 10} g_j \tilde{J}^s \tilde{f}_j \right\|_p &\lesssim \|(2^{-j(\frac{s-1}{2})} \tilde{J}^s \tilde{f}_j)_{l_j^2(j > 10)}\|_{2p} \|(2^{j(\frac{s-1}{2})} g_j)_{l_j^2(j > 10)}\|_{2p} \\ &\lesssim \|J^{s-1} \partial f\|_p^{1/2} \|\partial f\|_{\dot{B}_{\infty,\infty}^0}^{1/2} \|g\|_{\dot{B}_{\infty,\infty}^0}^{1/2} \|J^{s-2} \partial g\|_p^{1/2} \\ (6.6) \quad &\lesssim \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}_{\infty,\infty}^0} + \|\partial f\|_{\dot{B}_{\infty,\infty}^0} \|J^{s-2} \partial g\|_p. \end{aligned}$$

Estimate of (6.3). Clearly, by Lemma 6.1,

$$\left\| \sum_{j \leq 0} \tilde{J}^s (f_j \tilde{g}_j) \right\|_p \lesssim \sum_{j \leq 0} 2^{2j} \|f_j\|_p \|\tilde{g}_j\|_{\infty} \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}_{\infty,\infty}^0}.$$

Also,

$$\left\| \sum_{j > 0} \tilde{J}^s P_{\leq 0} (f_j \tilde{g}_j) \right\|_p \lesssim \sum_{j > 0} \|f_j\|_p \|g_j\|_{\infty} \lesssim \|J^{s-1} \partial f\|_p \cdot \|g\|_{\dot{B}_{\infty,\infty}^0}.$$

On the other hand,

$$\begin{aligned}
\left\| \sum_{j>0} \tilde{J}^s P_{>0}(f_j \tilde{g}_j) \right\|_p &\lesssim \|(2^{ks} P_k(\sum_{j>k-4} (f_{>-10})_j \tilde{g}_j))_{l_k^2}\|_p \\
&\lesssim \|(2^{ks} \sum_{j>k-4} |(f_{>-10})_j \tilde{g}_j|)_{l_k^2}\|_p \lesssim \|(2^{js} (f_{>-10})_j \tilde{g}_j)_{l_j^2}\|_p \\
&\lesssim \|(2^{js} (f_{>-10})_j)_{l_j^2}\|_p \cdot \|g\|_{\dot{B}_{\infty,\infty}^0} \lesssim \|J^{s-1} \partial f\|_p \cdot \|g\|_{\dot{B}_{\infty,\infty}^0}.
\end{aligned}$$

Similarly,

$$\left\| \sum_{j \leq 0} f_j \tilde{J}^s \tilde{g}_j \right\|_p \lesssim \sum_{j \leq 0} 2^{2j} \|f_j\|_p \|\tilde{g}_j\|_{\infty} \lesssim \|J^{s-1} \partial f\|_p \cdot \|g\|_{\dot{B}_{\infty,\infty}^0}.$$

Also,

$$\left\| \sum_{j>0} f_j \tilde{J}^s \tilde{g}_j \right\|_p \lesssim \|D^s f_{>-4}\|_p \|g\|_{\text{BMO}} \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\text{BMO}}.$$

For the case $s > 1$, by an estimate similar to (6.6), we have

$$\left\| \sum_{j>0} f_j \tilde{J}^s \tilde{g}_j \right\|_p \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\dot{B}_{\infty,\infty}^0} + \|\partial f\|_{\dot{B}_{\infty,\infty}^0} \|J^{s-2} \partial g\|_p.$$

This ends the estimate of the diagonal piece. \square

7. Counterexamples

In the previous section, we have proved several refined Kato–Ponce type inequalities for the operator J^s . In particular, we recall that for $1 < p < \infty$, $0 < s \leq 1$:

$$\|J^s(fg) - fJ^s g\|_p \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\infty} \lesssim \|J^s f\|_p \|g\|_{\infty};$$

and for $s > 1$,

$$\|J^s(fg) - fJ^s g\|_p \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\infty} + \|\partial f\|_{\infty} \|J^{s-2} \partial g\|_p.$$

In this section we collect several counterexamples which amounts to showing that in the above inequalities, the L^∞ -norms on the right-hand side cannot be replaced by the weaker BMO norm, not even partially. Roughly speaking, Proposition 7.3 shows that for $0 < s \leq 1$, one cannot hope any quantitative bound of the form

$$\|J^s(fg) - fJ^s g\|_p \leq F(\|J^s f\|_p, \|g\|_{\text{BMO}}),$$

where F is some function; similarly for $1 < s \leq 1 + d/p$, one *cannot* have (see Proposition 7.5)

$$\|J^s(fg) - fJ^s g\|_p \leq F(\|J^s f\|_p, \|g\|_{\text{BMO}}, \|\partial f\|_{\infty}, \|J^{s-1} g\|_p),$$

or (Proposition 7.7)

$$\|J^s(fg) - fJ^s g\|_p \leq F(\|J^s f\|_p, \|g\|_{\infty}, \|\partial f\|_{\text{BMO}}, \|J^{s-1} g\|_p).$$

For $s > 1 + d/p$, Proposition 7.8 and Proposition 7.9 show that (note below “ \lesssim ”!)

$$\begin{aligned} \|J^s(fg) - fJ^s g\|_p &\lesssim \|J^s f\|_p \|g\|_\infty + \|J^{s-1}g\|_p \|\partial f\|_{\text{BMO}}, \\ \|J^s(fg) - fJ^s g\|_p &\lesssim \|J^s f\|_p \|g\|_{\text{BMO}} + \|J^{s-1}g\|_p \|\partial f\|_\infty. \end{aligned}$$

In yet other words, L^∞ norms cannot be replaced by BMO norms even partially.

Lemma 7.1. *Let $s > 0$ and $1 < p < \infty$. Then for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ and $k \geq 1$,*

$$\begin{aligned} \|J^s(\phi(x)e^{ikx_1}) - \langle k \rangle^s \phi(x)e^{ikx_1}\|_p &\lesssim_{\phi,d,s} k^{s-1}, \\ \|J^{s-1}\partial_1(\phi(x)e^{ikx_1}) - \langle k \rangle^{s-1}(ik)\phi(x)e^{ikx_1}\|_p &\lesssim_{\phi,d,s} k^{s-1}, \\ \|D^s(\phi(x)e^{ikx_1}) - k^s \phi(x)e^{ikx_1}\|_p &\lesssim_{\phi,d,s} k^{s-1}. \end{aligned}$$

Also

$$\begin{aligned} \|J^s(\phi(x)e^{ikx_1}) - k^s \phi(x)e^{ikx_1}\|_p &\lesssim_{\phi,d,s} k^{s-1}, \\ \|J^{s-1}\partial_1(\phi(x)e^{ikx_1}) - ik^s \phi(x)e^{ikx_1}\|_p &\lesssim_{\phi,d,s} k^{s-1}. \end{aligned}$$

Moreover, if ϕ is not identically zero, then there is a constant $C_1 = C_1(\phi, p, d) > 0$, $k_0 = k_0(\phi, p, d) > 0$, such that if $k > k_0$, then

$$(7.1) \quad \|\phi(x) \cos(kx_1)\|_p \geq C_1.$$

Proof of Lemma 7.1. Denote $e_1 = (1, 0, \dots, 0)$. By definition, we have

$$\begin{aligned} J^s(\phi(x)e^{ikx_1}) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \langle \xi \rangle^s \hat{\phi}(\xi - ke_1) e^{i\xi \cdot x} d\xi \\ &= \frac{1}{(2\pi)^d} e^{ikx_1} \int_{\mathbb{R}^d} \langle \xi + ke_1 \rangle^s \hat{\phi}(\xi) e^{i\xi \cdot x} d\xi \\ &= \frac{1}{(2\pi)^d} e^{ikx_1} \langle k \rangle^s \int_{\mathbb{R}^d} \hat{\phi}(\xi) e^{i\xi \cdot x} d\xi \\ &\quad + \frac{1}{(2\pi)^d} e^{ikx_1} \langle k \rangle^s \int_{\mathbb{R}^d} \hat{\phi}(\xi) \left(\frac{\langle \xi + ke_1 \rangle^s}{\langle k \rangle^s} - 1 \right) e^{i\xi \cdot x} d\xi \\ &= \langle k \rangle^s \phi(x) e^{ikx_1} + \frac{1}{(2\pi)^d} e^{ikx_1} \langle k \rangle^s \int_{\mathbb{R}^d} \hat{\phi}(\xi) \left(\frac{\langle \xi + ke_1 \rangle^s}{\langle k \rangle^s} - 1 \right) e^{i\xi \cdot x} d\xi. \end{aligned}$$

Rewrite

$$\begin{aligned} \int_{\mathbb{R}^d} \hat{\phi}(\xi) \left(\frac{\langle \xi + ke_1 \rangle^s}{\langle k \rangle^s} - 1 \right) e^{i\xi \cdot x} d\xi &= \int_{\mathbb{R}^d} \hat{\phi}(\xi) \chi_{|\xi| \ll k} \left(\frac{\langle \xi + ke_1 \rangle^s}{\langle k \rangle^s} - 1 \right) e^{i\xi \cdot x} d\xi \\ &\quad + \int_{\mathbb{R}^d} \hat{\phi}(\xi) \chi_{|\xi| \gtrsim k} \left(\frac{\langle \xi + ke_1 \rangle^s}{\langle k \rangle^s} - 1 \right) e^{i\xi \cdot x} d\xi, \end{aligned}$$

where $\chi_{|\xi| \ll k}$ is a smooth cut-off function localized to the regime $|\xi| \ll k$, and $\chi_{|\xi| \gtrsim k} = 1 - \chi_{|\xi| \ll k}$. Consider first $|x| \lesssim 1$. In the regime $|\xi| \ll k$, one can use the

factor $(\langle \xi + ke_1 \rangle^s \langle k \rangle^{-s} - 1)$ to get $1/k$ decay. In the regime $|\xi| \gtrsim k$, one can use the decay of $\hat{\phi}(\xi)$ to get $1/k$ decay. For $|x| \gtrsim 1$, one can just do repeated integration by parts. It is then easy to check that

$$\left| \int_{\mathbb{R}^d} \hat{\phi}(\xi) \left(\frac{\langle \xi + ke_1 \rangle^s}{\langle k \rangle^s} - 1 \right) e^{i\xi \cdot x} d\xi \right| \lesssim \langle x \rangle^{-10d} \langle k \rangle^{-1}.$$

From this the desired result follows.

Now for the estimates of the operator D^s , we denote by $P_{|\xi| \sim k}$ the frequency projection to the block $\{\xi : ck < |\xi| < \frac{1}{c}k\}$ where $c > 0$ is a small constant. It is easy to check that

$$\|D^s((P_{|\xi| \sim k} \phi) \cdot e^{ikx_1})\|_p \lesssim k^s \|P_{|\xi| \sim k} \phi\|_p \lesssim 1 \quad \text{and} \quad \|k^s (P_{|\xi| \sim k} \phi) e^{ikx_1}\|_p \lesssim 1.$$

Thus one only needs to consider the regime $|\xi| \ll k$ and $|\xi| \gg k$ for which the factor $|\xi + ke_1|^s$ will cause no trouble when integrating by parts in ξ . This part of the argument is then similar to the J^s case. The estimate for $J^{s-1} \partial_1$ is also similar. Thus we omit details.

We now prove (7.1). Note that if $p = 2$, then

$$\|\phi(x) \cos kx_1\|_2^2 = \int_{\mathbb{R}^d} |\phi(x)|^2 \frac{1 + \cos(2kx_1)}{2} dx.$$

It is easy to check that for any integer $m \geq 1$,

$$\left| \int_{\mathbb{R}^d} |\phi(x)|^2 \cos(2kx_1) dx \right| \lesssim_m (k^2 + 1)^{-m}.$$

Thus, if k is sufficiently large, we have

$$\|\phi(x) \cos(kx_1)\|_2^2 \gtrsim 1.$$

Next, if $1 < p < 2$, then

$$\begin{aligned} \|\phi(x) \cos kx_1\|_2 &\lesssim \|\phi(x) \cos kx_1\|_p^{p/2} \|\phi(x) \cos kx_1\|_\infty^{1-p/2} \\ &\lesssim \|\phi\|_\infty^{1-p/2} \|\phi(x) \cos kx_1\|_p^{p/2}. \end{aligned}$$

Thus

$$\|\phi(x) \cos kx_1\|_p \gtrsim 1.$$

Similarly, the inequality also holds for $2 < p < \infty$. \square

Lemma 7.2. *Assume $1 < p < \infty$. For any $M > 0$, there exists $g \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$\|\langle \nabla \rangle^{d/p} g\|_p \leq 1, \quad \|g\|_{\text{BMO}} \leq 1,$$

but

$$\|g\|_\infty > M.$$

Proof of Lemma 7.2. Since $\|g\|_{\text{BMO}} \lesssim_{p,d} \|\langle \nabla \rangle^{d/p} g\|_p$, we only need to show the existence of $g \in \mathcal{S}(\mathbb{R}^d)$, such that $\|\langle \nabla \rangle^{d/p} g\|_p \ll 1$, and $\|g\|_\infty \gg 1$.

To this end, let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\text{supp}(\hat{\phi}) \subset \{\xi : 1/2 < |\xi| < 2\}$, and $\phi(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\phi}(\xi) d\xi \neq 0$. Define

$$g(x) = \sum_{j=1}^N \frac{1}{j} \phi(2^j x).$$

Clearly

$$\|g\|_\infty \geq |g(0)| > O(\log N)|\phi(0)| > M,$$

if N is taken sufficiently large.

On the other hand, if $1 < p \leq 2$, then

$$\begin{aligned} \|\langle \nabla \rangle^{d/p} g\|_p &\lesssim \left\| \left(\frac{1}{j} 2^{jd/p} \phi(2^j x) \right)_{l_j^2(j: 1 \leq j \leq N)} \right\|_p \lesssim \left\| \left(\frac{1}{j} 2^{jd/p} \phi(2^j x) \right)_{l_j^p(j: 1 \leq j \leq N)} \right\|_p \\ &\lesssim \left(\frac{1}{j} \right)_{l_j^p(j: 1 \leq j \leq N)} \cdot \|\phi\|_p \lesssim \|\phi\|_p \lesssim 1. \end{aligned}$$

If $2 < p < \infty$, then

$$\begin{aligned} \left\| \left(\frac{1}{j} \cdot 2^{jd/p} \phi(2^j x) \right)_{l_j^2(j: 1 \leq j \leq N)} \right\|_p &\lesssim \left(\frac{1}{j} \|2^{jd/p} \phi(2^j x)\|_p \right)_{l_j^2(j: 1 \leq j \leq N)} \\ &\lesssim \left(\frac{1}{j} \right)_{l_j^2(j: 1 \leq j \leq N)} \|\phi\|_p \lesssim \|\phi\|_p \lesssim 1. \end{aligned}$$

Thus multiplying g by a small constant if necessary, we can easily achieve $\|\langle \nabla \rangle^{d/p} g\|_p \leq 1$ with $\|g\|_\infty > M$. \square

Proposition 7.3. *Assume $0 < s \leq 1$ and $1 < p < \infty$. For any $M > 0$, there exist $f, g \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$\|J^s f\|_p + \|g\|_{\text{BMO}} \leq 1,$$

but

$$\|J^s(fg) - fJ^s g\|_p > M.$$

Proof of Proposition 7.3. By Theorem 6.2, and noting $\|J^{s-1} \partial f\|_p \lesssim \|J^s f\|_p$, we only need to choose $f, g \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|J^s f\|_p + \|g\|_{\text{BMO}} \leq 1,$$

but

$$\|g(J^s f - f)\|_p > M.$$

By Lemma 7.2, we can choose $g \in \mathcal{S}(\mathbb{R}^d)$, such that

$$\|g\|_{\text{BMO}} \leq \frac{1}{2},$$

and for some $x_0 \in \mathbb{R}^d$, $\delta_0 > 0$,

$$|g(x)| > N \gg 1, \quad \forall |x - x_0| \leq \delta_0.$$

Then we choose $\phi \in C_c^\infty(B(x_0, \delta_0/2))$, such that $\|\phi\|_p = 1$. Define

$$f(x) = \frac{1}{k^s} \phi(x) \cos(kx_1).$$

By Lemma 7.1, it is easy to check that

$$\|J^s f - \phi(x) \cos kx_1\|_p \lesssim_{\phi, d, s} k^{-1}.$$

On the other hand, by Lemma 7.1,

$$\|g(x)\phi(x) \cos kx_1\|_p \gtrsim N \|\phi(x) \cos kx_1\|_p \gtrsim N.$$

Clearly we then have

$$\|g(J^s f - f)\|_p \gtrsim N - O\left(\frac{1}{k}\right) - O\left(\frac{1}{k^s}\right) > M,$$

if N and k are sufficiently large. \square

The same construction used in Proposition 7.3 can be used to obtain the following corollary. In particular, it shows that the estimate

$$\|D^s(fg) - fD^s g\|_p \lesssim_{s, p, d} \|D^s f\|_p \|g\|_\infty,$$

for $0 < s \leq 1$, $1 < p < \infty$ is sharp.

Corollary 7.4. *Assume $1 < p < \infty$ and $0 < s \leq 1$. Then for any $M > 0$, there exist $f, g \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$\|J^s f\|_p + \|g\|_{\text{BMO}} \leq 1,$$

but

$$\|D^s(fg) - fD^s g\|_p > M.$$

Proposition 7.5. *Assume $1 < p < \infty$ and $1 < s \leq 1 + d/p$. Then for any $M > 0$, there exist $f, g \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$\|J^s f\|_p + \|g\|_{\text{BMO}} + \|\partial f\|_\infty + \|J^{s-1} g\|_p \leq 1,$$

but

$$\|J^s(fg) - fJ^s g\|_p > M.$$

Proof of Proposition 7.5. By Theorem 6.2, we only need to choose $f, g \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|J^s f\|_p + \|g\|_{\text{BMO}} + \|\partial f\|_\infty + \|J^{s-1} g\|_p \leq 1,$$

but

$$\|g \cdot (J^s f - f)\|_p > M.$$

By Lemma 7.2, we can find $g \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|J^{d/p} g\|_p \leq 1, \quad \text{but } \|g\|_\infty > N \gg 1.$$

Note that $\|g\|_{\text{BMO}} \lesssim \|J^{d/p}g\|_p \lesssim 1$, and for $1 < s \leq 1 + d/p$,

$$\|J^{s-1}g\|_p \lesssim \|J^{d/p}g\|_p \lesssim 1.$$

Since $\|g\|_\infty > N \gg 1$, we may assume for some $B(x_0, \delta_0)$,

$$|g(x)| > N, \quad \text{for all } x \in B(x_0, \delta_0).$$

We then choose $\phi \in C_c^\infty(B(x_0, \delta_0/2))$ with $\|\phi\|_p = 1$, and define

$$f(x) = \frac{1}{k^s} \phi(x) \cos kx_1.$$

Since $s > 1$, it is easy to check that $\|\partial f\|_\infty \lesssim k^{-(s-1)} \ll 1$ if k is large. Also by Lemma 7.1,

$$\begin{aligned} \|J^s f - \phi(x) \cos kx_1\|_p + \|f\|_p &\lesssim_{\phi, d, s, p} k^{-1}, \\ \|g(x)\phi(x) \cos kx_1\|_p &\geq N \|\phi(x) \cos kx_1\|_p \gtrsim N. \end{aligned}$$

Clearly we get

$$\|g(J^s f - f)\|_p > N - O\left(\frac{1}{k}\right) \gg M,$$

if N and k are taken sufficiently large. \square

Lemma 7.6. *Assume $1 < p < \infty$. For any $M > 0$, there exists $f \in \mathcal{S}(\mathbb{R}^d)$ such that*

$$\|\langle \nabla \rangle^{1+d/p} f\|_p \leq 1,$$

but

$$\|\partial f\|_\infty > M.$$

Proof of Lemma 7.6. This is similar to the proof of Lemma 7.2. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be such that $\text{supp}(\hat{\phi}) \subset \{\xi : 1/2 < |\xi| < 2\}$ and

$$\int_{\mathbb{R}^d} \hat{\phi}(\xi) \frac{\xi_1^2}{|\xi|^2} d\xi > 0.$$

Define

$$f(x) = \sum_{j=1}^N \frac{1}{j} (\Delta^{-1} \partial_1 \phi)(2^j x) \cdot 2^{-j}.$$

Then

$$|(\partial_1 f)(0)| \geq \left(\sum_{j=1}^N \frac{1}{j} \right) \cdot |(\Delta^{-1} \partial_1 \phi)(0)| \geq O(\log N) \gg M,$$

if N is taken sufficiently large. The rest of the argument now is similar to that in Lemma 7.2. We omit details. \square

Proposition 7.7. *Assume $1 < p < \infty$ and $1 < s \leq 1 + d/p$. Then for any $M > 0$, there exist $f, g \in \mathcal{S}(\mathbb{R}^d)$, such that*

$$\|J^s f\|_p + \|g\|_\infty + \|J^{s-1}g\|_p + \|\partial f\|_{\text{BMO}} \leq 1,$$

but

$$\|J^s(fg) - fJ^s g\|_p > M.$$

Proof of Proposition 7.7. Thanks to Theorem 6.2, we only need to choose $f, g \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|J^s f\|_p + \|g\|_\infty + \|J^{s-1}g\|_p + \|\partial f\|_{\text{BMO}} \leq 1,$$

but

$$\|\partial f \cdot J^{s-2}\partial g\|_p > M.$$

Now by Lemma 7.6, we can choose $f \in \mathcal{S}(\mathbb{R}^d)$, such that

$$\|J^s f\|_p \leq 1, \quad \text{but } \|\partial_1 f\|_\infty > N \gg 1.$$

Thus for some $B(x_0, \delta_0)$,

$$|(\partial_1 f)(x)| > N \gg 1, \quad \text{for all } x \in B(x_0, \delta_0).$$

Now choose $\phi \in C_c^\infty(B(x_0, \delta_0/2))$ such that $\|\phi\|_p = 1$. Define

$$g(x) = \frac{1}{k^{s-1}} \phi(x) \sin(kx_1).$$

By Lemma 7.1, we have (note $s - 1 > 0$ so the hypothesis of Lemma 7.1 is valid for $\tilde{s} = s - 1$)

$$\|J^{s-2}\partial_1 g - \phi(x) \cos kx_1\|_p \lesssim k^{-1} \quad \text{and} \quad \sum_{j=2}^d \|J^{s-2}\partial_j g\|_p \lesssim k^{-1}.$$

From these we get

$$\|\partial f \cdot J^{s-2}\partial g\|_p \gtrsim \|(\phi(x) \cos kx_1)\partial_1 f\|_p - O(k^{-1}) \gtrsim N - O(k^{-1}) > M,$$

if N and k are sufficiently large. \square

Proposition 7.8. *Assume $1 < p < \infty$ and $s > 1 + d/p$. Then for any $M > 0$, there exist $f, g \in \mathcal{S}(\mathbb{R}^d)$, such that*

$$\|J^s(fg) - fJ^s g\|_p > M(\|J^s f\|_p \|g\|_\infty + \|J^{s-1}g\|_p \|\partial f\|_{\text{BMO}}).$$

Proof of Proposition 7.8. By Theorem 6.2, we only need to find f and g such that

$$(7.2) \quad \|\partial f \cdot J^{s-2}\partial g\|_p > M(\|J^s f\|_p \|g\|_\infty + \|J^{s-1}g\|_p \|\partial f\|_{\text{BMO}}).$$

To this end, we first choose $f \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|\partial f\|_{\text{BMO}} \leq 1,$$

but

$$\|\partial f\|_{\infty} > M^2 \gg 1.$$

Without loss of generality we may assume that, for some $B(x_0, \delta_0)$,

$$|(\partial_1 f)(x)| > M^2, \quad \forall x \in B(x_0, \delta_0).$$

Let $\phi \in C_c^\infty(B(x_0, \delta_0/2))$ be such that $\|\phi\|_p = 1$, and define

$$g(x) = \frac{1}{k^{s-1}} \phi(x) \sin(kx_1).$$

Then by Lemma 7.1, we have

$$\|J^{s-2} \partial_1 g - \phi(x) \cos kx_1\|_p \lesssim_{\phi, d, s, p} k^{-1} \quad \text{and} \quad \sum_{j=2}^d \|J^{s-2} \partial_j g\|_p \lesssim_{\phi, d, s, p} k^{-1}.$$

Thus

$$\|\partial f \cdot J^{s-2} \partial g\|_p \gtrsim \|\phi(x) \cos kx_1 \cdot \partial_1 f(x)\|_p - O(k^{-1}) \gtrsim M^2 - O(k^{-1}).$$

On the other hand,

$$\begin{aligned} \|J^s f\|_p \|g\|_{\infty} &\lesssim \frac{1}{k^{s-1}} \|\phi\|_{\infty} \|J^s f\|_p, \\ \|J^{s-1} g\|_p \|\partial f\|_{\text{BMO}} &\lesssim 1 + O\left(\frac{1}{k}\right). \end{aligned}$$

Clearly, if k is sufficiently large, (7.2) follows. \square

Proposition 7.9. *Assume $1 < p < \infty$ and $s > 1 + d/p$. Then for any $M > 0$, there exist $f, g \in \mathcal{S}(\mathbb{R}^d)$, such that*

$$\|J^s(fg) - fJ^s g\|_p > M(\|J^s f\|_p \|g\|_{\text{BMO}} + \|J^{s-1} g\|_p \|\partial f\|_{\infty}).$$

Proof of Proposition 7.9. Again by Theorem 6.2, we only need to find $f, g \in \mathcal{S}(\mathbb{R}^d)$ such that

$$(7.3) \quad \|g(J^s f - f)\|_p > M(\|J^s f\|_p \|g\|_{\text{BMO}} + \|J^{s-1} g\|_p \|\partial f\|_{\infty}).$$

Choose $g \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\|g\|_{\text{BMO}} \leq 1,$$

and for some $x_0 \in \mathbb{R}^d$, $\delta_0 > 0$,

$$|g(x)| > M^2, \quad \forall |x - x_0| \leq \delta_0.$$

Let $\phi \in C_c^\infty(B(x_0, \delta_0/2))$ be such that $\|\phi\|_p = 1$. Define

$$f(x) = \frac{1}{k^s} \phi(x) \cos(kx_1).$$

Then by Lemma 7.1,

$$\|J^s f - f - \phi(x) \cos kx_1\|_p \lesssim_{\phi, d, s, p} k^{-1}.$$

Then

$$\|g(J^s f - f)\|_p \gtrsim \|g(x)\phi(x) \cos kx_1\|_p - O(k^{-1}) \gtrsim M^2 - O(k^{-1}).$$

On the other hand,

$$\begin{aligned} \|J^s f\|_p \|g\|_{\text{BMO}} &\lesssim 1 + O(k^{-1}), \\ \|J^{s-1} g\|_p \|\partial f\|_\infty &\lesssim \|J^{s-1} g\|_p \cdot \frac{1}{k^{s-1}} (\|\phi\|_\infty + \|\partial\phi\|_\infty). \end{aligned}$$

Thus (7.3) follows easily. \square

8. Proof of Theorem 1.9

Denote $\tilde{J}^s = J^s - I$ and write

$$\tilde{J}^s(fg) = \sum_j \tilde{J}^s(f_{<j-2}g_j) + \sum_j \tilde{J}^s(g_{<j-2}f_j) + \sum_j \tilde{J}^s(f_j\tilde{g}_j),$$

where $\tilde{g}_j = \sum_{a=-2}^2 g_{j+a}$.

The diagonal piece. By Lemma 6.1, we have

$$\left\| \sum_{j \leq 0} \tilde{J}^s(f_j\tilde{g}_j) \right\|_p \lesssim \sum_{j \leq 0} 2^{2j} \|f_j\tilde{g}_j\|_p \lesssim \|J^{s-1}\partial f\|_{\dot{B}_{p_1, \infty}^0} \|g\|_{p_2}.$$

Next,

$$\left\| \sum_{j > 0} \tilde{J}^s P_{\leq 0}(f_j\tilde{g}_j) \right\|_p \lesssim \sum_{j > 0} \|f_j\|_{p_1} \|\tilde{g}_j\|_{p_2} \lesssim \|J^{s-1}\partial f\|_{\dot{B}_{p_1, \infty}^0} \|g\|_{p_2}.$$

Then

$$\begin{aligned} \left\| \sum_{j > 0} \tilde{J}^s P_{>0}(f_j\tilde{g}_j) \right\|_p &\lesssim \left\| (P_k \tilde{J}^s \left(\sum_{j \geq k-4} f_j\tilde{g}_j \right)) \right\|_{l_k^2(k>0)} \\ &\lesssim \left\| (2^{ks} \left(\sum_{j \geq k-4} f_j\tilde{g}_j \right)) \right\|_{l_k^2(k>0)} \lesssim \|(2^{js} f_j \tilde{g}_j)_{l_j^2(j>-10)}\|_p. \end{aligned}$$

Now consider two cases.

If $p_1 < \infty$ and $p_2 \leq \infty$, then

$$\|(2^{js} f_j \tilde{g}_j)_{l_j^2(j>-10)}\|_p \lesssim \|(2^{js} f_j)_{l_j^2(j>-10)}\|_{p_1} \|g\|_{p_2} \lesssim \|J^{s-1}\partial f\|_{p_1} \|g\|_{p_2}.$$

If $p_1 = \infty$, $p_2 = p$, then

$$\|(2^{js} f_j \tilde{g}_j)_{l_j^2(j>-10)}\|_p \lesssim \|(2^{js} f_j)_{l_j^\infty(j>-10)}\|_\infty \|(\tilde{g}_j)_{l_j^2}\|_p \lesssim \|J^{s-1} \partial f\|_{\dot{B}_{\infty,\infty}^0} \|g\|_p.$$

Collecting the estimates, we get

$$\left\| \sum_j \tilde{J}^s(f_j \tilde{g}_j) \right\|_p \lesssim \begin{cases} \|J^{s-1} \partial f\|_{p_1} \|g\|_{p_2}, & \text{if } p_1 < \infty; \\ \|J^{s-1} \partial f\|_{\dot{B}_{\infty,\infty}^0} \|g\|_p, & \text{if } p_1 = \infty. \end{cases}$$

The low-high piece. We first write

$$\tilde{J}^s(f_{<j-2} g_j) = [\tilde{J}^s, f_{<j-2}] g_j + f_{<j-2} \tilde{J}^s g_j.$$

Clearly,

$$\left\| \sum_{j \leq 0} [\tilde{J}^s, f_{<j-2}] g_j \right\|_p \lesssim \sum_{j \leq 0} 2^j \|\partial f_{<0}\|_{p_1} \|g\|_{p_2} \lesssim \|J^{s-1} \partial f\|_{p_1} \|g\|_{p_2}.$$

Let $\tilde{P}_j = \sum_{l=-10}^{10} P_{j+l}$ and denote $K_j = \tilde{J}^s \tilde{P}_j \delta_0$. Then, in the same way as in (6.4)–(6.5), we have

$$(8.1) \quad \begin{aligned} & \tilde{J}^s(f_{<j-2} g_j) - f_{<j-2} \tilde{J}^s g_j \\ &= \int_{\mathbb{R}^d} K_j(y) (f_{<j-2}(x-y) - f_{<j-2}(x) + \partial f_{<j-2}(x) \cdot y) g_j(x-y) dy \end{aligned}$$

$$(8.2) \quad - \int_{\mathbb{R}^d} K_j(y) \partial f_{<j-2}(x) \cdot y g_j(x-y) dy.$$

Estimate of (8.1). First,

$$|f_{<j-2}(x-y) - f_{<j-2}(x) + \partial f_{<j-2}(x) \cdot y| \lesssim \mathcal{M}(\partial^2 f_{<j-2})(x) \cdot (1 + 2^j |y|)^d \cdot |y|^2.$$

So

$$\left\| \sum_{j>0} (8.1) \right\|_p \lesssim \|(2^{js} \cdot 2^{-2j} \cdot \mathcal{M}(\partial^2 f_{<j-2}) \cdot \mathcal{M} g_j)_{l_j^2(j>0)}\|_p.$$

If $0 < s \leq 1$, $p_1 < \infty$, and $p_2 \leq \infty$, then

$$\begin{aligned} & \|(2^{j(s-2)} \cdot \mathcal{M}(\partial^2 f_{<j-2}) \cdot \mathcal{M} g_j)_{l_j^2(j>0)}\|_p \\ & \lesssim \|(2^{j(s-2)} \cdot \mathcal{M}(\partial^2 f_{<j-2}))_{l_j^2(j>0)}\|_{p_1} \cdot \|(\mathcal{M} g_j)_{l_j^\infty}\|_{p_2} \\ & \lesssim \|J^{s-1} \partial f\|_{p_1} \|g\|_{p_2}. \end{aligned}$$

If $0 < s \leq 1$, $p_1 = \infty$, $p_2 = p$, then

$$\begin{aligned} & \|(2^{j(s-2)} \cdot \mathcal{M}(\partial^2 f_{<j-2}) \cdot \mathcal{M} g_j)_{l_j^2(j>0)}\|_p \\ & \lesssim \|(2^{j(s-2)} \cdot \mathcal{M}(\partial^2 f_{<j-2}))_{l_j^\infty(j>0)}\|_\infty \cdot \|(\mathcal{M} g_j)_{l_j^2}\|_p \\ & \lesssim \|J^{s-1} \partial f\|_\infty \|g\|_p. \end{aligned}$$

If $s > 1$, $p_3 \leq \infty$ and $p_4 < \infty$, we first note

$$|\partial^2 f_{<j-2}| \lesssim \sum_{k < j-2} 2^k |\tilde{P}_k(\partial f)| \lesssim 2^j \mathcal{M}(\partial f).$$

Then

$$\begin{aligned} \|(2^{j(s-2)} \cdot \mathcal{M}(\partial^2 f_{<j-2}) \cdot \mathcal{M}g_j)_{l_j^2(j>0)}\|_p &\lesssim \|\mathcal{M}(\partial f)\|_{p_3} \cdot \|(2^{j(s-1)}g_j)_{l_j^2(j>0)}\|_{p_4} \\ &\lesssim \|\partial f\|_{p_3} \|J^{s-2}\partial g\|_{p_4}. \end{aligned}$$

If $s > 1$, $p_3 = p$ and $p_4 = \infty$, then

$$\begin{aligned} \|(2^{j(s-2)} \cdot \mathcal{M}(\partial^2 f_{<j-2}) \cdot \mathcal{M}g_j)_{l_j^2(j>0)}\|_p &\lesssim \|(2^{-j}\mathcal{M}(\partial^2 f_{<j-2}))_{l_j^2(j>0)}\|_p \cdot \|(2^{j(s-1)}\mathcal{M}g_j)_{l_j^\infty(j>0)}\|_\infty \\ &\lesssim \|\partial f\|_p \|J^{s-2}\partial g\|_\infty. \end{aligned}$$

Thus (8.1) is OK for us.

Estimate of (8.2). In the same way as in (6.5), we have

$$\sum_{j>0} (8.2) = -s \sum_{j>0} \partial f_{<j-2} \cdot J^{s-2}\partial g_j.$$

If $0 < s < 1$, then

$$\begin{aligned} \|(\partial f_{<j-2} \cdot J^{s-2}\partial g_j)_{l_j^2(j>0)}\|_p &\lesssim \|\partial f_{\leq 0}\|_{p_1} \cdot \|(J^{s-2}\partial g_j)_{l_j^2(j>0)}\|_{p_2} + \|(\partial f_{0 < \cdot < j-2} \cdot J^{s-2}\partial g_j)_{l_j^2(j>0)}\|_p. \end{aligned}$$

If $p_2 < \infty$, then

$$\|\partial f_{\leq 0}\|_{p_1} \|(J^{s-2}\partial g_j)_{l_j^2(j>0)}\|_{p_2} \lesssim \|J^{s-1}\partial f\|_{p_1} \|g\|_{p_2};$$

and also

$$\begin{aligned} \|(\partial f_{0 < \cdot < j-2} \cdot J^{s-2}\partial g_j)_{l_j^2(j>0)}\|_p &\lesssim \|(2^{j(s-1)}\partial f_{0 < \cdot < j-2})_{l_j^\infty}\|_{p_1} \|(2^{j(1-s)}J^{s-2}\partial g_j)_{l_j^2(j>0)}\|_{p_2} \lesssim \|J^{s-1}\partial f\|_{p_1} \|g\|_{p_2}. \end{aligned}$$

If $p_2 = \infty$, then $p_1 = p$, and

$$\|\partial f_{\leq 0}\|_p \|(J^{s-2}\partial g_j)_{l_j^2(j>0)}\|_\infty \lesssim \|J^{s-1}\partial f\|_p \|g\|_\infty;$$

and also

$$\begin{aligned} \|(\partial f_{0 < \cdot < j-2} \cdot J^{s-2}\partial g_j)_{l_j^2(j>0)}\|_p &\lesssim \|(2^{j(s-1)}\partial f_{0 < \cdot < j-2})_{l_j^2}\|_p \|(2^{j(1-s)}J^{s-2}\partial g_j)_{l_j^\infty(j>0)}\|_\infty \lesssim \|J^{s-1}\partial f\|_p \|g\|_\infty. \end{aligned}$$

Next consider $s = 1$.

If $p_2 = \infty$, then by Lemma 3.1,

$$\left\| \sum_{j>0} \partial f_{<j-2} \cdot J^{-1} \partial g_j \right\|_p \lesssim \|\partial f\|_p \|J^{-1} \partial g\|_{\text{BMO}} \lesssim \|\partial f\|_p \|g\|_{\text{BMO}}.$$

If $p_2 < \infty$, then

$$\begin{aligned} \left\| \sum_{j>0} \partial f_{<j-2} \cdot J^{-1} \partial g_j \right\|_p &\lesssim \|(\partial f_{<j-2} \cdot J^{-1} \partial g_j)_{l_j^2}\|_p \\ &\lesssim \|\partial f\|_{p_1} \|J^{-1} \partial g\|_{p_2} \lesssim \|\partial f\|_{p_1} \|g\|_{p_2}. \end{aligned}$$

Next consider $s > 1$.

If $p_4 = \infty$, then again by Lemma 3.1,

$$\left\| \sum_{j>0} \partial f_{<j-2} \cdot J^{s-2} \partial g_j \right\|_p \lesssim \|\partial f\|_p \cdot \|J^{s-2} \partial g\|_{\text{BMO}}.$$

If $p_4 < \infty$, then clearly

$$\left\| \sum_{j>0} \partial f_{<j-2} J^{s-2} \partial g_j \right\|_p \lesssim \|\partial f\|_{p_3} \|J^{s-2} \partial g\|_{p_4}.$$

This ends the estimate of (8.2). We have finished the estimate of the commutator piece $[\tilde{J}^s, f_{<j-2}]g_j$.

To finish the estimate of the low-high piece, we still need to estimate the contribution of the piece $\sum_j f_{<j-2} \tilde{J}^s g_j$. Write

$$\sum_j f_{<j-2} \tilde{J}^s g_j = f \tilde{J}^s g - \sum_j f_{\geq j-2} \tilde{J}^s g_j = f \tilde{J}^s g - \sum_j f_j \tilde{J}^s g_{\leq j+2}.$$

Clearly,

$$\left\| \sum_{j \leq 10} f_j \tilde{J}^s g_{\leq j+2} \right\|_p \lesssim \sum_{j \leq 10} \|f_j\|_{p_1} \cdot 2^{2j} \|g\|_{p_2} \lesssim \|J^{s-1} \partial f\|_{p_1} \|g\|_{p_2}.$$

On the other hand,

$$\sum_{j>10} f_j \tilde{J}^s g_{\leq j+2} = \sum_{j>10} f_j \tilde{J}^s g_{<j-2} + \sum_{j>10} f_j \tilde{J}^s \tilde{g}_j,$$

where $\tilde{g}_j = g_{j-2 \leq \cdot \leq j+2}$.

By frequency localization, for $p_1 < \infty$, $p_2 \leq \infty$, we have

$$\begin{aligned} \left\| \sum_{j>10} f_j \tilde{J}^s g_{<j-2} \right\|_p &\lesssim \|(2^{js} f_j \cdot 2^{-js} \tilde{J}^s g_{<j-2})_{l_j^2(j>10)}\|_p \\ &\lesssim \|(2^{js} f_j)_{l_j^2(j>10)}\|_{p_1} \|(2^{-js} \tilde{J}^s g_{<j-2})_{l_j^\infty(j>10)}\|_{p_2} \\ &\lesssim \|J^{s-1} \partial f\|_{p_1} \|g\|_{p_2}. \end{aligned}$$

If $p_1 = \infty$, $p_2 = p$, then

$$\begin{aligned} \left\| \sum_{j>10} f_j \tilde{J}^s g_{<j-2} \right\|_p &\lesssim \left\| (2^{js} f_j)_{l_j^\infty(j>10)} \right\|_\infty \left\| (2^{-js} \tilde{J}^s g_{<j-2})_{l_j^2(j>10)} \right\|_p \\ &\lesssim \left\| J^{s-1} \partial f \right\|_{\dot{B}_{\infty,\infty}^0} \|g\|_p. \end{aligned}$$

For the piece $\sum_{j>10} f_j \tilde{J}^s \tilde{g}_j$, if $p_1 < \infty$ and $p_2 < \infty$, then

$$\begin{aligned} \left\| \sum_{j>10} f_j \tilde{J}^s \tilde{g}_j \right\|_p &\lesssim \left\| (2^{js} f_j)_{l_j^2(j>10)} \right\|_{p_1} \left\| (2^{-js} \tilde{J}^s \tilde{g}_j)_{l_j^2(j>10)} \right\|_{p_2} \\ &\lesssim \left\| J^{s-1} \partial f \right\|_{p_1} \|g\|_{p_2}. \end{aligned}$$

If $p_1 = p$, $p_2 = \infty$, then by Lemma 3.1,

$$\left\| \sum_{j>10} f_j \tilde{J}^s \tilde{g}_j \right\|_p \lesssim \|D^s f_{>0}\|_p \|D^{-s} \tilde{J}^s g_{>0}\|_{\text{BMO}} \lesssim \|J^{s-1} \partial f\|_p \|g\|_{\text{BMO}}.$$

Similarly if $p_1 = \infty$, $p_2 = p$, then

$$\left\| \sum_{j>10} f_j \tilde{J}^s \tilde{g}_j \right\|_p \lesssim \|J^{s-1} \partial f\|_{\text{BMO}} \|g\|_p.$$

The high-low piece. First,

$$\left\| \sum_{j \leq 0} \tilde{J}^s (g_{<j-2} f_j) \right\|_p \lesssim \sum_{j \leq 0} 2^{2j} \|g\|_{p_2} \|f_j\|_{p_1} \lesssim \|J^{s-1} \partial f\|_{p_1} \|g\|_{p_2}.$$

If $p_1 < \infty$, then

$$\left\| \sum_{j>0} \tilde{J}^s (g_{<j-2} f_j) \right\|_p \lesssim \left\| (2^{js} f_j g_{<j-2})_{l_j^2(j>0)} \right\|_p \lesssim \|J^{s-1} \partial f\|_{p_1} \|g\|_{p_2}.$$

If $p_1 = \infty$, then $p_2 = p$. In this case we write

$$\tilde{J}^s (g_{<j-2} f_j) = [\tilde{J}^s, g_{<j-2}] f_j + g_{<j-2} \tilde{J}^s f_j.$$

We first estimate the commutator as

$$\begin{aligned} \left\| \sum_{j>0} [\tilde{J}^s, g_{<j-2}] f_j \right\|_p &\lesssim \left\| (2^{js} \mathcal{M}(\partial g_{<j-2}) \cdot 2^{-j} \cdot \mathcal{M} f_j)_{l_j^2(j>0)} \right\|_p \\ &\lesssim \left\| (2^{js} \mathcal{M} f_j)_{l_j^\infty(j>0)} \right\|_\infty \left\| (2^{-j} \mathcal{M}(\partial g_{<j-2}))_{l_j^2(j>0)} \right\|_p \\ &\lesssim \|J^{s-1} \partial f\|_{\dot{B}_{\infty,\infty}^0} \|g\|_p. \end{aligned}$$

Finally, by Lemma 3.1,

$$\left\| \sum_{j>0} g_{<j-2} \tilde{J}^s f_j \right\|_p \lesssim \|\tilde{J}^s f_{>0}\|_{\text{BMO}} \|g\|_p \lesssim \|J^{s-1} \partial f\|_{\text{BMO}} \|g\|_p.$$

This concludes the proof of Theorem 1.9.

9. Further counterexamples

In this section we collect further counterexamples for the operator J^s on divergence-free velocity fields which are deeply connected with the investigation of norm estimates of incompressible Euler equations in Sobolev spaces (see [3], [4], [18] and the references therein). In typical energy-type estimates for Euler equations, we have for $s > 0$, $1 < p < \infty$, and divergence free u ,

$$\|J^s((u \cdot \nabla)u) - (u \cdot \nabla)J^s u\|_p \lesssim_{s,p,d} \|\partial u\|_\infty \|J^s u\|_p.$$

A natural question⁵ is whether $\|\partial u\|_\infty$ can be replaced by $\|\partial u\|_{\text{BMO}}$. If this is the case, it would yield single exponential in time growth of Sobolev norms of two-dimensional Euler thanks to conservation of vorticity. Proposition 9.3 and Proposition 9.4 disprove any such possibility. In particular, we show that for $1 < p < \infty$, $s > 1 + d/p$ and any Schwartz u with $\nabla \cdot u = 0$, one cannot hope any quantitative bound of the form

$$\|J^s((u \cdot \nabla)u) - (u \cdot \nabla)(J^s u)\|_p \leq F(\|\partial u\|_{\text{BMO}}) \cdot \|J^s u\|_p,$$

and also for $0 < s \leq 1 + d/p$, one cannot have

$$\|J^s((u \cdot \nabla)u) - (u \cdot \nabla)(J^s u)\|_p \leq F(\|J^s u\|_p, \|\partial u\|_{\text{BMO}}).$$

An enlightening feature of our construction is the incorporation of the divergence-free constraint.

Theorem 9.1. *Let $s > 0$ and $1 < p < \infty$. Fix an integer $l \in \{1, \dots, d\}$. Then the following holds for any $f, g \in \mathcal{S}(\mathbb{R}^d)$:*

$$\begin{aligned} \|J^s \partial_l(fg) - fJ^s \partial_l g - gJ^s \partial_l f - \partial f \cdot J^\partial g - \partial g \cdot J^\partial f\|_p \\ \lesssim_{s,p,d} \|J^s f\|_p \|\partial g\|_{\text{BMO}} + \|\partial f\|_{\text{BMO}} \|J^s g\|_p, \end{aligned}$$

where

$$\widehat{J^\partial g}(\xi) = \widehat{J^\partial}(\xi) \widehat{g}(\xi) \quad \text{and} \quad \widehat{J^\partial}(\xi) = -i \partial_\xi((\langle \xi \rangle^s - 1) \cdot i \xi_l).$$

Proof of Theorem 9.1. We shall only sketch the proof since it is similar to the proof of Theorem 6.2. Define $B = (J^s - 1)\partial_l$. Then

$$(9.1) \quad B(fg) - fBg = \sum_j (B(f_{<j-2}g_j) - f_{<j-2}Bg_j)$$

$$(9.2) \quad + \sum_j (B(f_j g_{<j-2}) - f_j Bg_{<j-2})$$

$$(9.3) \quad + \sum_j (B(f_j \tilde{g}_j) - f_j B\tilde{g}_j),$$

where $\tilde{g}_j = \sum_{a=-2}^2 g_{j+a}$.

⁵We thank Zhuan Ye for raising this question.

Estimate of (9.3). Clearly,

$$\begin{aligned} \left\| \sum_{j \leq 0} B(f_j \tilde{g}_j) \right\|_p &\lesssim \sum_{j \leq 0} 2^{3j} \|f_j\|_p \|\tilde{g}_j\|_\infty \lesssim \|J^s f\|_p \|\partial g\|_{\dot{B}_{\infty, \infty}^0}, \\ \left\| \sum_{j > 0} P_{\leq 0} B(f_j \tilde{g}_j) \right\|_p &\lesssim \sum_{j > 0} \|f_j\|_p \|\tilde{g}_j\|_\infty \lesssim \|J^s f\|_p \|\partial g\|_{\dot{B}_{\infty, \infty}^0}, \\ \left\| \sum_{j > 0} P_{> 0} B(f_j \tilde{g}_j) \right\|_p &\lesssim \left\| \left(2^{k(1+s)} \tilde{P}_k \left(\sum_{j > k-4} f_j \tilde{g}_j \right) \right)_{l_k^2(k > 0)} \right\|_p \\ &\lesssim \|(2^{j(1+s)} f_j \tilde{g}_j)_{l_j^2(j > -10)}\|_p \lesssim \|J^s f\|_p \|\partial g\|_{\dot{B}_{\infty, \infty}^0}. \end{aligned}$$

Similarly,

$$\left\| \sum_{j \leq 0} f_j B \tilde{g}_j \right\|_p \lesssim \|J^s f\|_p \|\partial g\|_{\dot{B}_{\infty, \infty}^0},$$

and by Lemma 3.1,

$$\begin{aligned} \left\| \sum_{j > 0} f_j B \tilde{g}_j \right\|_p &= \left\| \sum_j 2^{-js} \tilde{P}_j D^s f_{> 0} 2^{js} \tilde{P}_j ((J^s - 1) \partial_l D^{-s} g_{> -4}) \right\|_p \\ &\lesssim \|D^s f\|_p \|D^{-s} (J^s - 1) \partial g_{> -4}\|_{\text{BMO}} \lesssim \|J^s f\|_p \|\partial g\|_{\text{BMO}}. \end{aligned}$$

Thus (9.3) is OK for us.

Estimate of (9.1). Let $K_j = \sum_{a=-10}^{10} B P_{j+a} \delta_0$. Then

$$(9.4) \quad \begin{aligned} &B(f_{< j-2} g_j) - f_{< j-2} B g_j \\ &= \int K_j(y) (f_{< j-2}(x-y) - f_{< j-2}(x) - \partial f_{< j-2}(x) \cdot (-y)) g_j(x-y) dy \end{aligned}$$

$$(9.5) \quad + \int K_j(y) \partial f_{< j-2}(x) \cdot (-y) g_j(x-y) dy.$$

Estimate of (9.4). By using

$$\begin{aligned} &|f_{< j-2}(x-y) - f_{< j-2}(x) - \partial f_{< j-2}(x) \cdot (-y)| \\ &\lesssim \mathcal{M}(\partial^2 f_{< j-2})(x) \cdot (1 + 2^j |y|)^d \cdot |y|^2, \end{aligned}$$

we get

$$|(9.4)| \lesssim \mathcal{M}(\partial^2 f_{< j-2})(x) \cdot (\mathcal{M} g_j)(x) \cdot \begin{cases} 2^j, & \text{if } j \leq 0, \\ 2^{j(s-1)}, & \text{if } j > 0. \end{cases}$$

Therefore by frequency localization,

$$\begin{aligned} &\left\| \sum_j (9.4) \right\|_p \\ &\lesssim \left\| (\mathcal{M}(\partial^2 f_{< j-2}) \mathcal{M} g_j 2^j)_{l_j^2(j \leq 0)} \right\|_p + \left\| (\mathcal{M}(\partial^2 f_{< j-2}) \mathcal{M} g_j 2^{j(s-1)})_{l_j^2(j > 0)} \right\|_p \\ &\lesssim \|\partial f\|_{\dot{B}_{\infty, \infty}^0} \|J^s g\|_p. \end{aligned}$$

Estimate of (9.5). It is easy to check that

$$\int K_j(y)(-y)g_j(x-y) dy = \frac{1}{i} \mathcal{F}^{-1}(\partial_\xi(\widehat{K}_j(\xi)\widehat{g}_j(\xi))) = J^\partial g_j,$$

where we recall

$$\widehat{J^\partial}(\xi) = \frac{1}{i} \partial_\xi(\widehat{B}(\xi)) = -i\partial_\xi((\langle \xi \rangle^s - 1)i\xi).$$

Then

$$\sum_j (9.5) = \sum_j \partial f_{<j-2} \cdot J^\partial g_j = J^\partial g \cdot \partial f - \sum_j J^\partial g_{\leq j+2} \cdot \partial f_j.$$

Clearly, by Lemma 3.1,

$$\left\| \sum_j J^\partial g_{\leq j+2} \cdot \partial f_j \right\|_p \lesssim \|J^\partial g\|_p \|\partial f\|_{\text{BMO}} \lesssim \|J^s f\|_p \|\partial f\|_{\text{BMO}}.$$

Thus

$$\sum_j (9.5) = J^\partial g \cdot \partial f + \text{OK},$$

where

$$\|\text{OK}\|_p \lesssim \|J^s f\|_p \|\partial g\|_{\text{BMO}} + \|J^s g\|_p \|\partial f\|_{\text{BMO}}.$$

Estimate of (9.2). First note that by swapping f and g , and using the previous estimates, we have

$$\sum_j (B(f_j g_{<j-2}) - g_{<j-2} B f_j) = J^\partial f \cdot \partial g + \text{OK}.$$

On the other hand,

$$\begin{aligned} \sum_j g_{<j-2} B f_j &= g B f - \sum_j g_j B f_{\leq j+2}, \\ \left\| \sum_{j \leq 0} g_j B f_{\leq j+2} \right\|_p &\lesssim \sum_{j \leq 0} 2^{3j} \|g_j\|_\infty \|f\|_p \lesssim \|J^s f\|_p \|\partial g\|_{\dot{B}_{\infty, \infty}^0}, \\ \left\| \sum_{j > 0} g_j B f_{\leq j+2} \right\|_p &\lesssim \|\partial g\|_{\text{BMO}} \|J^s f\|_p. \end{aligned}$$

Also similarly,

$$\left\| \sum_j f_j B g_{<j-2} \right\|_p \lesssim \|\partial f\|_{\text{BMO}} \|J^s g\|_p.$$

Thus

$$(9.2) = J^\partial f \cdot \partial g + g B f + \text{OK}.$$

The desired estimates then follow from the simple identity that

$$B(fg) - fBg - gBf = J^s \partial_l(fg) - fJ^s \partial_l g - gJ^s \partial_l f. \quad \square$$

An immediate corollary of Theorem 9.1 is the following estimate.

Corollary 9.2. *Let $s > 0$ and $1 < p < \infty$. Then for any $u \in \mathcal{S}(\mathbb{R}^d)$ with $\nabla \cdot u = 0$, we have*

$$\|J^s((u \cdot \nabla)u) - (u \cdot \nabla)J^s u\|_p \lesssim_{s,p,d} \|\partial u\|_\infty \|J^s u\|_p.$$

Proof. Obvious. \square

Proposition 9.3. *Let $1 < p < \infty$ and $s > 1 + d/p$. Then for any $M > 0$, there exists $u \in \mathcal{S}(\mathbb{R}^d)$ with $\nabla \cdot u = 0$, such that*

$$\|\partial u\|_{\text{BMO}} \leq 1,$$

but

$$\|J^s((u \cdot \nabla)u) - (u \cdot \nabla)(J^s u)\|_p > M \|J^s u\|_p.$$

Proof of Proposition 9.3. We shall choose u in the form

$$u = (u_1, u_2, 0, \dots, 0).$$

Then we only need to show⁶

$$\left\| \sum_{l=1}^2 J^s \partial_l (u_l u_2) - \sum_{l=1}^2 u_l \partial_l J^s u_2 \right\|_p > M \|J^s u\|_p.$$

By Theorem 9.1, we have

$$\begin{aligned} \left\| \sum_{l=1}^2 J^s \partial_l (u_l u_2) - \sum_{l=1}^2 u_l J^s \partial_l u_2 - \sum_{l=1}^2 (J^s \partial_l u_l) u_2 - \sum_{l=1}^2 \partial u_l \cdot ((J^s - 1) \partial_l)^\partial u_2 \right. \\ \left. - \sum_{l=1}^2 ((J^s - 1) \partial_l)^\partial u_l \cdot \partial u_2 \right\|_p \lesssim \|J^s u\|_p \|\partial u\|_{\text{BMO}}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{F}(((J^s - 1) \partial_l)^\partial)(\xi) &= -i \partial_\xi ((\langle \xi \rangle^s - 1) i \xi_l) \\ &= s \langle \xi \rangle^{s-2} \xi \xi_l + (\langle \xi \rangle^s - 1) e_l, \quad e_l = \underbrace{(0, \dots, 0, 1, 0 \dots, 0)}_l, \end{aligned}$$

$$((J^s - 1) \partial_l)^\partial = -s J^{s-2} \partial \partial_l + e_l (J^s - 1).$$

Clearly by using $\nabla \cdot u = 0$, we get

$$\begin{aligned} \sum_{l=1}^2 \partial u_l \cdot ((J^s - 1) \partial_l)^\partial u_2 &= -s \sum_{l=1}^2 \partial u_l \cdot J^{s-2} \partial \partial_l u_2, \\ \sum_{l=1}^2 ((J^s - 1) \partial_l)^\partial u_l \cdot \partial u_2 &= \sum_{l=1}^2 ((J^s - 1) u_l) \partial_l u_2. \end{aligned}$$

⁶Here we choose u_2 for convenience only. One can of course choose u_1 as well.

Thus it suffices to show

$$\left\| s \sum_{l=1}^2 \partial u_l \cdot J^{s-2} \partial \partial_l u_2 - \sum_{l=1}^2 \partial_l u_2 (J^s - 1) u_l \right\|_p > M \|J^s u\|_p.$$

For convenience of notation, for $f = (f_1, f_2)$, $g = (g_1, g_2)$, denote

$$B(f, g) = s \sum_{l=1}^2 \partial f_l \cdot J^{s-2} \partial \partial_l g_2 - \sum_{l=1}^2 \partial_l f_2 (J^s - 1) g_l.$$

Now choose $\tilde{\phi} \in \mathcal{S}(\mathbb{R}^d)$, such that $\text{supp}(\tilde{\phi}) \subset \{\xi : 2/3 < |\xi| < 1\}$, and

$$\int_{\mathbb{R}^d} \tilde{\phi}(\xi) \xi_1 \xi_2 d\xi > 0.$$

The last condition is to ensure that $|(\partial_{12}\tilde{\phi})(0)| \neq 0$. Now define

$$\phi(x) = \sum_{\bar{l}=1}^m \tilde{\phi}(2^{3\bar{l}}x) 2^{-6\bar{l}} \cdot \frac{1}{\bar{l}},$$

where m is chosen sufficiently large such that $\sqrt{\log m} \gg M$. It is easy to check that

$$|(\partial_{12}\phi)(0)| \sim \log m \quad \text{and} \quad \sum_{i,j=1}^d \|\partial_i \partial_j \phi\|_{\text{BMO}} \lesssim \sum_{i,j=1}^d \|D^{d/2} \partial_i \partial_j \phi\|_2 \lesssim 1.$$

Set $u^\circ = (u_1^\circ, u_2^\circ, 0, \dots, 0)$, and

$$u_1^\circ = -\partial_2 \phi, \quad u_2^\circ = \partial_1 \phi.$$

Note that $(\partial_1 u_1^\circ)(0) = -(\partial_2 u_2^\circ)(0) = -(\partial_{12}\phi)(0) \sim \log m$. Now discuss two cases.

Case 1: $\|B(u^\circ, u^\circ)\|_p \geq \sqrt{\log m} \|J^s u^\circ\|_p$. In this case we set $u = u^\circ$ and no work is needed.

Case 2: $\|B(u^\circ, u^\circ)\|_p < \sqrt{\log m} \|J^s u^\circ\|_p$. In this case we shall do a further perturbation argument.

First by continuity, we can find $\delta_0 > 0$ such that

$$|(\partial_{12}\phi)(x)| > \frac{1}{2} |(\partial_{12}\phi)(0)| \sim \log m, \quad \forall |x| < \delta_0.$$

Now choose $b \in C_c^\infty(B(0, \delta_0/2))$ such that

$$\|b\|_p = \frac{1}{100} \|J^s u^\circ\|_p.$$

For $k \gg 1$, define

$$\begin{aligned}\phi^n &= \frac{1}{k^{1+s}}(\sin kx_1)b(x), \\ u^n &= (-\partial_2\phi^n, \partial_1\phi^n, 0, \dots, 0), \\ u &= u^o + u^n.\end{aligned}$$

Then by taking k large, it is easy to check that $\|J^s u^n\|_p \lesssim \|J^s u\|_p$, and

$$\begin{aligned}\|B(u^n, u^n)\|_p &\lesssim_{b,d,s,p} \frac{1}{k^{s-1}} \ll 1, \\ \|B(u^n, u^o)\|_p &= \left\| s \sum_{l=1}^2 \partial u_l^n \cdot J^{s-2} \partial \partial_l u_2^o - \sum_{l=1}^2 \partial_l u_2^n \cdot (J^s - 1) u_l^o \right\|_p \\ &\lesssim_{b,d,s,p,\phi} \frac{1}{k^{s-1}} \ll 1, \\ B(u^o, u^n) &= s \sum_{l=1}^2 \partial u_l^o \cdot J^{s-2} \partial \partial_l u_2^n - \sum_{l=1}^2 \partial_l u_2^o \cdot (J^s - 1) u_l^n \\ &= s \partial_1 u_1^o J^{s-2} \partial_1 \partial_1 u_2^n - \partial_2 u_2^o J^s u_2^n + \text{error},\end{aligned}$$

where

$$\|\text{error}\|_p \lesssim_{\bar{\phi},d,s,p} 1.$$

On the other hand,

$$\begin{aligned}\|J^{s-2} \partial_1 \partial_1 u_2^n - (-1) \cos(kx_1) b(x)\|_p &\lesssim_{b,d,s,p} \frac{1}{k}, \\ \|J^s u_2^n - \cos(kx_1) b(x)\|_p &\lesssim_{b,d,s,p} \frac{1}{k}.\end{aligned}$$

Thus

$$\|B(u^o, u^n)\|_p \gtrsim_{d,s,p} (s-1) \log m \|b\|_p,$$

and it follows easily that

$$\|B(u, u)\|_p \gtrsim_{d,s,p} ((s-1) \log m - \sqrt{\log m}) \|J^s u^o\|_p > M \|J^s u\|_p. \quad \square$$

Proposition 9.4. *Let $1 < p < \infty$ and $0 < s \leq 1 + d/p$. Then for any $M > 0$, there exists $u \in \mathcal{S}(\mathbb{R}^d)$ with $\nabla \cdot u = 0$ such that*

$$\|J^s u\|_p + \|\partial u\|_{\text{BMO}} \leq 1,$$

but

$$\|J^s((u \cdot \nabla)u) - (u \cdot \nabla)(J^s u)\|_p > M.$$

Proof of Proposition 9.4. We only need to alter slightly the construction in the proof of Proposition 9.3. We use the same notation as therein and define

$$\phi(x) = \sum_{l=1}^m \tilde{\phi}(2^{3l}x) 2^{-6l} \cdot \frac{1}{l}, \quad u^\circ = (u_1^\circ, u_2^\circ, 0, \dots, 0), \quad u_1^\circ = -\partial_2 \phi, \quad u_2^\circ = \partial_1 \phi.$$

It is easy to check that $\|J^{1+d/p}u^\circ\|_p \lesssim 1$.

Case 1: $\|B(u^\circ, u^\circ)\|_p \geq \sqrt{\log m}$. No work is needed and we can take $u = u^\circ$.

Case 2: $\|B(u^\circ, u^\circ)\|_p < \sqrt{\log m}$. In this case we just choose $b \in C_c^\infty(B(0, \delta_0/2))$ such that $\|b\|_p = 1$. The rest of the argument is then similar to that in the proof of Proposition 9.3. We omit details. \square

References

- [1] BÉNYI, Á. AND OH, T.: Smoothing of commutators for a Hörmander class of bilinear pseudodifferential operators. *J. Fourier Anal. Appl.* **20** (2014), no. 2, 282–300.
- [2] BOURGAIN, J. AND LI, D.: On an endpoint Kato–Ponce inequality. *Differential Integral Equations* **27** (2014), no. 11–12, 1037–1072.
- [3] BOURGAIN, J. AND LI, D.: Strong ill-posedness of the incompressible Euler equation in borderline Sobolev spaces. *Invent. Math.* **201** (2015), no. 1, 97–157.
- [4] BOURGAIN, J. AND LI, D.: Strong illposedness of the incompressible Euler equation in integer C^m spaces. *Geom. Funct. Anal.* **25** (2015), no. 1, 1–86.
- [5] COIFMAN, R. AND MEYER, Y.: *Au delà des opérateurs pseudo-différentiels*. Astérisque 57, Société Mathématique de France, Paris, 1978
- [6] CHAE, D., CONSTANTIN, P., CÓRDOBA, D., GANCEDO, F. AND WU, J.: Generalized surface quasi-geostrophic equations with singular velocities. *Comm. Pure Appl. Math.* **65** (2012), no. 8, 1037–1066.
- [7] DAWSON, L., MCGAHAGAN, H. AND PONCE, G.: On the decay properties of solutions to a class of Schrödinger equations. *Proc. Amer. Math. Soc.* **136** (2008), no. 6, 2081–2090.
- [8] FEFFERMAN, C., MCCORMICK, D., ROBINSON, J. AND RODRIGO, J.: Higher order commutator estimates and local existence for the non-resistive MHD equations and related models. *J. Funct. Anal.* **267** (2014), no. 4, 1035–1056.
- [9] FEFFERMAN, C. AND STEIN, E. M.: Some maximal inequalities. *Amer. J. Math.* **93** (1971), 107–116.
- [10] FEFFERMAN, C. AND STEIN, E. M.: H^p spaces of several variables. *Acta Math.* **129** (1972), no. 3–4, 137–193.
- [11] GRAFAKOS, L.: *Modern Fourier analysis*. Third edition. Graduate Texts in Mathematics 250, Springer, New York, 2014.
- [12] GRAFAKOS, L., MALDONADO, D. AND NAIBO, V.: A remark on an endpoint Kato–Ponce inequality. *Differential and Integral Equations* **27** (2014), No. 5–6, 415–424.
- [13] GRAFAKOS, L. AND OH, S.: The Kato–Ponce inequality. *Comm. Partial Differential Equations* **39** (2014), no. 6, 1128–1157.

- [14] KATO, T. AND PONCE, G.: Commutator estimates and the Euler and Navier–Stokes equations. *Comm. Pure Appl. Math.* **41** (1988), no. 7, 891–907.
- [15] KENIG, C., PONCE, G. AND VEGA, L.: Well-posedness and scattering results for the generalized Korteweg–de Vries equation via the contraction principle. *Comm. Pure Appl. Math.* **46** (1993), no. 4, 527–620.
- [16] KILLIP, R. AND VISAN, M.: Nonlinear Schrödinger equations at critical regularity. In *Evolution equations*, 325–437. Clay Math. Proc. 17, Amer. Math. Soc., Providence, RI, 2013.
- [17] KOZONO, H. AND TANIUCHI, Y.: Bilinear estimates in BMO and the Navier–Stokes equations. *Math. Z.* **235** (2000), no. 1, 173–194.
- [18] LEI, Z. AND SHI, J.: Infinite-time exponential growth of the Euler equation on two-dimensional torus. Preprint, arXiv: [1608.07010](https://arxiv.org/abs/1608.07010), 2016.
- [19] MUSCALU, C. AND SCHLAG, W.: *Classical and multilinear harmonic analysis. Vol. II.* Cambridge Studies in Advanced Mathematics 138, Cambridge University Press, Cambridge, 2013.
- [20] TRIEBEL, H.: *Theory of function spaces.* Monographs in Mathematics 78, Birkhäuser Verlag, Basel, 1983.
- [21] YE, Z.: Regularity criterion of the 2D Bénard equations with critical and supercritical dissipation. *Nonlinear Anal.* **156** (2017), 111–143.

Received September 6, 2016. Published online January 9, 2019.

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