GLOBAL WELL-POSEDNESS OF HEDGEHOG SOLUTIONS FOR THE (3 + 1) SKYRME MODEL

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Abstract

We consider hedgehog solutions in the $(3+1)$*-dimensional Skyrme model, which is an energy-supercritical problem. We introduce a new strategy to prove global well-posedness for arbitrarily large initial data.*

1. Introduction

In this article, we consider the $(3+1)$-dimensional Skyrme model in quantum field theory. This nonlinear sigma model was first proposed by Skyrme [24]–[26] to incorporate baryons as stable field configurations in the description of low energy interaction of pions. Let $U : \mathbb{R}^{3+1} \to \mathrm{SU}(2)$ be a map into the isospin group with signature $(+ - - -)$. Define the su(2)-valued connection 1-form A by (below $U^\dagger$ denotes the Hermitian adjoint)

$$A = U^\dagger \, dU = A_\mu \, dx^\mu,$$

where $x^0 = t$, $(x^j)_{1\le j\le 3} = x \in \mathbb{R}^3$. The Lagrangian density of the classical Skyrme model is given by

$$\mathcal{L} = -\frac{1}{4} f_\pi^2 \,\mathrm{Tr}(A_\mu A^\mu) + \frac{1}{4}\epsilon^2 \,\mathrm{Tr}\big([A_\mu, A_\nu][A^\mu, A^\nu]\big), \tag{1.1}$$

where f_π^2 is the pion decay constant and $\epsilon > 0$ is a coupling parameter. The actual value of f_π^2 does not play much of a role in our mathematical analysis and we will conveniently set it to be 2. Here $[\cdot,\cdot]$ is the usual Lie bracket on su(2) and $\mathrm{Tr}(\cdot)$ denotes the matrix trace.

The Euler–Lagrange equation of (1.1) takes the form

$$\partial_\mu\big(A^\mu - \epsilon^2\big[A_\nu, [A^\mu, A^\nu]\big]\big) = 0. \tag{1.2}$$

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Let I_2 be the identity matrix, and let σ_j, $1 \le j \le 3$ be the Pauli spin matrices. Introducing the angular variable $\omega = \omega(t,x)$ and the spin vector $\mathbf{n} = (n_j) \in \mathbb{S}^2$, we write the group element $U \in \mathrm{SU}(2)$ as

$$\begin{aligned} U(t,x) &= \exp\Big(\frac{\omega(t,x)}{2i}\sigma_j n_j(t,x)\Big) \\ &= I_2 \cos\frac{\omega(t,x)}{2} - i\big(\sigma_j n_j(t,x)\big)\sin\frac{\omega(t,x)}{2}. \end{aligned} \tag{1.3}$$

We will be mainly concerned with a special family of solutions known as *hedgehog solutions*. Under the hedgehog ansatz, we set $r = |x|$, $n_j(x) = \frac{x_j}{r}$, and $\omega(t,x) = 2f(r,t)$, where f is the unknown radial function. We then obtain from (1.2)–(1.3),

$$\begin{aligned} &\Big(1+\epsilon^2\frac{2\sin^2 f}{r^2}\Big)\Big(\partial_{tt}-\partial_{rr}-\frac{2}{r}\partial_r\Big)f \\ &\quad = -\epsilon^2\frac{4\sin^2 f}{r^3}\partial_r f - \epsilon^2\frac{\sin(2f)}{r^2}\big((\partial_t f)^2-(\partial_r f)^2\big) \\ &\qquad - \frac{\sin(2f)}{r^2} - \epsilon^2\frac{\sin^2 f\cdot\sin(2f)}{r^4}. \end{aligned} \tag{1.4}$$

Introduce the notation

$$\Delta_d = \partial_{rr} + \frac{d-1}{r}\partial_r$$

and

$$\Box_d = \partial_{tt} - \Delta_d = \partial_{tt} - \partial_{rr} - \frac{d-1}{r}\partial_r.$$

For radial functions on $\mathbb{R}^d$, Δ_d and $\Box_d$ are simply the usual Laplacian and d'Alembertian in polar coordinates. In our work, it will be useful to lift the function $f(r)$ to a radial function in $\mathbb{R}^d$ for some convenient choices of the dimension d.

Using the above notation, we write (1.4) compactly as

$$\begin{aligned} \Big(1+\epsilon^2\frac{2\sin^2 f}{r^2}\Big)\Box_3 f &= -\epsilon^2\frac{4\sin^2 f}{r^3}\partial_r f - \epsilon^2\frac{\sin(2f)}{r^2}\big((\partial_t f)^2-(\partial_r f)^2\big) \\ &\quad - \frac{\sin(2f)}{r^2} - \epsilon^2\frac{\sin^2 f\cdot\sin(2f)}{r^4}. \end{aligned} \tag{1.5}$$

The boundary conditions for f are

$$\lim_{r\to 0} f(t,r) = N_1\pi, \qquad \lim_{r\to\infty} f(t,r) = 0, \tag{1.6}$$

where $N_1 \ge 0$ is an integer.

The main result of this paper, roughly speaking, is that for smooth and arbitrarily large initial data the corresponding solution to (1.5)–(1.6) exists globally in time. The precise formulation of the results will be given in Section 2. The basic conservation law associated with (1.5) is given by the Skyrme energy

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^\infty \Big(1+\epsilon^2\frac{2\sin^2 f}{r^2}\Big)\big((\partial_t f)^2+(\partial_r f)^2\big)r^2\,dr \\ &\quad + \int_0^\infty \frac{\sin^2 f}{r^2}\Big(1+\epsilon^2\frac{\sin^2 f}{2r^2}\Big)r^2\,dr \qquad (1.7) \\ &= E_0, \quad \forall t>0. \end{aligned}$$

With respect to the Skyrme energy conservation, the main difficulty associated with the analysis of (1.5) is that it is *energy-supercritical* and no useful theory is readily available for such problems. We will introduce a new (and special) strategy to overcome this difficulty and prove global well-posedness for arbitrarily large initial data. As far as we know, this is the first unconditional result on a *physical* energy-supercritical problem.

We summarize below the main points of the proof.

Main steps of the proof

In our analysis, the value of ϵ does not play much of a role and we will henceforth set $\epsilon = 1$ in (1.5) for convenience.

Step 1. Local (in time) analysis and lifting to dimension 5.

The first step is to obtain a good local theory. Observe that the nonlinearity on the right-hand side of (1.5) has strong singularities near $r = 0$ which can only be balanced out by a good local asymptotics of f as $r \to 0$. To kill this singularity, we introduce $g = g(r,t)$ by the relation

$$f(r,t) = \phi(r,t) + rg(r,t), \qquad (1.8)$$

where ϕ is a smooth cutoff function such that $\phi(r) \equiv N_1\pi$ for $r \le 1$. We then regard g as a radial function on $\mathbb{R}^5$ and obtain from (1.4) and (1.8) an equation for g of the form

$$\Box_5 g = N(r, g, \partial_t g, \nabla g),$$

where N is a smooth nonlinearity and no longer contains any singularities near $r = 0$. Local well-posedness in $H^k_{\mathrm{rad}}(\mathbb{R}^5)$ then follows from energy estimates. From the local analysis, to continue the solution to all time, we only need to control the quantity

$$G(t) = \big\|\langle x\rangle g(t,x)\big\|_{L^\infty_x(\mathbb{R}^5)} + \big\|\langle x\rangle\big(|\partial_t g| + |\nabla g|\big)\big\|_{L^\infty_x(\mathbb{R}^5)}. \qquad (1.9)$$

We will achieve this in several steps.
Step 2. A nonlocal transformation and derivation of the Φ-equation.

The blowup/continuation criterion (1.9) is supercritical with respect to the Skyrme energy (1.7). To nail down global well-posedness, we analyze in a deeper way the structure of (1.5). For this purpose, we introduce a nonlocal transformation (see Section 3 for more details) of the form

$$\Phi(r,t) = \int_0^{g(r,t)} \Big(1 + \frac{2\sin^2(ry + \phi(r))}{r^2}\Big)^{\frac{1}{2}}\, dy + \frac{1}{r^3}\phi_{\gtrsim 1}(r), \tag{1.10}$$

where $\phi_{\gtrsim 1}$ is a smooth cutoff function localized to the regime $r \gtrsim 1$. Regard Φ as a radial function on $\mathbb{R}^5$. For Φ we then obtain from (1.5) and (1.10) a nonlocal equation of the form

$$\Box_5 \Phi = \frac{1}{r^3}\phi_{\gtrsim 1} - \frac{3}{2}\Phi + \frac{1}{2}\int_0^{g(r,t)} (3B^{\frac{3}{2}} + B^{-\frac{1}{2}} - B^{-\frac{3}{2}})\, dy, \tag{1.11}$$

where

$$B = 1 + \frac{2\sin^2(ry + \phi(r))}{r^2}.$$

The remarkable feature of this new system is that—at the cost of nonlocality—all derivative terms on the right-hand side of (1.4) have been eliminated.
Step 3. Control of the H^1-norm of Φ and a nonblowup argument.

This includes the estimates of $\|\Phi\|_{L_x^2(\mathbb{R}^5)}$, $\|\partial_t\Phi\|_{L_x^2(\mathbb{R}^5)}$, and $\|\nabla\Phi\|_{L_x^2(\mathbb{R}^5)}$. This is an important first step to beat energy supercriticality. Due to the particular structure in (1.10), it is not difficult to check that the Skyrme energy (1.7) is insufficient to give any control of $\|\nabla\Phi\|_{L_x^2(\mathbb{R}^5)}$ which is a manifestation of energy supercriticality at the lowest level. A heuristic analysis (see the beginning of Section 4) shows that in the worst case scenario the linear part of (1.11) could take the form

$$\Box_5 \Phi = -\frac{3}{2}\Phi + \frac{3}{r^2}\Phi,$$

which is a wave operator with *negative* inverse square potential. Since $d = 5$ and $3 > \frac{(d-2)^2}{4}$, we cannot use Strichartz (cf. [4]). To solve this problem, we resort to a nonlinear approach which exploits the fine structure of the equation. Let T be the first possible blowup time. By performing estimates directly on (1.10) and (1.11), we obtain

$$\int_{\mathbb{R}^5} \Big(\frac{1}{2}\big|\nabla\Phi(t)\big|^2 - \phi_{<r_0}(r)\cdot H(r,t)\Big)\, dx \le C(T), \quad \forall 0 \le t < T, \tag{1.12}$$

where $0 < C(T) < \infty$ is a constant depending on T, $r_0 < \frac{1}{2}$ is a small constant, $\phi_{<r_0}$ is a smooth cutoff function localized to $r \le r_0$, and

$$H(r,t) = \frac{3}{2}\int_0^{g(r,t)} \Big(1 + \frac{2\sin^2(rw)}{r^2}\Big)^{\frac{1}{2}} \cdot \Big(\int_0^w \Big(1 + \frac{2\sin^2(ry)}{r^2}\Big)^{\frac{1}{2}} \cdot \frac{2\sin^2(ry)}{r^2}\,dy\Big)\,dw.$$

By a detailed analysis on H, we show that H admits the sharp bound

$$H(r,t) \le \frac{9}{4}\cdot\frac{1}{2}\cdot\frac{|\Phi(r,t)|^2}{r^2}.$$

From this and (1.12), we obtain

$$0 \le \int_{\mathbb{R}^5} \Big(\big|\nabla\Phi(t)\big|^2 - \frac{9}{4}\cdot\frac{|\Phi(t)|^2}{r^2}\Big)\,dx \le C(T), \quad \forall 0 \le t < T, \tag{1.13}$$

where the positivity of the integral follows from Hardy's inequality (see Lemma 4.3) on $\mathbb{R}^5$. The estimate (1.13) is the sharpest available and yet it is *not coercive* enough to give control of the H^1-norm of Φ. The main reason is that there could exist a sequence

$$\big\|\nabla\Phi(t_n)\big\|_{L^2_x(\mathbb{R}^5)} \to +\infty, \qquad \Big\|\frac{\Phi(t_n)}{r}\Big\|_{L^2_x(\mathbb{R}^5)} \to +\infty,$$

but

$$\int_{\mathbb{R}^5} \Big(\big|\nabla\Phi(t_n)\big|^2 - \frac{9}{4}\cdot\frac{|\Phi(t_n)|^2}{r^2}\Big)\,dx \to C_1, \quad \text{as } t_n \to T,$$

where $C_1 \ge 0$ is a finite constant. To rule out this blowup scenario, we will analyze in detail the special structure of Φ and perform a delicate limiting and contradiction argument (see in particular (4.21)–(4.27) in the proof of Proposition 4.4). The technical details are contained in the proof of Proposition 4.4 and as a result we can control the H^1-norm of Φ.

Step 4. Nonlinear energy bootstrap and higher-order estimates.

In this final step, we upgrade the H^1-estimate of Φ to H^4-estimates which are sufficient to give an a priori bound of the quantity $G(t)$ defined in (1.9) (and yielding global well-posedness). The main task is to interweave the Sobolev estimates of g and Φ back and forth a number of times using in an essential way the structure of the nonlocal system (1.10)–(1.11). The estimates are organized in such a way that we first obtain temporal regularity and then use the structure of the equation to trade temporal regularity for spatial regularity. The technical details are given in Section 5.

The above four steps complete our proof of global well-posedness. To put things into perspective, we briefly review below some results connected with the Skyrme model.

Connection with other works.

(1) Prior to this work, progress has been slow on understanding the global dynamics of the Skyrme model. In [32], Wong gave a detailed analysis of the dominant energy condition and the breakdown of hyberbolicity for the Skyrme model (see also Gibbons [15], Crutchfield and Bell [8]). In particular, it follows that a small perturbation of a static skyrmion configuration yields local well-posedness. After our work was completed, the author learned that Geba, Nakanishi, and Rajeev [12] proved a small data global well-posedness and scattering result for the Skyrme wave map for initial data in critical Besov-type space.

(2) In [13] and [14], Geba and Rajeev considered a semilinear Skyrme model introduced by Adkins and Nappi [1]. The equivariant solutions satisfy

$$\partial_{tt} f - \partial_{rr} f - \frac{2}{r}\partial_r f + \frac{\sin(2f)}{r^2} + \frac{(f - \sin f \cos f)(1 - \cos 2f)}{r^4} = 0$$

and have conserved energy

$$E\big(f(t)\big) = \int_0^\infty \Big(\frac{1}{2}\big((\partial_t f)^2 + (\partial_r f)^2\big) + \frac{\sin^2 f}{r^2} + \frac{(f - \sin f \cos f)^2}{2r^4}\Big) r^2\, dr.$$

They proved that near the first possible blowup time, the energy does not concentrate. But the issue of global well-posedness is still open.

(3) If $\epsilon = 0$ in (1.5), then we recover the equivariant wave map from $\mathbb{R}^{3+1}$ to $\mathbb{S}^3$ which is also an *energy-supercritical* problem. Generally, smooth solutions will blow up in finite time. Indeed, Shatah [22] constructed finite-time blowup solutions which are self-similar and have finite energy. This was extended to other target manifolds in [23] and higher dimensions $d \geq 4$ in [5]. In [2], Bizoń constructed a countable family of spherically symmetric self-similar wave maps from the $(3+1)$ Minkowski space-time into the 3-sphere. These constructions all rely on the existence of a nontrivial harmonic map.

(4) The $(2+1)$-dimensional analogue of the Skyrme model is known as a *baby Skyrme model*. The technique developed here can also be used to prove global well-posedness of corresponding hedgehog solutions. The details will be given in a future publication. In contrast, the $\epsilon = 0$ limit of the baby Skyrme model gives rise to the $(2+1)$-dimensional *energy-critical* equivariant wave map

$$\partial_{tt} f - \partial_{rr} f - \frac{\partial_r f}{r} + \frac{k^2 \sin(2f)}{2r^2} = 0,$$

where $k \geq 1$ is an integer giving the homotopy index. It is known that (cf. [6], [23], [30]) for smooth initial data with energy $E < E(Q)$, where $Q(r) =$

$2\arctan(r^k)$, the corresponding solution is global. Also, by an argument of Struwe, there is no blowup of self-similar type. The existence (and dynamics) of finite-time blowup solutions was obtained in [21] ($k \geq 4$) and [20] ($k = 1$) using different techniques and giving different blowup rates. For results and some recent developments on *energy-critical* wave maps from (2 + 1) Minkowski space-time to general target manifolds, we refer to [3], [16]–[19], [27], [28], [31] and references therein.

(5) The technique introduced in this paper has been recently generalized and extended to many other important physical models. In [10], Geba and Grillakis improved and streamlined the result of this paper to Sobolev H^s, $s > 7/2$. Creek [7] and Geba–Grillakis [11] obtained large data global regularity for the (2 + 1)-dimensional equivariant Faddeev model. We refer to the monograph [9] for an extensive overview of more recent developments.

2. Reformulation and main results

As was already mentioned, the value of ϵ will not play much of a role in our analysis as long as $\epsilon > 0$. In the rest of this article, we will set $\epsilon = 1$ in (1.5).

Denote

$$A_1 = 1 + \frac{2\sin^2 f}{r^2}.$$

Then

$$\begin{aligned}\Box_3 f &= -\frac{1}{A_1}\cdot\frac{4\sin^2 f}{r^3}\cdot\partial_r f - \frac{1}{A_1}\cdot\frac{\sin(2f)}{r^2}\cdot\big((\partial_t f)^2 - (\partial_r f)^2\big) \\ &\quad -\frac{1}{A_1}\cdot\frac{\sin(2f)}{r^2} - \frac{1}{A_1}\cdot\frac{\sin^2 f\cdot\sin(2f)}{r^4} \\ &=: N(r, f, f'),\end{aligned} \tag{2.1}$$

with boundary condition (1.6).

Let ϕ be a smooth cutoff function such that $\phi(r) = N_1\pi$ for $r \leq 1$ and $\phi(r) = 0$ for $r \geq 2$. Define $g(r,t)$ by

$$f(r,t) = \phi(r) + rg(r,t). \tag{2.2}$$

No boundary condition is needed for g at $r = 0$.[1] Note that

$$\begin{aligned}\Box_3 f &= -\Delta_3\phi + \Box_3(rg) \\ &= -\Delta_3\phi + r\Box_5 g - \frac{2}{r}g.\end{aligned} \tag{2.3}$$

[1] We will regard g as a radial function on $\mathbb{R}^5$ and construct a classical solution $g \in H^k(\mathbb{R}^5)$. By radial Sobolev embedding, $|g(r,t)| \lesssim r^{-2}$ as $r \to \infty$. Hence the boundary condition $f(\infty,t) = 0$ causes no trouble either.

By (2.1), (2.2), and (2.3), the equation for g then takes the form

$$\Box_5 g = \frac{2}{r^2} g + \frac{1}{r}\Delta_3\phi + \frac{1}{r}\phi_{<1}\cdot N\big(r, rg, (rg)'\big) + \frac{1}{r}\phi_{>1}\cdot N\big(r, \phi + rg, (\phi+rg)'\big), \tag{2.4}$$

where $\phi_{>1} = 1 - \phi_{<1}$, and $\phi_{<1}$ is a smooth cutoff function such that $\phi_{<1}(r) = 1$ for $r < \frac{1}{2}$ and $\phi_{<1}(r) = 0$ for $r \geq 1$. In more detail,

$$\begin{aligned}\Box_5 g = \frac{\phi_{<1}}{1+\tilde{F}_0(rg)g^2}\big(&\tilde{F}_1(rg)g^3 + \tilde{F}_2(rg)g^5 \\ &- \tilde{F}_3(rg)\cdot g\cdot\big((\partial_t g)^2 - (\partial_r g)^2\big) \\ &+ \tilde{F}_4(rg)\cdot g^4\cdot r\partial_r g\big) \\ &+ \phi_{>1}\cdot\frac{2}{r^2}g + \frac{1}{r}\Delta_3\phi \\ &+ \frac{1}{r}\phi_{>1}\cdot N\big(r, \phi+rg, (\phi+rg)'\big),\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}\tilde{F}_0(x) &= 2\Big(\frac{\sin x}{x}\Big)^2, \\ \tilde{F}_1(x) &= \frac{2}{x^2} - \frac{\sin(2x)}{x^3}, \\ \tilde{F}_2(x) &= \frac{\sin(2x)}{x^3} - \frac{\sin^2 x\sin(2x)}{x^5}, \\ \tilde{F}_3(x) &= \frac{\sin(2x)}{x}, \\ \tilde{F}_4(x) &= -\frac{4\sin^2 x}{x^4} + \frac{2\sin(2x)}{x^3}.\end{aligned}$$

It is not difficult to check that $\tilde{F}_i(x)$, $0 \leq i \leq 4$ are well defined for all $x \in \mathbb{R}$ with the help of power series expansion. Observe that the functions $\tilde{F}_i$ can all be written as

$$\tilde{F}_i(x) = F_i(x^2), \quad i = 0, \ldots, 4,$$

where the F_i's are smooth functions satisfying

$$\Big\|\frac{d^k}{dx^k}F_i(x)\Big\|_{L_x^\infty} \leq C_k, \quad \forall k \geq 0, \tag{2.6}$$

where C_k are constants depending only on k.

The reason that we write $\tilde{F}_i(rg) = F_i(r^2g^2)$ is that we will regard $F_i(r^2g^2) = F_i(|x|^2g^2)$ for $x \in \mathbb{R}^5$ which is smooth in x. This will help local energy estimates in the local theory.

Now we lift g to be a radial function on $\mathbb{R}^5$; clearly then,

$$r\partial_r g = \sum_{i=1}^{5} x_i \cdot \partial_{x_i} g = x \cdot \nabla g.$$

Thus we rewrite (2.5) as

$$\begin{aligned}\Box_5 g = \frac{\phi_{<1}}{1 + F_0(r^2g^2)g^2}\big(&F_1(r^2g^2)g^3 + F_2(r^2g^2)g^5 \\ &- F_3(r^2g^2)\cdot g\cdot\big((\partial_t g)^2 - (\nabla g)^2\big) \\ &+ F_4(r^2g^2)\cdot g^4\cdot(x\cdot\nabla g)\big) \\ &+ \phi_{>1}\cdot\frac{2}{r^2}g + \frac{1}{r}\Delta_3\phi \\ &+ \frac{1}{r}\phi_{>1}\cdot N\big(r, \phi + rg, (\phi + rg)'\big).\end{aligned} \tag{2.7}$$

For any integer k, we will denote by $H^k_{\text{rad}}(\mathbb{R}^5)$ the usual H^k Sobolev space restricted to radial functions on $\mathbb{R}^5$.

PROPOSITION 2.1 (Local well-posedness and continuation criterion)
Let $k > \frac{5}{2} + 1$ be an integer. Assume that

$$(g, \partial_t g)|_{t=0} = (g_0, g_1) \in H^k_{\text{rad}}(\mathbb{R}^5) \times H^{k-1}_{\text{rad}}(\mathbb{R}^5).$$

Then there exist $T > 0$ and a local solution $g \in C([0,T), H^k_{\text{rad}}(\mathbb{R}^5)) \cap C^1([0,T), H^{k-1}_{\text{rad}}(\mathbb{R}^5))$ to (2.7). Furthermore, the solution can be continued past any $T_1 \geq T$ as long as

$$\sup_{0\leq t<T_1} G(t) < \infty, \tag{2.8}$$

where

$$G(t) = \big\|\langle x\rangle g(t)\big\|_{L^\infty_x(\mathbb{R}^5)} + \big\|\langle x\rangle\big(|\partial_t g| + |\nabla g|\big)\big\|_{L^\infty_x(\mathbb{R}^5)}. \tag{2.9}$$

The proof of Proposition 2.1 uses standard energy estimates and will be omitted here. Our main result is the following.

THEOREM 2.2 (Global well-posedness for large data)
Let $k \geq 4$ be an integer, and assume that

$$(g, \partial_t g)|_{t=0} = (g_0, g_1) \in H^k_{\mathrm{rad}}(\mathbb{R}^5) \times H^{k-1}_{\mathrm{rad}}(\mathbb{R}^5).$$

Then the corresponding solution in Proposition 2.1 is global.

By Proposition 2.1, the proof of Theorem 2.2 reduces to showing that (2.8) holds for any $T > 0$. We will achieve this by devising a new nonlinear energy bootstrap method.

3. Nonlinear energy bootstrap: Preliminary transformations

Recall that (2.1) has the basic energy conservation

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^\infty \Big(1 + \frac{2\sin^2 f}{r^2}\Big)\big((\partial_t f)^2 + (\partial_r f)^2\big) r^2\, dr \\ &\quad + \int_0^\infty \frac{\sin^2 f}{r^2}\Big(1 + \frac{\sin^2 f}{2r^2}\Big) r^2\, dr \\ &= E_0, \quad \forall t > 0. \end{aligned} \tag{3.1}$$

The continuation criterion (2.8) is supercritical with respect to this basic energy conservation. To prove global well-posedness of (2.1), one certainly needs a new strategy. In this section, we explain the setup of our nonlinear energy bootstrap argument.

Define $\tilde{\Phi}_1 : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{\Phi}_1(\rho, z) = \int_{N_1\pi}^{z} \Big(1 + \frac{2\sin^2 y}{\rho^2}\Big)^{\frac{1}{2}} dy. \tag{3.2}$$

The definition of $\tilde{\Phi}_1$ takes into consideration the boundary condition (1.6), especially when $N_1 \neq 0$.

Define

$$\Phi_1(r, t) = \tilde{\Phi}_1\big(r, f(r, t)\big). \tag{3.3}$$

Then

$$\begin{aligned} \Box_3 \Phi_1 &= (\partial_{zz}\tilde{\Phi}_1)\big(r, f(r,t)\big)\big((\partial_t f)^2 - (\partial_r f)^2\big) + (\partial_z \tilde{\Phi}_1)\big(r, f(r,t)\big)\Box_3 f \\ &\quad - (\Delta_{3,\rho}\tilde{\Phi}_1)\big(r, f(r,t)\big) - 2(\partial_\rho \partial_r \tilde{\Phi}_1)\big(r, f(r,t)\big)\partial_r f. \end{aligned} \tag{3.4}$$

Here $\Delta_{3,\rho}$ is the 3-dimensional radial Laplacian in the ρ variable, that is,

$$(\Delta_{3,\rho}\tilde{\Phi}_1)(\rho, z) = (\partial_\rho^2 \tilde{\Phi}_1)(\rho, z) + \frac{2}{\rho}(\partial_\rho \tilde{\Phi}_1)(\rho, z).$$

Recall that

$$A_1 = 1 + \frac{2\sin^2 f}{r^2}.$$

It is easy to check that

$$\begin{aligned} &(\partial_{zz}\tilde{\Phi}_1)\big(r, f(r,t)\big) + (\partial_z\tilde{\Phi}_1)\big(r, f(r,t)\big)\cdot\Big(-\frac{1}{A_1}\Big)\cdot\frac{\sin(2f)}{r^2} = 0,\\ &-2(\partial_\rho\partial_z\tilde{\Phi}_1)\big(r, f(r,t)\big) - \frac{(\partial_z\tilde{\Phi}_1)(r, f(r,t))}{A_1}\cdot\frac{4\sin^2 f}{r^3} = 0. \end{aligned} \tag{3.5}$$

Therefore, by (3.4), (2.1), and (3.5), we obtain

$$\Box_3\Phi_1 = -A_1^{-\frac{1}{2}}\cdot\frac{\sin(2f)}{r^2} - A_1^{-\frac{1}{2}}\cdot\frac{\sin^2 f\cdot\sin(2f)}{r^4} - (\Delta_{3,\rho}\tilde{\Phi}_1)\big(r, f(r,t)\big). \tag{3.6}$$

Denote

$$B_1 = 1 + \frac{2\sin^2 y}{r^2}. \tag{3.7}$$

By a simple computation,

$$\Delta_{3,r}(B_1^{\frac{1}{2}}) = \frac{1}{r^2}(B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}}).$$

Hence

$$(\Delta_{3,\rho}\tilde{\Phi}_1)\big(r, f(r,t)\big) = \frac{1}{r^2}\int_{N_1\pi}^{f(r,t)}(B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}})\,dy. \tag{3.8}$$

By a tedious calculation, we have

$$\begin{aligned} A_1^{-\frac{1}{2}}\cdot\frac{\sin(2f)}{r^2} &= \frac{1}{r^2}\int_{N_1\pi}^{f(r,t)}\partial_y\big(B_1^{-\frac{1}{2}}\cdot\sin(2y)\big)\,dy\\ &= \frac{1}{r^2}\int_{N_1\pi}^{f(r,t)}B_1^{-\frac{3}{2}}\big(2 - r^2(B_1^2 - 1)\big)\,dy. \end{aligned} \tag{3.9}$$

Similarly,

$$\begin{aligned} A_1^{-\frac{1}{2}}\cdot\frac{\sin^2 f\cdot\sin(2f)}{r^4} = &\frac{1}{r^2}\cdot\int_{N_1\pi}^{f(r,t)}(2B_1^{\frac{1}{2}} - B_1^{-\frac{3}{2}} - B_1^{-\frac{1}{2}})\,dy\\ &+ \frac{1}{2}\int_{N_1\pi}^{f(r,t)}B_1^{-\frac{3}{2}}(-3B_1^3 + 5B_1^2 - B_1 - 1)\,dy. \end{aligned} \tag{3.10}$$

Plugging (3.8), (3.9), and (3.10) into (3.6), we obtain

$$\Box_3\Phi_1 = -\frac{2}{r^2}\Phi_1 + \frac{1}{2}\int_{N_1\pi}^{f(r,t)} (3B_1^{\frac{3}{2}} - 3B_1^{\frac{1}{2}} + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}})\,dy. \tag{3.11}$$

Equation (3.11) is still not very satisfactory since it contains terms of inverse square potential type. To remove such terms, one more transformation is needed.

Define $\Phi_2(r,t)$ by

$$\Phi_1(r,t) = r\Phi_2(r,t). \tag{3.12}$$

Then

$$\begin{aligned}\Box_3\Phi_1 &= \Box_3(r\Phi_2)\\ &= r\Box_5\Phi_2 - \frac{2}{r^2}\Phi_1.\end{aligned} \tag{3.13}$$

By (3.13), equation (3.11) expressed in the Φ_2 variable now takes the form

$$\Box_5\Phi_2 = -\frac{3}{2}\Phi_2 + \frac{1}{2r}\int_{N_1\pi}^{f(r,t)} (3B_1^{\frac{3}{2}} + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}})\,dy. \tag{3.14}$$

Although formally the right-hand side of (3.14) still contains $1/r$ terms which may be singular when $r \to 0$, it actually causes no trouble in our energy bootstrap estimates later. To see this, we bring back the g-function used in the local analysis.

Recall that

$$f(r,t) = \phi(r) + rg(r,t), \tag{3.15}$$

where $\phi(r) \equiv N_1\pi$ for $r < 1$ and $\phi(r) = 0$ for $r \geq 2$. Define

$$B_2 = 1 + \frac{2\sin^2(ry)}{r^2}. \tag{3.16}$$

Observe that B_2 is a smooth function (see the discussion preceding the estimate (2.6)).

Let $\phi_{<1}$ be a smooth cutoff function such that $\phi_{<1}(r) = 1$ for $r \leq \frac{1}{2}$ and $\phi_{<1}(r) = 0$ for $r > 1$. By (3.15) and (3.16), we have

$$\begin{aligned}&\frac{1}{2r}\phi_{<1}(r)\int_{N_1\pi}^{f(r,t)} (3B_1^{\frac{3}{2}} + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}})\,dy\\ &\quad= \frac{1}{2r}\phi_{<1}(r)\int_{N_1\pi}^{N_1\pi + rg(r,t)} (3B_1^{\frac{3}{2}} + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}})\,dy\\ &\quad= \frac{1}{2}\phi_{<1}(r)\int_0^{g(r,t)} (3B_2^{\frac{3}{2}} + B_2^{-\frac{1}{2}} - B_2^{-\frac{3}{2}})\,dy.\end{aligned} \tag{3.17}$$

In the second equality above, we have performed a change of variable $y \to N_1\pi + ry$. Clearly, (3.17) is smooth as long as g is smooth since it has no singular terms in r.

By using (3.17), we rewrite (3.14) as

$$\Box_5 \Phi_2 = -\frac{3}{2}\Phi_2 + \frac{1}{2}\phi_{<1}\int_0^{g(r,t)} (3B_2^{\frac{3}{2}} + B_2^{-\frac{1}{2}} - B_2^{-\frac{3}{2}})\,dy$$
$$+ \frac{1}{2r}\phi_{>1}\int_{N_1\pi}^{f(r,t)} (3B_1^{\frac{3}{2}} + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}})\,dy, \tag{3.18}$$

where $\phi_{>1} = 1 - \phi_{<1}$ is localized to $r \gtrsim 1$.

Equation (3.18) is almost good for us since it no longer contains any derivative terms or singularities in r. However, there is one more problem.

By (3.2), (3.3), and (3.12), we have

$$\Phi_2(r,t) = \frac{1}{r}\int_{N_1\pi}^{f(r,t)} \Big(1 + \frac{2\sin^2 y}{r^2}\Big)^{\frac{1}{2}}\,dy. \tag{3.19}$$

By (3.18), it is not difficult to check that Φ_2 has no singularity near $r \sim 0$. However, for $r \geq 2$, by using energy conservation (3.1) and radial Sobolev embedding, we obtain $|f(r,t)| \lesssim r^{-1}$. If $N_1 > 0$, then (3.19) asserts that

$$\Phi_2(r,t) \sim \frac{\text{Const}}{r}, \quad \text{as } r \to \infty.$$

In particular, $\Phi_2 \notin L_x^2(\mathbb{R}^5)$ when we regard Φ_2 as a radial function on $\mathbb{R}^5$. We therefore need to introduce one more transformation to kill this divergence.

To this end, we define

$$\Phi(r,t) = \Phi_2(r,t) + \frac{1}{3}\phi_{>1}\cdot\frac{1}{r}\int_0^{N_1\pi} (3B_1^{\frac{3}{2}} + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}})\,dy \tag{3.20}$$
$$= \frac{1}{r}\phi_{<1}\int_{N_1\pi}^{f(r,t)} B_1^{\frac{1}{2}}\,dy \tag{3.21}$$
$$+ \frac{1}{r}\phi_{>1}\int_0^{f(r,t)} B_1^{\frac{1}{2}}\,dy \tag{3.22}$$
$$+ \frac{1}{r}\phi_{>1}\int_0^{N_1\pi} \Big(B_1^{\frac{3}{2}} - B_1^{\frac{1}{2}} + \frac{1}{3}B_1^{-\frac{1}{2}} - \frac{1}{3}B_1^{-\frac{3}{2}}\Big)\,dy. \tag{3.23}$$

Since $\phi(r) \equiv N_1\pi$ for $r < 1$, by (3.15), we have

$$(3.21) = \phi_{<1}\int_0^{g(r,t)} \Big(1 + \frac{2\sin^2(ry + \phi(r))}{r^2}\Big)^{\frac{1}{2}}\,dy. \tag{3.24}$$

For (3.22), we have

$$(3.22) = \phi_{>1}\int_0^{g(r,t)}\Big(1+\frac{2\sin^2(ry+\phi(r))}{r^2}\Big)^{\frac{1}{2}}\,dy + \frac{1}{r}\phi_{>1}\int_0^{\phi(r)} B_1^{\frac{1}{2}}\,dy. \tag{3.25}$$

Note that $\phi(r)=0$ for $r\ge 2$. Therefore, we can write

$$\frac{1}{r}\phi_{>1}\int_0^{\phi(r)} B_1^{\frac{1}{2}}\,dy = \phi_{\sim 1}(r), \tag{3.26}$$

where $\phi_{\sim 1}$ is a smooth cutoff function localized to $r\sim 1$.

For (3.23), observe that by (3.7)

$$B_1^{\frac{3}{2}} - B_1^{\frac{1}{2}} = O\Big(\frac{1}{r^2}\Big), \quad r\gtrsim 1,$$

and similarly

$$\frac{1}{3}B_1^{-\frac{1}{2}} - \frac{1}{3}B_1^{-\frac{3}{2}} = O\Big(\frac{1}{r^2}\Big), \quad r\gtrsim 1.$$

Therefore, we will write

$$(3.23) = \frac{1}{r^3}\phi_{\gtrsim 1}(r), \tag{3.27}$$

where $\phi_{\gtrsim 1}(r)$ is a smooth cutoff function localized to $r\gtrsim 1$ and can vary from place to place.

By using (3.21)–(3.27), we obtain

$$\Phi(r,t) = \int_0^{g(r,t)}\Big(1+\frac{2\sin^2(ry+\phi(r))}{r^2}\Big)^{\frac{1}{2}}\,dy + \phi_{\sim 1} + \frac{1}{r^3}\phi_{\gtrsim 1}(r).$$

We can further include $\phi_{\sim 1}(r)$ into $\phi_{\gtrsim 1}(r)$ and simply write

$$\phi_{\sim 1}(r) + \frac{1}{r^3}\phi_{\gtrsim 1}(r) = \frac{1}{r^3}\phi_{\gtrsim 1}(r).$$

Then

$$\Phi(r,t) = \int_0^{g(r,t)}\Big(1+\frac{2\sin^2(ry+\phi(r))}{r^2}\Big)^{\frac{1}{2}}\,dy + \frac{1}{r^3}\phi_{\gtrsim 1}(r). \tag{3.28}$$

On the other hand, by (3.20) and a simple computation, we have

$$\Phi(r,t) = \Phi_2(r,t) + \frac{1}{r}\cdot\phi_{\gtrsim 1}(r). \tag{3.29}$$

Plugging (3.29) into (3.18) and using (3.20), we obtain

$$\Box_5 \Phi = \frac{1}{r^3} \cdot \phi_{\gtrsim 1} - \frac{3}{2}\Phi + \frac{1}{2}\phi_{<1} \int_0^{g(r,t)} (3B_2^{\frac{3}{2}} + B_2^{-\frac{1}{2}} - B_2^{-\frac{3}{2}})\, dy + \frac{1}{2} \cdot \phi_{>1} \cdot \frac{1}{r} \int_0^{f(r,t)} (3B_1^{\frac{3}{2}} + B_1^{-\frac{1}{2}} - B_1^{-\frac{3}{2}})\, dy. \tag{3.30}$$

By using an argument similar to the derivation of (3.28), we further simplify (3.30) as

$$\Box_5 \Phi = \frac{1}{r^3}\phi_{\gtrsim 1} - \frac{3}{2}\Phi + \frac{1}{2} \int_0^{g(r,t)} (3B^{\frac{3}{2}} + B^{-\frac{1}{2}} - B^{-\frac{3}{2}})\, dy, \tag{3.31}$$

where

$$B = 1 + \frac{2\sin^2(ry + \phi(r))}{r^2}. \tag{3.32}$$

Formula (3.28) then takes the form

$$\Phi(r,t) = \int_0^{g(r,t)} B^{\frac{1}{2}}\, dy + \frac{1}{r^3}\phi_{\gtrsim 1}(r). \tag{3.33}$$

We analyze (3.31)–(3.33) in the next section.

4. Nonblowup of the H^1-norm of Φ

The first step in our analysis is to control the H^1-norm of Φ. This includes $\|\Phi\|_{L^2_x(\mathbb{R}^5)}$, $\|\partial_t \Phi\|_{L^2_x(\mathbb{R}^5)}$, and $\|\nabla\Phi\|_{L^2_x(\mathbb{R}^5)}$. By (3.33) and (2.2), we have

$$\partial_t \Phi = \frac{1}{r} \cdot \partial_t f \cdot \Big(1 + \frac{2\sin^2 f}{r^2}\Big)^{\frac{1}{2}},$$

and therefore, by (3.1), we obtain

$$\|\partial_t \Phi\|_{L^2_x(\mathbb{R}^5)} \lesssim 1. \tag{4.1}$$

By (3.32) and (3.33), it is easy to see that

$$\big|\Phi(r,t)\big| \lesssim \big|g(r,t)\big| + \big|g(r,t)\big|^2 + \frac{1}{r^3}\big|\phi_{\gtrsim 1}(r)\big|. \tag{4.2}$$

By the assumption of Proposition 2.1 and Sobolev embedding, we have $\|g(0)\|_{L^4_x(\mathbb{R}^5)} \lesssim 1$. By (4.2), this gives $\|\Phi(0)\|_{L^2_x(\mathbb{R}^5)} \lesssim 1$. Using (4.1), we then have

$$\big\|\Phi(t)\big\|_{L^2_x(\mathbb{R}^5)} \le \text{Const} \cdot t, \quad \forall t > 0. \tag{4.3}$$

By (3.1), we have

$$\|\partial_t f\|_{L^2_x(\mathbb{R}^3)} + \|\partial_r f\|_{L^2_x(\mathbb{R}^3)} \lesssim 1.$$

Since $\|f(0)\|_{L^2_x(\mathbb{R}^3)} \lesssim 1$, we obtain

$$\|f(t)\|_{H^1_x(\mathbb{R}^3)} \leq \mathrm{Const}\cdot t, \quad \forall t > 0. \tag{4.4}$$

By (2.2) and Hardy's inequality (see (4.13)), we obtain

$$\|g(t)\|_{H^1_x(\mathbb{R}^5)} \leq \mathrm{Const}\cdot t, \quad \forall t > 0. \tag{4.5}$$

However, it is not difficult to check that (3.1) and (4.5) are insufficient to bound $\|\nabla\Phi\|_{L^2_x(\mathbb{R}^5)}$. One may try to do Strichartz. But there is one problem as we now explain.

Imagine that

$$g(r,t) \sim \frac{1}{r} \tag{4.6}$$

for a range of values of $r \ll 1$.[2]

Then by (3.33),

$$\Phi(r,t) \sim \frac{\sqrt{2}}{r} g(r,t)$$

and

$$\frac{3}{2}\int_0^{g(r,t)} B^{\frac{3}{2}}\,dy \sim \frac{3\sqrt{2}}{r^3} g(r,t) \sim \frac{3}{r^2}\Phi(r,t).$$

Therefore for a range of values of $r \ll 1$, the linear part of (3.31) takes the form

$$\square_5 \Phi = -\frac{3}{2}\Phi + \frac{3}{r^2}\Phi. \tag{4.7}$$

Equation (4.7) is a wave operator with *negative* inverse square potential. Since $d = 5$ and

$$3 > \frac{(d-2)^2}{4},$$

no Strichartz is available (cf. [4]). This destroys the hope of employing good linear estimates.

Therefore a new idea is required to establish the H^1-norm bound of Φ. In particular, we will use a nonlinear approach which exploits in an essential way the structure of the equation.

[2]Certainly (4.6) cannot hold for all $r \to 0$ since g is assumed to be regular at $r = 0$.

LEMMA 4.1
There exists $r_0 > 0$ sufficiently small such that for any $0 \le r \le r_0$, we have

$$F(\beta) = \int_0^\beta (r^2 + 2\sin^2 y)^{\frac{1}{2}} \Big(\frac{3}{4} - \sin^2 y\Big)\, dy \ge 0, \quad \forall \beta \ge 0.$$

If $r > 0$, then the equality holds if and only if $\beta = 0$.

Proof
By a simple calculation, we have

$$\int_0^\pi (\sin y) \cdot \Big(\frac{3}{4} - \sin^2 y\Big)\, dy = \frac{1}{6}.$$

Clearly there exists $r_1 > 0$ sufficiently small such that

$$\int_0^\pi (r^2 + 2\sin^2 y)^{\frac{1}{2}} \Big(\frac{3}{4} - \sin^2 y\Big)\, dy \ge \frac{1}{12}, \quad \forall 0 \le r < r_1. \tag{4.8}$$

Consider $m\pi \le \beta < (m+1)\pi$, and assume that m is large. Then by (4.8), for $0 \le r < r_1$, we have

$$\begin{aligned}
&\int_0^\beta (r^2 + 2\sin^2 y)^{\frac{1}{2}} \Big(\frac{3}{4} - \sin^2 y\Big)\, dy \\
&\quad \ge \frac{1}{12} m + \int_{m\pi}^\beta (r^2 + 2\sin^2 y)^{\frac{1}{2}} \Big(\frac{3}{4} - \sin^2 y\Big)\, dy \\
&\quad \ge \frac{m}{12} - O(1) > \frac{1}{12},
\end{aligned}$$

if m is taken to be sufficiently large.

Therefore, we only need to consider $F(\beta)$ on a compact interval $[0, m\pi]$. Observe that $F(0) = 0$, $F(m\pi) > \frac{1}{12}$. It suffices to consider critical points of F in $(0, m\pi)$ and prove the positivity of F at these points. Solving $F'(\beta) = 0$ yields

$$\sin(\beta) = \pm\frac{\sqrt{3}}{2}.$$

Hence

$$\beta = j\pi + \frac{\pi}{3} \qquad \text{or} \qquad j\pi + \frac{2\pi}{3}, \quad j \ge 0, j \in \mathbb{Z}.$$

If $\beta = j\pi + \frac{\pi}{3}$, then for $0 \le r < r_1$, by (4.8),

$$F\Big(j\pi + \frac{\pi}{3}\Big) = \int_0^{j\pi+\frac{\pi}{3}} (r^2 + 2\sin^2 y)^{\frac{1}{2}} \Big(\frac{3}{4} - \sin^2 y\Big)\, dy$$
$$\ge \frac{j}{12} + \int_0^{\frac{\pi}{3}} (r^2 + 2\sin^2 y)^{\frac{1}{2}} \Big(\frac{3}{4} - \sin^2 y\Big)\, dy$$
$$> 0.$$

If $\beta = j\pi + \frac{2\pi}{3}$, then for $0 \le r < r_1$,

$$F\Big(j\pi + \frac{2\pi}{3}\Big) \ge \frac{j}{12} + \int_0^{\frac{2\pi}{3}} (r^2 + 2\sin^2 y)^{\frac{1}{2}} \Big(\frac{3}{4} - \sin^2 y\Big)\, dy$$
$$\ge \int_0^{\frac{2\pi}{3}} (r^2 + 2\sin^2 y)^{\frac{1}{2}} \Big(\frac{3}{4} - \sin^2 y\Big)\, dy.$$

Define

$$\tilde{F}(\rho) = \int_0^{\frac{2\pi}{3}} (\rho + \sin^2 y)^{\frac{1}{2}} \Big(\frac{3}{4} - \sin^2 y\Big)\, dy.$$

It is easy to check that $\tilde{F}(0) = 0$. On the other hand,

$$\frac{\tilde{F}(\rho) - \tilde{F}(0)}{\rho} = \int_0^{\frac{2\pi}{3}} \frac{1}{\sqrt{\rho + \sin^2 y} + \sqrt{\sin^2 y}} \cdot \Big(\frac{3}{4} - \sin^2 y\Big)\, dy$$
$$> 0, \quad \text{for } \rho \text{ sufficiently small.}$$

Hence $F(j\pi + \frac{2\pi}{3}) > 0$ for $0 < r \le r_0$, where r_0 is sufficiently small. ☐

Define $G_i : (0,\infty) \times \mathbb{R} \to \mathbb{R}$, $i = 0, 1, 2$ by

$$G_0(r, w) = \int_0^w \Big(1 + \frac{2\sin^2(ry)}{r^2}\Big)^{\frac{1}{2}} \cdot \frac{2\sin^2(ry)}{r^2}\, dy, \tag{4.9}$$

$$G_1(r, z) = \frac{3}{2} \int_0^z G_0(r, w)\Big(1 + \frac{2\sin^2(rw)}{r^2}\Big)^{\frac{1}{2}}\, dw, \tag{4.10}$$

$$G_2(r, w) = \int_0^w \Big(1 + \frac{2\sin^2(ry)}{r^2}\Big)^{\frac{1}{2}}\, dy. \tag{4.11}$$

COROLLARY 4.2

For any $0 < r \le r_0$, $z \in \mathbb{R}$, we have

$$\big|G_1(r, z)\big| \le \frac{9}{4} \cdot \frac{1}{2} \cdot \frac{(G_2(r, z))^2}{r^2}. \tag{4.12}$$

Proof
Since $G_1(r,z)$ is an even function of z, it suffices to consider the case $z > 0$. By Lemma 4.1 for $w \geq 0$,

$$0 \leq \frac{3}{2} G_0(r,w) \leq \frac{9}{4} \cdot \frac{1}{r^2} \cdot G_2(r,w).$$

Therefore,

$$\begin{aligned} 0 \leq G_1(r,z) &\leq \frac{9}{4} \cdot \frac{1}{r^2} \cdot \int_0^z G_2(r,w) \cdot \Big(1 + \frac{2\sin^2(rw)}{r^2}\Big)^{\frac{1}{2}} \, dw \\ &= \frac{9}{4} \cdot \frac{1}{r^2} \int_0^z G_2(r,w) \cdot (\partial_w G_2)(r,w) \, dw \\ &= \frac{9}{4} \cdot \frac{1}{2} \cdot \frac{(G_2(r,z))^2}{r^2}. \end{aligned}$$

□

LEMMA 4.3 (Hardy's inequality)
Let $d \geq 3$. Then

$$\int_{\mathbb{R}^d} \frac{f^2}{|x|^2} \, dx \leq \frac{4}{(d-2)^2} \int_{\mathbb{R}^d} |\nabla f|^2 \, dx, \quad \forall f \in C_0^\infty(\mathbb{R}^d). \tag{4.13}$$

The constant $\frac{4}{(d-2)^2}$ is sharp.

The goal of this section is to prove the following.

PROPOSITION 4.4 (Nonblowup of the H^1-norm of Φ)
Let $T > 0$ be the maximal lifespan of the local solution g constructed in Proposition 2.1. If $T < \infty$, then

$$\sup_{0 \leq t < T} \Big(\big\| \Phi(t) \big\|_{H_x^1(\mathbb{R}^5)} + \big\| \partial_t \Phi(t) \big\|_{L_x^2(\mathbb{R}^5)} \Big) < \infty. \tag{4.14}$$

Before we begin the proof of Proposition 4.4, we set up some notation.

Notation
Throughout the rest of the present work, unless explicitly mentioned, we will suppress the dependence of constants on the initial data or on the time T. For example, we will write (4.14) simply as

$$\big\| \Phi(t) \big\|_{H_x^1(\mathbb{R}^5)} + \big\| \partial_t \Phi(t) \big\|_{L_x^2(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \leq t < T.$$

Proof of Proposition 4.4
By (4.1) and (4.3), we only need to show that

$$\left\|\nabla\Phi(t)\right\|_{L_x^2(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \le t < T.$$

Let $\psi \in C_c^\infty(\mathbb{R}^5)$, $0 \le \psi \le 1$, be a radial smooth cutoff function such that $\psi(x) = 1$ for $|x| \le \frac{1}{2}$ and $\psi(x) = 0$ for $|x| \ge 1$. Choose $r_0 \le \frac{1}{2}$ as in Lemma 4.1, and define

$$\phi_{<r_0}(x) = \psi\Big(\frac{x}{r_0}\Big),$$
$$\phi_{>r_0}(x) = 1 - \phi_{<r_0}(x).$$

By (3.31)–(3.33), we have

$$\begin{aligned}
\Box_5\Phi &= \frac{1}{r^3}\phi_{\gtrsim 1} + \frac{1}{2}\int_0^{g(r,t)} (B^{-\frac{1}{2}} - B^{-\frac{3}{2}})\,dy + \frac{3}{2}\int_0^{g(r,t)} (B^{\frac{3}{2}} - B^{\frac{1}{2}})\,dy \\
&= \frac{1}{r^3}\phi_{\gtrsim 1} + \frac{1}{2}\int_0^{g(r,t)} (B^{-\frac{1}{2}} - B^{-\frac{3}{2}})\,dy \\
&\quad + \frac{3}{2}\phi_{>r_0}\int_0^{g(r,t)} (B^{\frac{3}{2}} - B^{\frac{1}{2}})\,dy \\
&\quad + \frac{3}{2}\phi_{<r_0}\int_0^{g(r,t)} \Big(1 + \frac{2\sin^2(ry)}{r^2}\Big)^{\frac{1}{2}} \cdot \frac{2\sin^2(ry)}{r^2}\,dy.
\end{aligned} \tag{4.15}$$

Multiplying both sides of (4.15) by $\partial_t\Phi$ and integrating by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^5}\Big(\frac{1}{2}(\partial_t\Phi)^2 + \frac{1}{2}|\nabla\Phi|^2 - \phi_{<r_0}(x)\cdot G_1\big(r, g(r,t)\big)\Big)\,dx \\
&\qquad \lesssim \|\partial_t\Phi\|_{L_x^2(\mathbb{R}^5)} \cdot \big(1 + \left\|g(t)\right\|_{L_x^2(\mathbb{R}^5)}\big).
\end{aligned} \tag{4.16}$$

Plugging (4.1) and (4.5) into (4.16) and integrating in time, we obtain

$$\sup_{0\le t<T}\int_{\mathbb{R}^5}\Big(\frac{1}{2}(\partial_t\Phi)^2 + \frac{1}{2}|\nabla\Phi|^2 - \phi_{<r_0}(x)\cdot G_1\big(r, g(r,t)\big)\Big)\,dx \lesssim 1. \tag{4.17}$$

In particular, this yields

$$\int_{\mathbb{R}^5}\Big(\frac{1}{2}\left|\nabla\Phi(t)\right|^2 - \phi_{<r_0}(x)\cdot G_1\big(r, g(r,t)\big)\Big)\,dx \lesssim 1, \quad \forall 0 \le t < T. \tag{4.18}$$

Using Corollary 4.2 and (4.3), we obtain

$$\int_{\mathbb{R}^5}\Big(\left|\nabla\Phi(t)\right|^2 - \frac{9}{4}\cdot\frac{|\Phi(t)|^2}{r^2}\Big)\,dx \lesssim 1, \quad \forall 0 \le t < T. \tag{4.19}$$

By Hardy's inequality (Lemma 4.3), we have

$$\int_{\mathbb{R}^5} \left(\left|\nabla \Phi(t)\right|^2 - \frac{9}{4} \cdot \frac{|\Phi(t)|^2}{r^2} \right) dx \geq 0. \tag{4.20}$$

There is no hope of obtaining (4.14) by using only (4.19) and (4.20) since there could possibly exist a sequence $\Phi(t_n)$ with the property that

$$\left\|\nabla \Phi(t_n)\right\|_{L_x^2(\mathbb{R}^5)} \to \infty, \qquad \left\|\frac{\Phi(t_n)}{r}\right\|_{L_x^2(\mathbb{R}^5)} \to \infty,$$

but

$$\int_{\mathbb{R}^5} \left(\left|\nabla \Phi(t_n)\right|^2 - \frac{9}{4} \cdot \frac{|\Phi(t_n)|^2}{r^2} \right) dx \to C_1, \quad \text{as } t_n \to T,$$

where $C_1 \geq 0$ is a finite constant.

Certainly, a new argument is needed here. To solve this problem, we will proceed by exploiting in more detail the structure of Φ.

Assume that (4.14) does not hold. By (4.1) and (4.3), there exists $t_n \to T$ such that

$$\lim_{n\to\infty} \left\|\nabla \Phi(t_n)\right\|_{L_x^2(\mathbb{R}^5)} = +\infty. \tag{4.21}$$

Define

$$\tilde{\Phi}(t_n) = \frac{\Phi(t_n)}{\|\nabla \Phi(t_n)\|_{L_x^2(\mathbb{R}^5)}}. \tag{4.22}$$

Then

$$\left\|\nabla \tilde{\Phi}(t_n)\right\|_{L_x^2(\mathbb{R}^5)} = 1, \tag{4.23}$$

and by (4.21), (4.19), and (4.20),

$$\left\|\frac{\tilde{\Phi}(t_n)}{r}\right\|_{L_x^2(\mathbb{R}^5)} \to \frac{2}{3}, \quad \text{as } t_n \to T. \tag{4.24}$$

Next consider (3.33). If $r \gtrsim 1$, then

$$\left|\partial_r \Phi(r,t)\right| \lesssim |\partial_r g| + \left|\frac{g}{r}\right| + \left|\frac{1}{r^3}\phi_{\gtrsim 1}(r)\right|.$$

Therefore, by (4.5) and (4.3),

$$\left\|\nabla \Phi(t)\right\|_{L_x^2(|x|>\frac{1}{2}, x\in\mathbb{R}^5)} + \left\|\frac{2}{r}\Phi(t)\right\|_{L_x^2(|x|>\frac{1}{2}, x\in\mathbb{R}^5)} \lesssim 1. \tag{4.25}$$

For $r \le \frac{1}{2}$, by (3.33) and a short computation, we have

$$\partial_r \Phi(r,t) + \frac{2}{r}\Phi(r,t) = \Big(1+\frac{2\sin^2 f}{r^2}\Big)^{\frac{1}{2}} \cdot \frac{\partial_r f}{r} + \frac{1}{r}\int_0^{g(r,t)} \Big(1+\frac{2\sin^2(ry)}{r^2}\Big)^{-\frac{1}{2}}\,dy. \tag{4.26}$$

By (3.1),

$$\Big\| \Big(1+\frac{2\sin^2 f}{r^2}\Big)^{\frac{1}{2}} \cdot \frac{\partial_r f}{r} \Big\|_{L_x^2(\mathbb{R}^5)} \lesssim 1.$$

Hence, (4.25) and (4.26) give

$$\Big\| \partial_r \Phi(t) + \frac{2}{r}\Phi(t) \Big\|_{L_x^2(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \le t < T. \tag{4.27}$$

By (4.21), (4.22), and (4.27), we obtain

$$\Big\| \partial_r \tilde{\Phi}(t_n) + \frac{2}{r}\tilde{\Phi}(t_n) \Big\|_{L_x^2(\mathbb{R}^5)} \to 0, \quad \text{as } t_n \to T.$$

But this contradicts (4.23) and (4.24). □

Remark 4.5

In the above derivation, the contradiction (blowup) argument is actually not needed. One can directly use both (4.19) and (4.27) to then obtain the desired uniform bound on $\|\partial_r \Phi(t)\|_{L_x^2(\mathbb{R}^5)}$. (We thank one of the anonymous referees for pointing this out.)

5. Nonlinear energy bootstrap: More estimates

Let $T > 0$ be the same as in Proposition 4.4. Our goal in this section is to prove

$$\sum_{|\alpha|+|\beta|\le 4} \big\| \partial_x^\alpha \partial_t^\beta \Phi(t) \big\|_{L_x^2(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \le t < T, \tag{5.1}$$

and eventually

$$\sup_{0\le t<T} G(t) < \infty, \tag{5.2}$$

where $G(t)$ is defined in (2.9). By Proposition 2.1, this implies global well-posedness.

We will prove (5.1) in several steps. First we obtain some decay estimates of Φ and g.

By Proposition 4.4 and radial Sobolev embedding, we have

$$\big|\Phi(r,t)\big| \lesssim \min\{r^{-\frac{3}{2}}, r^{-2}\}, \quad \forall r > 0, 0 \le t < T. \tag{5.3}$$

We claim that

$$\left|g(r,t)\right| \lesssim \min\{r^{-\frac{3}{4}}, r^{-2}\}, \quad \forall r > 0, 0 \le t < T. \tag{5.4}$$

By (3.28), it suffices to prove that

$$\left|g(r,t)\right| \lesssim r^{-\frac{3}{4}}, \quad \forall 0 < r \ll 1, 0 \le t < T.$$

For $r \ll 1$, (3.33) gives

$$\Phi(r,t) = \int_0^{g(r,t)} \left(1 + \frac{2\sin^2(ry)}{r^2}\right)^{\frac{1}{2}} dy. \tag{5.5}$$

Suppose that for some $0 < r \ll 1$, $|g(r,t)| \gtrsim \frac{1}{r}$. Then clearly

$$\Phi(r,t) \sim \frac{g}{r}.$$

By (5.3), this would imply that

$$\left|g(r,t)\right| \lesssim r^{-\frac{1}{2}},$$

which contradicts the assumption $|g(r,t)| \gtrsim \frac{1}{r}$.

Therefore, $|g(r,t)| \lesssim \frac{1}{r}$ for all $r \ll 1$. By (5.5), we obtain

$$\left|\Phi(r,t)\right| \sim \left|\int_0^{g(r,t)} (1+y^2)^{\frac{1}{2}} dy\right| \gtrsim g(r,t)^2.$$

Hence by (5.3),

$$g(r,t)^2 \lesssim r^{-\frac{3}{2}}, \quad \forall 0 < r \ll 1, 0 \le t < T.$$

Therefore, (5.4) is proved.

Before we continue, we need to introduce standard Strichartz for the wave operator.

Definition 5.1

Let $d \ge 2$. A pair (q,r) is said to be *wave admissible* if

$$2 \le q \le \infty, \qquad 2 \le r < \infty, \qquad \text{and} \qquad \frac{1}{q} + \frac{d-1}{2r} \le \frac{d-1}{4}.$$

Note that the case $(q,r,d) = (2,\infty,3)$ is not admissible.

LEMMA 5.2
Let $d \geq 2$. Suppose that $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ solves

$$\begin{cases} \partial_{tt} u - \Delta u = F, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1). \end{cases}$$

Let (q, r), $(\tilde{q}, \tilde{r})$ be wave admissible and satisfy the scaling condition

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - \gamma = \frac{1}{\tilde{q}'} + \frac{d}{\tilde{r}'} - 2.$$

Then on the space-time slab $[0, T] \times \mathbb{R}^d$, we have

$$\begin{aligned} &\|u\|_{L_t^q L_x^r} + \|u\|_{C_t \dot{H}_x^\gamma} + \|\partial_t u\|_{C_t \dot{H}_x^{\gamma-1}} \\ &\qquad \lesssim \|u_0\|_{\dot{H}_x^\gamma} + \|u_1\|_{\dot{H}_x^{\gamma-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}. \end{aligned}$$

Here $(\tilde{q}', \tilde{r}')$ are the conjugates of $(\tilde{q}, \tilde{r})$, that is, $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = \frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$.

To simplify the presentation, we introduce more notation.

Notation
For any $z \in \mathbb{R}^d$, we use the Japanese bracket notation $\langle z \rangle := (1 + |z|^2)^{\frac{1}{2}}$. For any space-time slab $[0, T_1] \times \mathbb{R}^5$, we will use the notation

$$\|u\|_{L_t^q L_x^r([0,T_1])}$$

to denote

$$\|u\|_{L_t^q L_x^r([0,T_1]\times\mathbb{R}^5)}.$$

We will need to use the standard Littlewood–Paley projection operators. Let $\tilde{\phi} \in C_c^\infty(\mathbb{R}^5)$ be a radial bump function supported in the ball $\{x \in \mathbb{R}^5 : |x| \leq \frac{25}{24}\}$ and equal to 1 on the ball $\{x \in \mathbb{R}^5 : |x| \leq 1\}$. For any constant $C > 0$, denote $\tilde{\phi}_{\leq C}(x) := \tilde{\phi}(\frac{x}{C})$ and $\tilde{\phi}_{>C} := 1 - \tilde{\phi}_{\leq C}$. For each dyadic $N > 0$, define the Littlewood–Paley projectors

$$\begin{aligned} \widehat{P_{\leq N} f}(\xi) &:= \tilde{\phi}_{\leq N}(\xi) \hat{f}(\xi), \\ \widehat{P_{>N} f}(\xi) &:= \tilde{\phi}_{>N}(\xi) \hat{f}(\xi), \\ \widehat{P_N f}(\xi) &:= (\tilde{\phi}_{\leq N} - \tilde{\phi}_{\leq \frac{N}{2}}) \hat{f}(\xi), \end{aligned}$$

and similarly $P_{<N}$ and $P_{\geq N}$.

Now we are ready to continue our estimates. Taking the time derivative on both sides of (3.31), we obtain

$$\Box_5(\partial_t\Phi) = -\frac{3}{2}\partial_t\Phi + \frac{3}{2}A^{\frac{3}{2}}\partial_t g + \frac{1}{2}(A^{-\frac{1}{2}} - A^{-\frac{3}{2}})\partial_t g, \tag{5.6}$$

where

$$A = 1 + \frac{2\sin^2(rg(r,t)+\phi(r))}{r^2}. \tag{5.7}$$

By (5.4), we have

$$|A-1| \lesssim \min\{r^{-\frac{3}{2}}, r^{-4}\}. \tag{5.8}$$

From (3.33), one has

$$\partial_t\Phi = A^{\frac{1}{2}}\partial_t g. \tag{5.9}$$

Substituting (5.9) into (5.6), we obtain

$$\Box_5(\partial_t\Phi) = \Big(\frac{3}{2} + \frac{1}{2}A^{-2}\Big)(A-1)\partial_t\Phi. \tag{5.10}$$

By Strichartz (Lemma 5.2) and (5.8), we have, for any $0 < T_1 < T$,

$$\begin{aligned}\|P_{\geq 1}\partial_t\Phi\|_{L_t^3L_x^3([0,T_1])} &\lesssim \big\|P_{\geq 1}\partial_t\Phi(0)\big\|_{\dot H_x^{\frac{1}{2}}} + \big\|P_{\geq 1}\partial_{tt}\Phi(0)\big\|_{\dot H_x^{-\frac{1}{2}}} \\ &\quad + \big\|(A-1)\partial_t\Phi\big\|_{L_t^{\frac{3}{2}}L_x^{\frac{3}{2}}([0,T_1])} \\ &\lesssim 1 + \big\|(A-1)\big\|_{L_t^3L_x^3([0,T_1])}\cdot\|\partial_t\Phi\|_{L_t^3L_x^3([0,T_1])} \\ &\lesssim 1 + T_1^{\frac{1}{3}}\|\partial_t\Phi\|_{L_t^3L_x^3([0,T_1])}.\end{aligned} \tag{5.11}$$

Obviously,

$$\|P_{<1}\partial_t\Phi\|_{L_t^3L_x^3([0,T_1])} \lesssim \|\partial_t\Phi\|_{L_t^\infty L_x^2} \lesssim 1. \tag{5.12}$$

Using (5.11), (5.12), and a continuity argument (see Appendix B) yields

$$\|\partial_t\Phi\|_{L_t^3L_x^3([0,T))} \lesssim 1. \tag{5.13}$$

Therefore,

$$\|\partial_t\Phi\|_{L_t^\infty\dot H_x^{\frac{1}{2}}([0,T))} \lesssim 1. \tag{5.14}$$

Using (5.10), we have

$$\Box_5(\partial_{tt}\Phi) = \Big(\frac{3}{2} + \frac{1}{2}A^{-2}\Big)(A-1)\partial_{tt}\Phi + \Big(-\frac{1}{2}A^{-2} + A^{-3} + \frac{3}{2}\Big)\partial_t A\partial_t\Phi. \tag{5.15}$$

By (5.7), observe that

$$|\partial_t A| \lesssim |\partial_t \Phi|. \tag{5.16}$$

Therefore by (5.13),

$$\Big\|\Big(-\frac{1}{2}A^{-2} + A^{-3} + \frac{3}{2}\Big)\partial_t A\partial_t\Phi\Big\|_{L_t^{\frac{3}{2}}L_x^{\frac{3}{2}}([0,T))} \lesssim \|\partial_t\Phi\|^2_{L_t^3 L_x^3([0,T))} \lesssim 1.$$

Denote

$$G_3(r,t) = \frac{1}{2}\int_0^{g(r,t)} (3B^{\frac{3}{2}} + B^{-\frac{1}{2}} - B^{-\frac{3}{2}})\,dy. \tag{5.17}$$

Then by (3.32) and (5.4), we have

$$\big|G_3(r,t)\big| \lesssim \begin{cases} |g(r,t)| & \text{if } r \gtrsim 1, \\ |\Phi(r,t)|^2 + |\Phi(r,t)| & \text{if } r \ll 1. \end{cases} \tag{5.18}$$

Hence by (4.14),

$$\big\|P_{<1}G_3(t)\big\|_{L_x^2(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \le t < T. \tag{5.19}$$

By essentially repeating the derivation of (5.13) and (5.14) with $\partial_t\Phi$ replaced by $\partial_{tt}\Phi$, we obtain

$$\|\partial_{tt}\Phi\|_{L_t^\infty H_x^{\frac{1}{2}}([0,T))} \lesssim 1. \tag{5.20}$$

Note that the low frequency part of $\partial_{tt}\Phi$ causes no trouble since it can be controlled by $\|P_{\le 1}\Delta\Phi\|_{L_x^2} \lesssim \|\Phi\|_{L_x^2}$ using equation (3.31) together with (5.19).

Now by (3.31), we have

$$-\Delta\Phi = -\partial_{tt}\Phi + \frac{1}{r^3}\phi_{\gtrsim 1} - \frac{3}{2}\Phi + G_3, \tag{5.21}$$

where $G_3(r,t)$ was already defined in (5.17). By (5.18) and (4.5),

$$\big\|G_3(t)\big\|_{L_x^2(|x|>\frac{1}{4}, x\in\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \le t < T. \tag{5.22}$$

By (5.18) and (4.14), we obtain

$$\left\|\phi_{\le\frac{1}{2}}G_3(t)\right\|_{L_x^{\frac{5}{3}}(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \le t < T, \tag{5.23}$$

where $\phi_{\le\frac{1}{2}}$ is a smooth cutoff function localized to $r \le \frac{1}{2}$.

By (5.20), (5.21), (5.22), and (5.23), we have

$$\begin{aligned}\left\|P_{>1}|\nabla|^{-\frac{1}{2}}\Delta\Phi(t)\right\|_{L_x^2(\mathbb{R}^5)} &\lesssim \left\|P_{>1}\partial_{tt}\Phi(t)\right\|_{\dot{H}_x^{\frac{1}{2}}(\mathbb{R}^5)} + 1 + \left\||\nabla|^{-\frac{1}{2}}P_{>1}G_3\right\|_{L_x^2(\mathbb{R}^5)} \\ &\lesssim 1 + \left\|(1-\phi_{\le\frac{1}{2}})G_3(t)\right\|_{L_x^2(\mathbb{R}^5)} + \left\|\phi_{\le\frac{1}{2}}G_3(t)\right\|_{L_x^{\frac{5}{3}}(\mathbb{R}^5)} \\ &\lesssim 1, \quad \forall 0 \le t < T.\end{aligned}$$

Hence

$$\left\||\nabla|^{\frac{3}{2}}\Phi(t)\right\|_{L_x^2(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \le t < T.$$

By Sobolev embedding,

$$\left\|\Phi(t)\right\|_{L_x^5(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \le t < T. \tag{5.24}$$

By (5.18), (5.22), and (5.24), we obtain

$$\left\|G_3(t)\right\|_{L_x^2(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \le t < T.$$

Hence by (5.21) and (5.20), we obtain

$$\left\|\Phi(t)\right\|_{H_x^2(\mathbb{R}^5)} \lesssim 1, \quad \forall 0 \le t < T. \tag{5.25}$$

By radial Sobolev embedding, we have

$$\left\|r^{\frac{1}{2}}\Phi(t)\right\|_{L_x^\infty} \lesssim \|\Delta\Phi\|_{L_x^2} \lesssim 1.$$

Therefore, (5.3), (5.4), and (5.8) can be refined to

$$\left|\Phi(r,t)\right| \lesssim \min\{r^{-\frac{1}{2}}, r^{-2}\}, \tag{5.26}$$

$$\left|g(r,t)\right| \lesssim \min\{r^{-\frac{1}{4}}, r^{-2}\}, \tag{5.27}$$

$$|A-1| \lesssim \min\{r^{-\frac{1}{2}}, r^{-4}\}. \tag{5.28}$$

By (5.15), (5.16), and Strichartz, we have, for any $T_1 < T$,

$$
\begin{aligned}
&\|\partial_{tt}\Phi\|_{L_t^\infty \dot H_x^1([0,T_1])} + \|\partial_{ttt}\Phi\|_{L_t^\infty L_x^2([0,T_1])} \\
&\quad\lesssim \big\|\partial_{tt}\Phi(0)\big\|_{\dot H_x^1} + \big\|\partial_{ttt}\Phi(0)\big\|_{L_x^2} + \big\|(A-1)\partial_{tt}\Phi\big\|_{L_t^1 L_x^2([0,T_1])} \\
&\qquad + \|\partial_t A\cdot\partial_t\Phi\|_{L_t^1 L_x^2([0,T_1])} \\
&\quad\lesssim 1 + T_1\big\|(A-1)\big\|_{L_t^\infty L_x^5([0,T_1])}\cdot\|\partial_{tt}\Phi\|_{L_t^\infty \dot H_x^1([0,T_1])} \\
&\qquad + \|\partial_t\Phi\|^2_{L_t^2 L_x^4([0,T_1])}.
\end{aligned}
\tag{5.29}
$$

By (5.28),

$$
\big\|(A-1)\big\|_{L_t^\infty L_x^5}\lesssim 1. \tag{5.30}
$$

By (5.10), (5.28), and Strichartz, it is not difficult to check that

$$
\|\partial_t\Phi\|_{L_t^2 L_x^4([0,T))} + \|\partial_t\Phi\|_{L_t^\infty \dot H_x^1([0,T))}\lesssim 1. \tag{5.31}
$$

Plugging (5.30) and (5.31) into (5.29), a simple continuity argument then shows that

$$
\|\partial_{tt}\Phi\|_{L_t^\infty \dot H_x^1([0,T))} + \|\partial_{ttt}\Phi\|_{L_t^\infty L_x^2([0,T))}\lesssim 1. \tag{5.32}
$$

By (5.10), (5.32), (5.28), and Hardy's inequality, we then have

$$
\begin{aligned}
\|\partial_t\Delta\Phi\|_{L_t^\infty L_x^2([0,T))} &\lesssim \|\partial_{ttt}\Phi\|_{L_t^\infty L_x^2([0,T))} \\
&\qquad + \big\|(A-1)\partial_t\Phi\big\|_{L_t^\infty L_x^2([0,T))} \\
&\lesssim 1 + \|\nabla\partial_t\Phi\|_{L_t^\infty L_x^2([0,T))} \\
&\lesssim 1.
\end{aligned}
\tag{5.33}
$$

We can write (5.32) and (5.33) collectively as

$$
\|\partial_{ttt}\Phi\|_{L_t^\infty L_x^2([0,T))} + \|\partial_{tt}\nabla\Phi\|_{L_t^\infty L_x^2([0,T))} + \|\partial_t\Delta\Phi\|_{L_t^\infty L_x^2([0,T))}\lesssim 1. \tag{5.34}
$$

By (3.31) and (5.34), we have

$$
\begin{aligned}
&\|\nabla\Delta\Phi\|_{L_t^\infty L_x^2([0,T))} \\
&\quad\lesssim 1 + \left\|\partial_r\Big(\int_0^{g(r,t)}(3B^{\frac32}+B^{-\frac12}-B^{-\frac32})\,dy\Big)\right\|_{L_t^\infty L_x^2([0,T))} \\
&\quad\lesssim 1 + \|A^{\frac32}\partial_r g\|_{L_t^\infty L_x^2([0,T))} \\
&\qquad + \left\|\int_0^{g(r,t)}\Big(\frac92 B^{\frac12}-\frac12 B^{-\frac32}+\frac32 B^{-\frac52}\Big)\partial_r B\,dy\right\|_{L_t^\infty L_x^2([0,T))}.
\end{aligned}
\tag{5.35}
$$

Observe that, for $r \leq \frac{1}{2}$,

$$\left|(\partial_r B)(r, y)\right| \lesssim |y|^3.$$

Therefore by (5.27),

$$\begin{aligned}\Big\| \int_0^{g(r,t)} \Big(\frac{9}{2} B^{\frac{1}{2}} - \frac{1}{2} B^{-\frac{3}{2}} + \frac{3}{2} B^{-\frac{5}{2}}\Big) \partial_r B \, dy \Big\|_{L_t^\infty L_x^2([0,T))} &\lesssim \|g\|^5_{L_t^\infty L_x^{10}([0,T))} + \|g\|_{L_t^\infty L_x^2([0,T))} \\ &\lesssim 1. \end{aligned} \tag{5.36}$$

On the other hand, by (5.27), (5.7), and (4.5),

$$\begin{aligned}\|A^{\frac{3}{2}} \partial_r g\|_{L_t^\infty L_x^2([0,T))} &\lesssim \|\partial_r g\|_{L_t^\infty L_x^2([0,T))} + \|\phi_{<\frac{1}{2}} r^{-\frac{3}{4}} \partial_r g\|_{L_t^\infty L_x^2([0,T))} \\ &\lesssim 1 + \|\phi_{<\frac{1}{2}} \cdot r^{-\frac{3}{4}} \partial_r g\|_{L_t^\infty L_x^2([0,T))}. \end{aligned} \tag{5.37}$$

Plugging (5.36) and (5.37) into (5.35), we obtain

$$\|\nabla \Delta \Phi\|_{L_t^\infty L_x^2([0,T))} \lesssim 1 + \|\phi_{<\frac{1}{2}} \cdot r^{-\frac{3}{4}} \cdot \partial_r g\|_{L_t^\infty L_x^2([0,T))}. \tag{5.38}$$

By (4.26), (5.26), (5.27), and Hardy's inequality (see Appendix C), we have

$$\begin{aligned}\|\phi_{<\frac{1}{2}} \cdot r^{-\frac{3}{4}} \cdot \partial_r g\|_{L_t^\infty L_x^2([0,T))} &\lesssim 1 + \|\phi_{<\frac{1}{2}} \cdot r^{-\frac{3}{4}} \partial_r \Phi\|_{L_t^\infty L_x^2} \\ &\lesssim 1 + \Big\|\frac{1}{r} \nabla \Phi\Big\|_{L_t^\infty L_x^2} \\ &\lesssim 1. \end{aligned}$$

Substituting it into (5.38), we obtain

$$\|\nabla \Delta \Phi\|_{L_t^\infty L_x^2([0,T))} \lesssim 1.$$

Hence together with (5.34), we have

$$\begin{aligned}&\|\partial_{ttt} \Phi\|_{L_t^\infty L_x^2([0,T))} + \|\partial_{tt} \nabla \Phi\|_{L_t^\infty L_x^2([0,T))} \\ &\quad + \|\partial_t \Delta \Phi\|_{L_t^\infty L_x^2([0,T))} + \|\nabla \Delta \Phi\|_{L_t^\infty L_x^2([0,T))} \lesssim 1. \end{aligned} \tag{5.39}$$

By Sobolev embedding, we obtain

$$\|\Phi\|_{L_t^\infty L_x^\infty([0,T))} \lesssim 1.$$

Therefore, we refine (5.26), (5.27), and (5.28) to

$$\left|\Phi(r,t)\right| \lesssim \langle r\rangle^{-2}, \tag{5.40}$$

$$\left|g(r,t)\right| \lesssim \langle r\rangle^{-2}, \tag{5.41}$$

$$|A-1| \lesssim \langle r\rangle^{-4}. \tag{5.42}$$

By (4.26) (see Appendix C), we obtain

$$\|\partial_r g\|_{L_t^\infty L_x^4([0,T))} \lesssim 1. \tag{5.43}$$

By (5.9) and (5.39), we have

$$\|\partial_t g\|_{L_t^\infty L_x^4([0,T))} + \|\partial_{tt} g\|_{L_t^\infty L_x^2([0,T))} \lesssim 1. \tag{5.44}$$

Using (5.43), (5.44), and (2.7), we obtain

$$\|\Delta g\|_{L_t^\infty L_x^2([0,T))} \lesssim 1. \tag{5.45}$$

Also, by Hardy's inequality, we obtain $\|\frac{1}{r}\partial_r g\|_{L_t^\infty L_x^2([0,T))} \lesssim 1$ and hence

$$\|\partial_{rr} g\|_{L_t^\infty L_x^2([0,T))} \lesssim 1. \tag{5.46}$$

By (5.7), (5.41), (5.43), (5.45), and (5.46), it follows that

$$\|\nabla A\|_{L_t^\infty L_x^4([0,T))} + \|\Delta A\|_{L_t^\infty L_x^2([0,T))} \lesssim 1. \tag{5.47}$$

Also, it is not difficult to check that

$$\left\|\Delta\Big(\int_0^{g(r,t)} (3B^{\frac{3}{2}} + B^{-\frac{1}{2}} - B^{-\frac{3}{2}})\,dy\Big)\right\|_{L_t^\infty L_x^2([0,T))} \lesssim 1. \tag{5.48}$$

From (5.7), (5.9), and (5.39), we obtain

$$\|\partial_{tt} A\|_{L_t^\infty L_x^2([0,T))} + \|\partial_{tt} A\|_{L_t^\infty L_x^{\frac{10}{3}}([0,T))} \lesssim 1. \tag{5.49}$$

Differentiating (5.15) in time, we have

$$\begin{aligned}
\Box_5(\partial_{ttt}\Phi) = {} & \Big(\frac{3}{2}A - \frac{3}{2} + \frac{1}{2}A^{-1} - \frac{1}{2}A^{-2}\Big)\partial_{ttt}\Phi \\
& + (2A^{-3} - A^{-2} + 3)\partial_t A \cdot \partial_{tt}\Phi \\
& + \Big(A^{-3} - \frac{1}{2}A^{-2} + \frac{3}{2}\Big)\partial_{tt} A \partial_t \Phi \\
& + (-3A^{-4} + A^{-3})(\partial_t A)^2 \partial_t \Phi.
\end{aligned} \tag{5.50}$$

By Strichartz, (5.39), (5.49), and Sobolev, we obtain

$$\begin{aligned}
&\|\partial_{ttt}\Phi\|_{L_t^\infty \dot{H}_x^1([0,T))} + \|\partial_{tttt}\Phi\|_{L_t^\infty L_x^2([0,T))} \\
&\quad \lesssim \big\|\partial_{ttt}\Phi(0)\big\|_{\dot{H}_x^1(\mathbb{R}^5)} + \big\|\partial_{tttt}\Phi(0)\big\|_{L_x^2(\mathbb{R}^5)} \\
&\qquad + \big\|(A-1)\partial_{ttt}\Phi\big\|_{L_t^1 L_x^2([0,T))} + \|\partial_t\Phi\cdot\partial_{tt}\Phi\|_{L_t^1 L_x^2([0,T))} \\
&\qquad + \|\partial_{tt}A\cdot\partial_t\Phi\|_{L_t^1 L_x^2([0,T))} + \|\partial_t\Phi\|^3_{L_t^3 L_x^6([0,T])} \\
&\quad \lesssim 1. \qquad (5.51)
\end{aligned}$$

By (5.51) and (3.31), we obtain

$$\begin{aligned}
\|\partial_{tt}\Delta\Phi\|_{L_t^\infty L_x^2([0,T))} &\lesssim 1 + \big\|\partial_t\big((3A^2 + A^{-1} - A^{-2})\partial_t\Phi\big)\big\|_{L_t^\infty L_x^2([0,T))} \\
&\lesssim 1. \qquad (5.52)
\end{aligned}$$

Using (3.31) again with the estimates (5.52) and (5.48), we finally obtain

$$\|\Delta^2\Phi\|_{L_t^\infty L_x^2([0,T))} \lesssim 1.$$

In a similar way we have the estimate

$$\|\partial_t\nabla\Delta\Phi\|_{L_t^\infty L_x^2([0,T))} \lesssim 1.$$

Hence we have established

$$\begin{aligned}
&\|\partial_{ttt}\Phi\|_{L_t^\infty H_x^1([0,T))} + \|\partial_{tttt}\Phi\|_{L_t^\infty L_x^2([0,T))} \\
&\quad + \|\partial_{tt}\Phi\|_{L_t^\infty H_x^2([0,T))} + \|\Phi\|_{L_t^\infty H_x^4([0,T))} \\
&\quad + \|\partial_t\Phi\|_{L_t^\infty H_x^3([0,T))} \lesssim 1. \qquad (5.53)
\end{aligned}$$

This proves (5.1).

We are now ready to prove (5.2). By (5.41)

$$\big\|\langle x\rangle g(t)\big\|_{L_t^\infty L_x^\infty([0,T))} \lesssim 1. \qquad (5.54)$$

By (5.9), (5.53), Sobolev embedding, and radial Sobolev embedding, we have

$$\begin{aligned}
\big\|\langle x\rangle\partial_t g\big\|_{L_t^\infty L_x^\infty([0,T))} &\lesssim \big\|\langle x\rangle\partial_t\Phi\big\|_{L_t^\infty L_x^\infty([0,T))} \\
&\lesssim \|\partial_t\Phi\|_{L_t^\infty H_x^3([0,T))} \\
&\lesssim 1. \qquad (5.55)
\end{aligned}$$

In a similar way, by using (4.26), we obtain

$$\big\|\langle x\rangle\partial_r g\big\|_{L_t^\infty L_x^\infty([0,T))} \lesssim 1. \qquad (5.56)$$

Now (5.2) clearly follows from (5.54)–(5.56).

Appendix A. Some technical estimates

In this Appendix, we collect some useful technical estimates. Some of these estimates are rather pedestrian. Nevertheless, we include all the details here for the sake of completeness.

The following radial Sobolev embedding is well known and dates back to Strauss [29]. We will often use it without explicit mentioning.

LEMMA A.1 (Radial Sobolev embedding)
Suppose that $d \geq 2$ *and* $h : \mathbb{R}^d \to \mathbb{R}$ *is radial. If* $h \in C_c^\infty(\mathbb{R}^n)$*, then for some constant* $C_d > 0$ *depending only on the dimension* d*, we have*

$$r^{\frac{d-1}{2}} \left| h(r) \right| \leq C_d \|h\|_{H^1(\mathbb{R}^d)}, \quad \forall r > 0,$$

Proof
Use the identity $h(r)^2 = -2 \int_r^\infty h(\rho)\partial_\rho h(\rho)\, d\rho$, and observe that $r^{d-1} \leq \rho^{d-1}$. □

In the rest of this section, we will show that at $t = 0$, under the assumption that $(g_0, g_1) \in H^4_{\mathrm{rad}}(\mathbb{R}^5) \times H^4_{\mathrm{rad}}(\mathbb{R}^5)$, we have

$$\sum_{j=0}^{3} \left\| \partial_t^j \Phi(t=0) \right\|_{H^1(\mathbb{R}^5)} + \left\| \partial_t^4 \Phi(t=0) \right\|_{L_x^2(\mathbb{R}^5)} < \infty.$$

These were used in Sections 4 and 5.

We now give the details. We will proceed in eight steps.

Recall that

$$\Phi(r,t) = \int_0^{g(r,t)} \Big(1 + \frac{2\sin^2(ry + \phi(r))}{r^2}\Big)^{\frac{1}{2}} dy + r^{-3}\phi_{\gtrsim 1}(r),$$

where $\phi(r) = N_1\pi$ for $r \leq 1$ and $\phi(r) = 0$ for $r \geq 2$. Recall that $f(r,t) = \phi(r) + rg(r,t)$ and

$$E(t) = \frac{1}{2}\int_0^\infty \Big(1 + \frac{2\sin^2 f}{r^2}\Big)\big((\partial_t f)^2 + (\partial_r f)^2\big) r^2\, dr + \int_0^\infty \frac{\sin^2 f}{r^2}\Big(1 + \frac{\sin^2 f}{2r^2}\Big) r^2\, dr.$$

Assume that $(g, \partial_t g)_{t=0} = (g_0, g_1) \in H^4_{\mathrm{rad}}(\mathbb{R}^5) \times H^3_{\mathrm{rad}}(\mathbb{R}^5)$. By Hardy, we have $\| \frac{g_0}{r} \|_{L_x^2(\mathbb{R}^5)} < \infty$. By Sobolev embedding, we have $\|g_0\|_\infty + \|\nabla g_0\|_\infty + \|g_1\|_\infty < \infty$.

Step (1): $\|(\partial_r f)(t=0)\|_{L^2(\mathbb{R}^3)} + \|(\partial_t f)(t=0)\|_{L^2_x(\mathbb{R}^3)} < \infty$.
Since $\partial_r f|_{t=0} = \phi'(r) + r\partial_r g_0 + g_0$, we have

$$\big\|\partial_r f(t=0)\big\|_{L^2_x(\mathbb{R}^3)} \lesssim 1 + \|\partial_r g_0\|_{L^2_x(\mathbb{R}^5)} + \Big\|\frac{g_0}{r}\Big\|_{L^2_x(\mathbb{R}^5)} \lesssim 1 + \|g_0\|_{H^1(\mathbb{R}^5)} < \infty.$$

The estimate for $\|\partial_t f\|_{L^2(\mathbb{R}^3)}$ is similar and therefore omitted.

Step (2): $E(0) < \infty$.
We first consider the term $\int_0^\infty \frac{\sin^2 f}{r^2}(1 + \frac{\sin^2 f}{2r^2})r^2\,dr$. Clearly, the contribution of the part $r \sim 1$ is bounded. Therefore, we only need to consider $0 < r \le 1$ and $r \ge 2$. Then (below for simplicity of notation f, g, and $\partial_t f$ will be evaluated at $t = 0$)

$$\begin{aligned}\int_0^\infty \frac{\sin^2 f}{r^2}\Big(1 + \frac{\sin^2 f}{2r^2}\Big)r^2\,dr &\lesssim 1 + \int_0^\infty g^2 r^2\,dr + \int_0^\infty g^4 r^2\,dr \\ &\lesssim 1 + \Big\|\frac{g}{r}\Big\|^2_{L^2_x(\mathbb{R}^5)} + \Big\|\frac{g^2}{r}\Big\|^2_{L^2_x(\mathbb{R}^5)} \\ &\lesssim 1 + \|\nabla g\|^2_{L^2_x(\mathbb{R}^5)} + \big\|\nabla(g^2)\big\|^2_{L^2_x(\mathbb{R}^5)} \\ &\lesssim 1 + \|g\|^4_{H^4(\mathbb{R}^5)} < \infty. \end{aligned} \tag{A.1}$$

Next, for the first term in E, we first deal with $r \ge 1$:

$$\int_1^\infty \Big(1 + \frac{2\sin^2 f}{r^2}\Big)\big((\partial_t f)^2 + (\partial_r f)^2\big)r^2\,dr \lesssim \|\partial_t f\|^2_{L^2(\mathbb{R}^3)} + \|\partial_r f\|^2_{L^2(\mathbb{R}^3)} < \infty.$$

For the part $0 < r \le 1$, we have

$$\begin{aligned}&\int_0^1 \Big(1 + \frac{2\sin^2 f}{r^2}\Big)\big((\partial_t f)^2 + (\partial_r f)^2\big)r^2\,dr \\ &\quad\lesssim \int_0^1 (1+g^2)\big(r^2(\partial_t g)^2 + r^2(\partial_r g)^2 + g^2\big)r^2\,dr \\ &\quad\lesssim 1 + \int_0^1 g^2(1+g^2)r^2\,dr + \big(1 + \|g\|^2_\infty\big)\big(\|\partial_t g\|^2_{L^2_x(\mathbb{R}^5)} + \|\partial_r g\|^2_{L^2_x(\mathbb{R}^5)}\big) \\ &\quad< \infty.\end{aligned}$$

Step (3): $\|\nabla\Phi(t=0)\|_{L^2_x(\mathbb{R}^5)} < \infty$.
First, observe that[3] $|\Phi(r,0)| \lesssim |g(r,0)| + |g(r,0)|^2 + r^{-3}|\phi_{\gtrsim 1}(r)|$ and

$$\Big\|\frac{\Phi}{r}\Big\|_{L^2_x(\mathbb{R}^5)} \lesssim 1 + \Big\|\frac{g}{r}\Big\|_{L^2_x(\mathbb{R}^5)} + \Big\|\frac{g^2}{r}\Big\|_{L^2_x(\mathbb{R}^5)} < \infty,$$

where we have used (A.1).

[3]Recall that $\phi_{\gtrsim 1}$ can vary from line to line.

Next, by a simple change of variable $ry \to y$, we rewrite Φ as

$$\Phi(r,t) = r^{-2} \int_0^{rg(r,t)} \big(r^2 + 2\sin^2\big(y + \phi(r)\big)\big)^{\frac{1}{2}}\, dy + r^{-3}\phi_{\gtrsim 1}(r).$$

Then

$$\begin{aligned}\partial_r \Phi = &-\frac{2}{r}\Phi + r^{-3}\phi_{\gtrsim 1}(r) + r^{-2}\partial_r(rg)\big(r^2 + 2\sin^2\big(rg + \phi(r)\big)\big)^{\frac{1}{2}} \\ & r^{-2}\int_0^{rg} \big(r^2 + 2\sin^2(y+\phi)\big)^{-\frac{1}{2}} \big(r + \sin(2y + 2\phi)\phi'(r)\big)\, dy.\end{aligned}$$

If $r \geq \frac{1}{2}$, then it follows that

$$|\partial_r \Phi| \lesssim r^{-1}|\Phi| + r^{-3}\big|\phi_{\gtrsim 1}(r)\big| + |\partial_r g| + r^{-1}|g|.$$

If $0 < r < \frac{1}{2}$, then

$$\begin{aligned}|\partial_r \Phi| &\lesssim r^{-1}|\Phi| + r^{-2}\big|\partial_r(rg)\big| \cdot \big(r + r|g|\big) + r^{-1}|g| \\ &\lesssim r^{-1}|\Phi| + r^{-1}|g| + r^{-1}g^2 + |g||\partial_r g| + |\partial_r g|.\end{aligned}$$

Thus we have $\|\partial_r \Phi\|_{L_x^2(\mathbb{R}^5)} < \infty$.

Step (4): $\|\partial_t \Phi(t=0)\|_{H^1(\mathbb{R}^5)} < \infty$.
First, observe that

$$\partial_t \Phi = \frac{1}{r}\partial_t f \Big(1 + \frac{2\sin^2 f}{r^2}\Big)^{\frac{1}{2}}. \tag{A.2}$$

Thus

$$\partial_t \Phi|_{t=0} = g_1 \cdot \Big(1 + \frac{2\sin^2(rg_0 + \phi(r))}{r^2}\Big)^{\frac{1}{2}}. \tag{A.3}$$

Since $g_1 \in H^3(\mathbb{R}^5)$ and $g_0 \in H^4(\mathbb{R}^5)$, we clearly have $\| \frac{2\sin^2(rg_0+\phi(r))}{r^2} \|_\infty \lesssim 1$, and

$$\big\|\partial_t \Phi(t=0)\big\|_{L_x^2(\mathbb{R}^5)} < \infty.$$

To bound the $\dot{H}^1$-norm, we first consider the regime $r \geq \frac{1}{2}$. Denote

$$B_0 = 1 + \frac{2\sin^2(rg_0 + \phi(r))}{r^2}.$$

Clearly for $r \geq \frac{1}{2}$, using $\|g_0\|_\infty + \|\nabla g_0\|_\infty \lesssim 1$, we have

$$|\partial_r B_0| \lesssim 1 + r^{-2}\big(\big|\partial_r(rg_0)\big| + \big|\phi'(r)\big|\big) \lesssim 1. \tag{A.4}$$

Next for $r < \frac{1}{2}$, we have

$$B_0 = 1 + \frac{2\sin^2(rg_0)}{(rg_0)^2} g_0^2 = 1 + \tilde{G}(rg_0) g_0^2,$$

where $\tilde{G}(z) = 2z^{-2}\sin^2(z)$ has bounded derivatives of all orders. It follows that

$$\begin{aligned} |\partial_r B_0| &\lesssim \left|\partial_r (rg_0)\right| g_0^2 + |\partial_r g_0| \cdot |g_0| \\ &\lesssim r|\partial_r g_0| g_0^2 + |g_0|^3 + |\partial_r g_0| \cdot |g_0| \lesssim 1, \end{aligned} \tag{A.5}$$

where we again used the fact that $\|g_0\|_\infty + \|\nabla g_0\|_\infty \lesssim 1$. It follows that

$$\left\|\partial_r(\partial_t \Phi)\right\|_{L_x^2(\mathbb{R}^5)} \lesssim \|\partial_r g_1\|_{L_x^2(\mathbb{R}^5)} + \|g_1 B_0^{-\frac{1}{2}} \partial_r B_0\|_{L_x^2(\mathbb{R}^5)} \lesssim 1.$$

Step (5):

Denote $A = 1 + \frac{2\sin^2 \tilde{f}_0}{r^2}$, where $\tilde{f}_0 = \phi(r) + rg_0$. Then

$$\|A\|_\infty + \|\partial_r A\|_\infty \lesssim 1, \qquad \|\Delta_5 A\|_{L_x^2(\mathbb{R}^5)} + \|\Delta_5 A\|_{L_x^{10}(\mathbb{R}^5)} \lesssim 1. \tag{A.6}$$

For $r \geq \frac{1}{2}$, we use (A.4). For $r < \frac{1}{2}$, we use (A.5). Thus the first inequality is obvious. The second inequality follows from a similar computation. One should note that by Sobolev embedding,

$$\|\Delta_5 g_0\|_{L_x^{10}(\mathbb{R}^5)} \lesssim \|g_0\|_{H^4(\mathbb{R}^5)} < \infty.$$

Step (6): $\|(\partial_{tt}\Phi)(t=0)\|_{H^1(\mathbb{R}^5)} < \infty$.
First, observe that

$$\partial_{tt}\Phi = \frac{1}{r}\partial_{tt} f \Big(1 + \frac{2\sin^2 f}{r^2}\Big)^{\frac{1}{2}} + \frac{1}{r^3}(\partial_t f)^2 \Big(1 + \frac{2\sin^2 f}{r^2}\Big)^{-\frac{1}{2}} \sin(2f). \tag{A.7}$$

We first consider the second term on the right-hand side. Since $\partial_t f|_{t=0} = rg_1$, we have

$$\begin{aligned} &\frac{1}{r^3}(\partial_t f)^2 \Big(1 + \frac{2\sin^2 f}{r^2}\Big)^{-\frac{1}{2}} \sin(2f)\Big|_{t=0} \\ &\quad = r^{-1} g_1^2 \Big(1 + \frac{2\sin^2(\phi + rg_0)}{r^2}\Big)^{-\frac{1}{2}} \sin(2\phi + 2rg_0). \end{aligned}$$

For any $r > 0$, it is not difficult to check that (note below that for $r \leq \frac{1}{2}$, one can write $r^{-1}(1 + \frac{2\sin^2(\phi + rg_0)}{r^2})^{-\frac{1}{2}} \sin(2\phi + 2rg_0) = \tilde{F}(rg_0) g_0$, where $\tilde{F}$ has bounded derivatives)

$$\Big|\Big(r^{-1}\Big(1+\frac{2\sin^2(\phi+rg_0)}{r^2}\Big)^{-\frac{1}{2}}\sin(2\phi+2rg_0)\Big)\Big|\lesssim 1+\big|g_0(r)\big|,$$

$$\Big|\partial_r\Big(r^{-1}\Big(1+\frac{2\sin^2(\phi+rg_0)}{r^2}\Big)^{-\frac{1}{2}}\sin(2\phi+2rg_0)\Big)\Big|$$
$$\lesssim 1+\big|g_0(r)\big|+|g_0|\big|\partial_r\big(rg_0(r)\big)\big|+\big|\partial_r g_0(r)\big|.$$

It follows easily that

$$\Big\|g_1^2r^{-1}\Big(1+\frac{2\sin^2(\phi+rg_0)}{r^2}\Big)^{-\frac{1}{2}}\sin(2\phi+2rg_0)\Big\|_{H^1(\mathbb{R}^5)}\lesssim 1.$$

It remains for us to check the first term on the right-hand side in (A.7). Note that by (A.6) the factor $(1+\frac{2\sin^2 f}{r^2})^{\frac{1}{2}}$ is harmless for us when estimating the H^1-norm. Therefore, we only need to focus on the estimate of $\|\frac{1}{r}\partial_{tt}f\|_{H^1(\mathbb{R}^5)}$. Observe that

$$\frac{1}{r}\partial_{tt}f=\partial_{tt}g=\Box_5 g+\Delta_5 g. \tag{A.8}$$

Clearly, $\|\Delta_5 g_0\|_{H^1(\mathbb{R}^5)}\lesssim 1$. For $\Box_5 g$, we use (2.5):

$$\begin{aligned}\Box_5 g=\ &\frac{\phi_{<1}}{1+\tilde F_0(rg)g^2}\big(\tilde F_1(rg)g^3+\tilde F_2(rg)g^5\\ &-\tilde F_3(rg)\cdot g\cdot\big((\partial_t g)^2-(\partial_r g)^2\big)\\ &+\tilde F_4(rg)\cdot g^4\cdot r\partial_r g\big)\\ &+\phi_{>1}\cdot\frac{2}{r^2}g+\frac{1}{r}\Delta_3\phi\\ &+\frac{1}{r}\phi_{>1}\cdot N\big(r,\phi+rg,(\phi+rg)'\big),\end{aligned} \tag{A.9}$$

where $\tilde F_i(x)=F_i(x^2)$, and F_i has bounded derivatives of all orders. Clearly by using radial Sobolev embedding and $\|g_0\|_{H^4}\lesssim 1$, we have

$$\big\|\partial_r\big(F_i(r^2g_0^2)\big)\big\|_\infty\lesssim\big\|\partial_r(r^2g_0^2)\big\|_\infty\lesssim 1.$$

By a tedious calculation, it is not difficult to check then that the right-hand side of (A.9) all have bounded $H^1(\mathbb{R}^5)$-norm. Thus $\|\Box_5 g_0\|_{H^1(\mathbb{R}^5)}\lesssim 1$ and, consequently, $\|(\partial_{tt}\Phi)(t=0)\|_{H^1(\mathbb{R}^5)}<\infty$.

Step (7): $\|(\partial_{ttt}\Phi)(t=0)\|_{H^1(\mathbb{R}^5)}<\infty$.
Here we use (3.31):

$$\begin{aligned}\partial_{ttt}\Phi|_{t=0}&=\Delta_5\partial_t\Phi|_{t=0}-\frac{3}{2}\partial_t\Phi|_{t=0}+\frac{1}{2}\partial_t g(3B^{\frac{3}{2}}|_{t=0}+B^{-\frac{1}{2}}|_{t=0}-B^{-\frac{3}{2}}|_{t=0}),\\ &=\Delta_5\partial_t\Phi|_{t=0}-\frac{3}{2}\partial_t\Phi|_{t=0}+\frac{1}{2}g_1(3A^{\frac{3}{2}}+A^{-\frac{1}{2}}-A^{-\frac{3}{2}}),\end{aligned}$$

where A is the same as in (A.6). By (A.6), the last term above clearly is bounded in $H^1(\mathbb{R}^5)$. Also, in Step (4) we have shown that $\|\partial_t \Phi|_{t=0}\|_{H^1(\mathbb{R}^5)} < \infty$. By (A.3), we have

$$\Delta_5(\partial_t \Phi|_{t=0}) = \Delta_5(g_1 A^{\frac{1}{2}}) = \Delta_5(g_1)A^{\frac{1}{2}} + 2\partial_r g_1 \cdot \partial_r(A^{\frac{1}{2}}) + g_1 \Delta_5(A^{\frac{1}{2}}).$$

Clearly by (A.6), it follows that $\|\Delta_5(\partial_t \Phi|_{t=0})\|_{L^2_x(\mathbb{R}^5)} \lesssim 1$.

Step (8): $\|(\partial_{tttt}\Phi)(t=0)\|_{L^2_x(\mathbb{R}^5)} < \infty$.
Here we again use (3.31):

$$\begin{aligned}\partial_{tttt}\Phi|_{t=0} = \Delta_5 \partial_{tt}\Phi|_{t=0} - \frac{3}{2}\partial_{tt}\Phi|_{t=0} + \frac{1}{2}\partial_{tt} g|_{t=0}(3A^{\frac{3}{2}} + A^{-\frac{1}{2}} - A^{-\frac{3}{2}}) \\ + \frac{1}{2} g_1\Big(\frac{9}{2}A^{\frac{1}{2}} - \frac{1}{2}A^{-\frac{3}{2}} + \frac{3}{2}A^{-\frac{5}{2}}\Big) \cdot \frac{\sin(2rg_0 + 2\phi)}{r^2} \cdot (2rg_1).\end{aligned}$$

By the calculation in Step (6), we have $\|\partial_{tt} g|_{t=0}\|_{L^2_x(\mathbb{R}^5)} \lesssim 1$. The last three terms above are clearly $L^2_x(\mathbb{R}^5)$-bounded.

We now only to need to estimate $\|\Delta_5 \partial_{tt}\Phi|_{t=0}\|_{L^2_x(\mathbb{R}^5)}$. By (A.2), we have

$$\partial_{tt}\Phi|_{t=0} = \frac{1}{r}\partial_{tt} f|_{t=0} A^{\frac{1}{2}} + r^{-1} g_1^2 A^{-\frac{1}{2}} \sin 2\tilde{f}_0,$$

where $\tilde{f}_0 = \phi + rg_0$. Clearly by (A.6) and $\|g_0\|_{H^4} + \|g_1\|_{H^3} \lesssim 1$, we have

$$\Big\|\Delta_5\Big(g_1^2 A^{-\frac{1}{2}} \frac{\sin 2\tilde{f}_0}{r}\Big)\Big\|_{L^2_x(\mathbb{R}^5)} \lesssim 1.$$

By (A.8), we have

$$\frac{1}{r}\partial_{tt} f|_{t=0} A^{\frac{1}{2}} = A^{\frac{1}{2}}(\Box_5 g + \Delta_5 g)|_{t=0}.$$

By (A.6), we have

$$\big\|\Delta_5(A^{\frac{1}{2}}\Delta_5 g_0)\big\|_{L^2_x(\mathbb{R}^5)} \lesssim \|g_0\|_{H^4} + \big\|\Delta_5(A^{\frac{1}{2}})\big\|_{L^4_x(\mathbb{R}^5)} \|\Delta_5 g_0\|_{L^4_x(\mathbb{R}^5)} < \infty.$$

Similarly, we have (below we used the simple inequality $\|h\|_{L^4_x(\mathbb{R}^5)} \lesssim \|h\|_{L^2_x(\mathbb{R}^5)} + \|\Delta_5 h\|_{L^2_x(\mathbb{R}^5)}$)

$$\begin{aligned}\big\|\Delta_5(A^{\frac{1}{2}}\Box_5 g_0)\big\|_{L^2_x(\mathbb{R}^5)} &\lesssim \|\Box_5 g_0\|_{L^4_x(\mathbb{R}^5)} + \|\nabla\Box_5 g_0\|_{L^2_x(\mathbb{R}^5)} + \|\Delta_5\Box_5 g_0\|_{L^2_x(\mathbb{R}^5)} \\ &\lesssim \|\Box_5 g_0\|_{L^2_x(\mathbb{R}^5)} + \|\Delta_5\Box_5 g_0\|_{L^2_x(\mathbb{R}^5)}.\end{aligned}$$

In Step (6) (see the estimates near (A.8)), we have estimated $\|\Box_5 g_0\|_{H^1(\mathbb{R}^5)}$. Thus we only need to deal with the term $\|\Delta_5\Box_5 g_0\|_{L^2_x(\mathbb{R}^5)}$. By using (A.9) and a tedious computation, it is not difficult to check that the right-hand side of (A.9) has finite $H^2(\mathbb{R}^5)$-norm. This then completes the estimate of $\|\partial_{tttt}\Phi|_{t=0}\|_{L^2_x(\mathbb{R}^5)}$.

Appendix B. The continuity argument

In this appendix we give more details of the continuity argument in the derivation of (5.13). Recall the main equation

$$\Box_5(\partial_t \Phi) = \Big(\frac{3}{2} + \frac{1}{2}A^{-2}\Big)(A-1)\partial_t \Phi, \tag{B.1}$$

and

$$|A-1| \lesssim \min\{r^{-\frac{3}{2}}, r^{-4}\}. \tag{B.2}$$

Now denote $u = \partial_t \Phi$. Our goal is to show that on the interval $[0, T_1]$ ($T_1 < T$ can be arbitrarily close to T), we have

$$\|u\|_{L_t^3 L_x^3([0,T_1])} + \|u\|_{C_t^0 \dot{H}_x^{\frac{1}{2}}([0,T_1])} \lesssim 1, \tag{B.3}$$

where the implied constant is independent of T_1.

To this end we decompose $[0, T_1] = \bigcup_{i=0}^{N_0} [t_i, t_{i+1}]$, where $t_0 = 0$, $t_{N_0+1} = T_1$, and $t_{i+1} - t_i$ will be taken sufficiently small. The needed smallness will become clear in the argument below.

First observe that by using the estimates in Section 4, we have

$$\|P_{<1}u\|_{L_t^3 L_x^3([0,T_1])} \lesssim \|u\|_{L_t^\infty L_x^2([0,T_1])} = \|\partial_t \Phi\|_{L_t^\infty L_x^2([0,T_1])} \lesssim 1. \tag{B.4}$$

By Strichartz (Lemma 5.2) and (B.2), we have on each $[t_i, t_{i+1}]$,

$$\begin{aligned} &\|P_{\geq 1}u\|_{L_t^3 L_x^3([t_i,t_{i+1}])} + \|P_{\geq 1}u\|_{C_t^0 \dot{H}_x^{\frac{1}{2}}([t_i,t_{i+1}])} \\ &\qquad + \|P_{\geq 1}\partial_t u\|_{C_t^0 \dot{H}_x^{-\frac{1}{2}}([t_i,t_{i+1}])} \qquad\qquad (\text{B.5}) \\ &\quad \lesssim \big\|P_{\geq 1}u(t_i)\big\|_{\dot{H}_x^{\frac{1}{2}}} + \big\|P_{\geq 1}\partial_t u(t_i)\big\|_{\dot{H}_x^{-\frac{1}{2}}} + \big\|(A-1)u\big\|_{L_t^{\frac{3}{2}} L_x^{\frac{3}{2}}([t_i,t_{i+1}])} \\ &\quad \lesssim \big\|P_{\geq 1}u(t_i)\big\|_{\dot{H}_x^{\frac{1}{2}}} + \big\|P_{\geq 1}\partial_t u(t_i)\big\|_{\dot{H}_x^{-\frac{1}{2}}} \\ &\qquad + \big\|(A-1)\big\|_{L_t^3 L_x^3([t_i,t_{i+1}])} \cdot \|u\|_{L_t^3 L_x^3([t_i,t_{i+1}])} \\ &\quad \lesssim \big\|P_{\geq 1}u(t_i)\big\|_{\dot{H}_x^{\frac{1}{2}}} + \big\|P_{\geq 1}\partial_t u(t_i)\big\|_{\dot{H}_x^{-\frac{1}{2}}} \\ &\qquad + (t_{i+1} - t_i)^{\frac{1}{3}} \|u\|_{L_t^3 L_x^3([t_i,t_{i+1}])}. \qquad\qquad (\text{B.6}) \end{aligned}$$

Clearly if $(t_{i+1} - t_i)$ is sufficiently small, then we have (using (B.4))

$$\begin{aligned} &\|P_{\geq 1}u\|_{L_t^3 L_x^3([t_i,t_{i+1}])} + \|P_{\geq 1}u\|_{C_t^0 \dot{H}_x^{\frac{1}{2}}([t_i,t_{i+1}])} + \|P_{\geq 1}\partial_t u\|_{C_t^0 \dot{H}_x^{-\frac{1}{2}}([t_i,t_{i+1}])} \\ &\quad \lesssim \big\|P_{\geq 1}u(t_i)\big\|_{\dot{H}_x^{\frac{1}{2}}} + \big\|P_{\geq 1}\partial_t u(t_i)\big\|_{\dot{H}_x^{-\frac{1}{2}}} + 1. \end{aligned}$$

Clearly by using the above estimate and iterating from $i = 0$ to $i = N_0$ (for the base step $i = 0$, one can use the estimates in Appendix A to obtain $\|P_{\geq 1}u(t = 0)\|_{\dot{H}_x^{\frac{1}{2}}} + \|P_{\geq 1}\partial_t u(t = 0)\|_{\dot{H}_x^{-\frac{1}{2}}} \lesssim 1$), we can obtain the estimate (B.3).

Appendix C. Additional estimates

This appendix is for the estimates in (5.38) and (5.43).

For $r \leq \frac{1}{2}$, we have $f = N_1\pi + rg$. By using (4.26), we have

$$\begin{aligned}\partial_r\Phi(r,t) &+ \frac{2}{r}\Phi(r,t)\\ &= \Big(1 + \frac{2\sin^2 f}{r^2}\Big)^{\frac{1}{2}} \cdot \Big(\partial_r g + \frac{g}{r}\Big)\\ &\quad + \frac{1}{r}\int_0^{g(r,t)} \Big(1 + \frac{2\sin^2(ry)}{r^2}\Big)^{-\frac{1}{2}}\, dy. \end{aligned} \tag{C.1}$$

By (5.26) and (5.27), we have for $r \leq \frac{1}{2}$,

$$\big|\Phi(r)\big| \lesssim r^{-\frac{1}{2}}, \qquad \big|g(r)\big| \lesssim r^{-\frac{1}{4}}. \tag{C.2}$$

Thus for $r \leq \frac{1}{2}$, plugging (C.2) into (C.1), we obtain

$$|\partial_r g| \lesssim |\partial_r\Phi| + r^{-\frac{3}{2}}.$$

This estimate is used in (5.38).

Next we turn to (5.43). By (5.40) and (5.41), we have

$$\big|\Phi(r)\big| \lesssim \langle r\rangle^{-2}, \qquad \big|g(r)\big| \lesssim \langle r\rangle^{-2}.$$

By (C.1), we then have

$$|\partial_r g| \lesssim |\partial_r\Phi| + r^{-1}\langle r\rangle^{-2}.$$

Thus, $\|\partial_r g\|_{L_t^\infty L_x^4} \lesssim 1$.

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