Commun. Math. Phys. 372, 261–280 (2019)
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-019-03373-z

Communications in
Mathematical Physics

Galilean Boost and Non-uniform Continuity for Incompressible Euler

Jean Bourgain[1], **Dong Li**[2]

[1] School of Mathematics, Institute for Advanced Study, Princeton, NJ 08544, USA.
E-mail: bourgain@math.ias.edu
[2] Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong. E-mail: madli@ust.hk

Received: 9 August 2018 / Accepted: 18 December 2018
Published online: 23 February 2019

Abstract: By using an idea of localized Galilean boost, we show that the data-to-solution map for incompressible Euler equations is not uniformly continuous in $H^s(\mathbb{R}^d)$, $s \geq 0$. This settles the end-point case ($s = 0$) left open in Himonas–Misiołek (Commun Math Phys 296(1):285–301, 2010) and gives a unified treatment for all H^s. We also show the solution map is nowhere uniformly continuous.

1. Introduction

We consider the initial value problem for the incompressible Euler equation posed on a d-dimensional domain Ω:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = 0, & (t, x) \in \mathbb{R} \times \Omega, \\ \nabla \cdot u = 0, \\ u(0, x) = u_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

In [1], Himonas and Misiołek showed that the solution operator for (1.1) is not uniformly continuous in $H^s(\Omega)$ for any $s \in \mathbb{R}$ if Ω is the flat torus $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$. For the whole space case $\mathbb{R}^d$, they showed that the same conclusion holds for $H^s(\mathbb{R}^d)$ with $s > 0$. We refer to the introduction in [1] and the references therein for related literatures. The purpose of this note is to settle the end-point case $s = 0$, i.e. $L^2(\mathbb{R}^d)$ case. Moreover, as we shall explain below shortly, we will give a unifying treatment for the case $s > 0$ as well. The main result reads as follows.

Theorem 1.1. *Let $d \geq 2$, $s \geq 0$ and $T > 0$. Then the solution map $u_0 \to u$ for (1.1) is not uniformly continuous from $H^s(\mathbb{R}^d)$ into $C([0, T], H^s(\mathbb{R}^d))$. More precisely, for any $R_0 > 0$, there exist two sequences of solutions $u_n^1 = u_n^1(t, x)$, $u_n^2 = u_n^2(t, x)$, $n = 1, 2, 3, \ldots$ such that the following hold:*

- *For each finite n, u_n^1, u_n^2 has lifespan at least $[0, T]$, and $u_n^1, u_n^2 \in C_t^0 H^m$, for all $m \geq 0$.*

- *At time* $t = 0$, $\|u_n^1(0,\cdot)\|_{H^s} = R_0$, $\|u_n^2(0,\cdot)\|_{H^s} = R_0$, *and*

$$\lim_{n\to\infty} \|u_n^1 - u_n^2\|_{H^s} = 0.$$

- *At time* $t = T$,

$$\liminf_{n\to\infty} \|u_n^1(T,\cdot) - u_n^2(T,\cdot)\|_{\dot H^s} \ge c_0 R_0 > 0,$$

where c_0 *is a constant depending only on* (s,d).

Remark 1.2. If we denote the solution map $u_0 \to u(T,\cdot)$ as $S(T,u_0)$, then clearly for any $\lambda > 0$, $S(T,\lambda u_0) = \lambda S(\lambda T, u_0)$. Theorem 1.1 amounts to finding initial data u_0, $\tilde u_0$, with $\|u_0\|_{H^s} \sim R_0$, $\|\tilde u_0\|_{H^s} \sim R_0$, $\|u_0 - \tilde u_0\|_{H^s} \ll 1$, such that

$$\frac{\|S(T,u_0) - S(T,\tilde u_0)\|_{\dot H^s}}{\|u_0\|_{H^s} + \|\tilde u_0\|_{H^s}} \gtrsim 1.$$

By using the scaling symmetry of the solution operator, it suffices to prove the statement for $R_0 = 1$ and any $T > 0$. Alternatively it suffices to prove the statement for $T = 1$ and any $R_0 > 0$.

Remark. As a matter of fact, it follows from our proof that for any $0 < t \le \min\{T, 1\}$,

$$\liminf_{n\to\infty} \frac{\|u_n^1(t,\cdot) - u_n^2(t,\cdot)\|_{\dot H^s}}{\|u_n^1(0,\cdot)\|_{H^s} + \|u_n^2(0,\cdot)\|_{H^s}} \gtrsim t > 0.$$

In [2], Inci gave a strengthening of [1] and proved the solution map is nowhere uniformly continuous in H^s, $s > d/2 + 1$ topology. Our next result pushes this all the way down to $s \ge 0$.

Theorem 1.3. *Let* $d \ge 2$, $s \ge 0$ *and* $T > 0$. *Then the solution map* $u_0 \to u$ *for* (1.1) *is nowhere uniformly continuous from* $H^s(\mathbb{R}^d)$ *to* $C([0,T], H^s(\mathbb{R}^d))$. *More precisely, let* $u^{(g)} = u^{(g)}(t,x)$ *be a given classical solution to* (1.1) *in the class* $C_t^0([0,T], H^{s_0}(\mathbb{R}^d) \cap H^s(\mathbb{R}^d))$ *for some* $s_0 > \frac{d}{2} + 1$. *For any* $0 < \eta \le 1$, *one can find two sequences of (nearby* $u^{(g)}$*) solutions* $u_n^1 = u_n^1(t,x)$, $u_n^2 = u_n^2(t,x)$, $n = 1,2,3,\ldots$ *with the following properties:*

- *For each finite* n, u_n^1, u_n^2 *has lifespan at least* $[0,T]$, *and* $u_n^1 - u^{(g)}$, $u_n^2 - u^{(g)} \in C_t^0 H^m$, *for all* $m \ge 0$.
- *At time* $t = 0$, $\|u_n^1(0,\cdot) - u^{(g)}(0,\cdot)\|_{H^s} \le \eta$, $\|u_n^2(0,\cdot) - u^{(g)}(0,\cdot)\|_{H^s} \le \eta$, *and*

$$\lim_{n\to\infty} \|u_n^1(0,\cdot) - u_n^2(0,\cdot)\|_{H^s} = 0.$$

- *At time* $t = T$,

$$\liminf_{n\to\infty} \|u_n^1(T,\cdot) - u_n^2(T,\cdot)\|_{\dot H^s} \ge c_0 \cdot \eta > 0,$$

where c_0 *is a constant depending only on* (s,d).

Remark. A classical solution $u \in C_t^0 H^{s_0}$ for $s_0 > \frac{d}{2} + 1$ solving (1.1) can be understood as a solution to the integral equation:

$$u(t) = u_0 + \int_0^t \mathbb{P}((u \cdot \nabla u)(\tau)) d\tau,$$

where $\mathbb{P}$ is the usual Leray projection operator. Note that the kinematic transport term $(u \cdot \nabla)u$ only has regularity $C_t^0 H^{s_0 - 1}$ which formally lose one derivative and can cause difficulty in the construction of local solutions (if without using mollification or using semigroup techniques à la Kato). This artifact is only a matter of representation and can be remedied by exploiting a Lagrangian formulation.[1] More precisely one starts with the representation

$$-\nabla p = -\nabla(-\Delta)^{-1} \nabla \cdot ((u \cdot \nabla) u) =: \mathcal{F}(u, u),$$

where (by a div-curl type argument) the bilinear map $\mathcal{F}$ is a bounded operator from $H^{s_0} \times H^{s_0}$ (with divergence-free conditions) to H^{s_0}. Now introduce the characteristic lines $\phi_t(\alpha)$ solving

$$\begin{cases} \partial_t \phi_t(\alpha) = u(t, \phi_t(\alpha)) =: v_t(\alpha), \\ \phi_0(\alpha) = \alpha \in \mathbb{R}^d. \end{cases}$$

Then by writing $\phi_t(\alpha) = \alpha + g_t(\alpha)$, one gets

$$\begin{cases} \partial_t g_t = v_t, \\ \partial_t v_t = [\mathcal{F}(v_t \circ \phi_t^{-1}, v_t \circ \phi_t^{-1})] \circ \phi_t, \\ g_0 = 0, \quad v_0(\alpha) = u_0(\alpha), \ \alpha \in \mathbb{R}^d. \end{cases} \tag{1.2}$$

The above can then be easily solved as a Banach space (H^{s_0}) valued ODE locally in time. In this way the solution map for (1.1) can be realized as the map $u_0 \to v_t \circ \phi_t^{-1}$.

Remark. For dimension $d = 2$ there is no issue with lifespan and one can deal with any initial data $u_0^{(g)} \in H^{s_0} \cap H^s$. However for dimension $d \geq 3$ we must work with a given classical solution having known lifespan $[0, T]$. One subtle issue in this case is to guarantee that the perturbed solution sequences still have the same lifespan so that the solution map is well-defined. We get around this issue by constructing perturbations that have very little interaction with the given solution $u^{(g)}$. A crucial fact exploited here is the finite speed propagation.

Our proof of Theorem 1.1 is deeply inspired from the Himonas–Misiołek construction and can be viewed as a slight improvement of their method. In the following discussion we shall assume for simplicity $T = 1$ and work with the unit time interval. The strategy undertaken in [1] is different for the torus case and the whole space case. For the torus case, the elegant idea is to use the explicit solutions of the form

$$u^{\omega,n}(t, x_1, x_2) = (\omega n^{-1} + n^{-s} \cos(n x_2 - \omega t), \omega n^{-1} + n^{-s} \cos(n x_1 - \omega t)), \quad \omega = \pm 1.$$

[1] In some deeper sense this is equivalent to Kato's idea of using $\mathbb{P}((u \cdot \nabla) \cdot)$ as a quasi-linear opeator.

One can then check that for $n \to \infty$, $\|u^{1,n}(0,\cdot) - u^{-1,n}(0,\cdot)\|_{H^s} \lesssim 1/n$, whereas $\|u^{1,n}(t,\cdot) - u^{-1,n}(t,\cdot)\|_{H^s} \gtrsim \sin t$. On the other hand, for the whole space $\mathbb{R}^2$ case, their idea is to use approximate solutions of the form

$$u^{\omega,\lambda}(t,x) = u^l(t,x) + u^h(t,x),$$

where u^h is a high frequency piece having frequency localized to $|\xi| \sim \lambda$, and the low frequency term $u^l(t,x)$ is a solution to the incompressible Euler equation with almost constant initial data. To complete the construction one has to show the above approximate solutions are close (in H^s norm) to the true solutions. The technical condition $s > 0$ comes from controlling the residual error terms in this part of analysis (see Section 4 in [1]). The point of departure of our analysis is to revisit the periodic case and try to understand in a deeper way why the explicit solution works. In some sense we try to build a universal strategy for both the periodic case and the whole space case. As it turns out, what is crucial in the argument is the Galilean invariance of Euler equations: namely if $u(t,x)$ is a solution to the incompressible Euler equation, then for any constant vector $c \in \mathbb{R}^d$,

$$u_c(t,x) = c + u(t, x - ct)$$

is also a solution. In the periodic $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ case, one can begin with the explicit steady state solution (below $\nabla^\perp = (-\partial_2, \partial_1)$)

$$u^{0,n}(x) = \frac{1}{n^s}\nabla^\perp(\sin(nx_1)) = \frac{1}{n^s}(0, \cos nx_1).$$

Applying Galilean boost with $c = (1/n, 0)$, we then obtain a second solution

$$u^{c,n}(t,x) = (\frac{1}{n}, \frac{1}{n^s}\cos(nx_1 - t)).$$

Due to the phase shift $O(t)$, it follows that as $n \to \infty$

$$\|u^{0,n} - u^{c,n}(t,\cdot)\|_{H^s} \gtrsim \sin t, \quad \forall\, 0 < t \le 1.$$

Now we turn to the whole space case $\mathbb{R}^2$ taking $s = 0$ (i.e. $L^2(\mathbb{R}^2)$) as the main case. Our idea is to mimic the periodic case above by introducing suitable spatial cut-offs. We first build an approximate steady state solution of the form

$$u^{(0)}(x) = \nabla^\perp\Phi^{(0)} = \nabla^\perp(\frac{1}{\mu\lambda}\phi_0(\frac{x}{\mu})\sin\lambda x_1),$$

where $\phi_0 \in C_c^\infty(\mathbb{R}^2)$ is a bump function, $\mu = \lambda/\sqrt{\log\lambda}$ and $\lambda \gg 1$. Note that in the main order,

$$u^{(0)}(x) \sim \frac{1}{\mu}\phi_0(\frac{x}{\mu})(0, \cos\lambda x_1).$$

We then apply *localized* Galilean boost and obtain a second approximate solution of the form

$$u^a(t,x) = u^{(0)}(x_1 - c_1 t, x_2) + \underbrace{\nabla^\perp((-c_1 x_2)\psi(\frac{x}{\mu}))}_{=:\beta(x)},$$

where $c_1 = 1/\lambda$, and ψ is a bump function having support slightly larger than ϕ_0. The function β is chosen in a way such that $\beta(x) \equiv (c_1, 0)$ on the support of u^a on the unit time interval [0, 1]. In $O(1)$ time interval, the function β then induces an $O(1)$ phase shift in the x_1 variable just as in the periodic case. Thanks to this phase shift, the approximate solution pair $(u^{(0)}, u^a)$ then has the desired property that the L^2 distance inflates from $O(\lambda/\mu) = O(1/\sqrt{\log \lambda})$ at time $t = 0$ to $O(1)$ at time $t = 1$. Let u^1 and u^2 be exact solutions corresponding to the initial data $u^{(0)}$ and $u^a(0, \cdot)$ respectively. To complete the construction, we then use a simple L^2 estimate to show that

$$\max_{0 \le t \le 1} \|u^1 - u^{(0)}\|_2 \lesssim e^{\|\nabla u^{(0)}\|_\infty} \cdot O(\lambda^{-2}) \lesssim e^{\mathrm{const} \cdot \sqrt{\log \lambda}} \cdot \lambda^{-2} \lesssim \lambda^{-1},$$

and similarly

$$\max_{0 \le t \le 1} \|u^2 - u^a\|_2 \lesssim \lambda^{-1}.$$

These estimates then show that the pair (u^1, u^2) inherits the desired properties from $(u^{(0)}, u^a)$. The argument above can be adapted for the H^s, $s > 0$ case by choosing suitable μ, λ together with H^s stability estimates (see Section 3 for details). Before we conclude this introduction, we should mention that this simple method seems quite robust and can likely be applied to a large class of equations admitting Galilean invariance or other symmetries such as Lorentz symmetry. In its most general form, the method is as follows:

Step 1. Find an explicit approximate solution (say frequency localized to $|\xi| \sim \lambda \gg 1$) which is close to the true exact solution.

Step 2. Apply suitably chosen symmetry group (such as Galilean boost with appropriate cut-off if needed) to produce another explicit approximate solution. This second solution should also remain close to a true exact solution.

Step 3. Produce norm inflation for the two approximate solutions and then show it for the nearby exact solutions.

Needless to say the delicacy of the matter often lies in the choice of parameters (i.e. construction of a good approximate solution) and the associated stability estimates. Nevertheless it would be interesting to find more examples of this kind and generalize the method.[2] A closely related deeper problem is to investigate the correct functional set up for the solution operator for problems having mild or singular drifts. For such problems it might be useful to pursue both Lagrangian and Eulerian point of views, and find a possible connection between weak and strong solutions (cf. a recent nice survey by Wiedemann [6]). In any case this line of research certainly merits further investigations.

Remark 1.4. It is worthwhile pointing out that analogous non-uniform continuity results (of the solution operator) is already exhibited in the 2D inviscid Burgers equation $\partial_t u + (u \cdot \nabla) u = 0$. This fact is implicitly contained in the proof for the incompressible Euler equation as we now explain. For simplicity consider the periodic $\mathbb{T}^2$ case. Take

$$u^{0,n}(x) = \frac{1}{n^s} \nabla^\perp (\sin(nx_1) = \frac{1}{n^s}(0, \cos nx_1).$$

One notes that $u^{0,n}$ actually satisfies:

$$(u^{0,n} \cdot \nabla) u^{0,n} = 0.$$

[2] An interesting question is to compare this with the usual high and low splitting method.

In yet other words $u^{0,n}$ solves the 2D Euler equation with pressure $p = 0$, i.e. the Burgers equation. Applying the Galilean boost with $c = (1/n, 0)$, we then obtain a second solution

$$u^{c,n}(t,x) = (\frac{1}{n}, \frac{1}{n^s}\cos(nx_1 - t))$$

which also solves 2D Burgers equation. It follows that as $n \to \infty$

$$\|u^{0,n} - u^{c,n}(t,\cdot)\|_{H^s} \gtrsim \sin t, \quad \forall\, 0 < t \le 1.$$

Similarly Theorem 1.1 also holds for the 2D Burgers case.

Remark 1.5. We do not address in this work the closely related issue of non-continuity extension in L^2 for the solution operator. Yet another closely related and well-known problem is the (lack of) wellposedness in L^p (for vorticity) in the spirit of classical Yudovich theory. In this connection it is worthwhile noting that recently Vishik ([4,5]) has showed non-uniqueness of weak solutions (with radially symmetric locally integrable external forcing) and studied the associated stability questions. These problems (and their connections) certainly merit further study and we plan to address some of these issues in the future.

We now gather below some notation and preliminaries used in this note.

Notation and preliminaries.

- For any function $f : \mathbb{R}^d \to \mathbb{R}$, we use $\|f\|_{L^p}$ or sometimes $\|f\|_p$ to denote the usual Lebesgue L^p norm for $1 \le p \le \infty$. For vector valued function $f = (f_1, \ldots, f_m)$, $m \ge 1$, we use the notation $\|f\|_p = (\sum_{i=1}^m \|f_i\|_p^p)^{\frac{1}{p}}$ for $1 \le p < \infty$ and $\|f\|_\infty = \max_{1\le i\le m} \|f_i\|_\infty$ for $p = \infty$.
- For any two non-negative quantities X and Y, we denote $X \lesssim Y$ if $X \le CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \ge CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. The dependence of the constant C on other parameters or constants are usually clear from the context and we will often suppress this dependence.
- For any two non-negative quantities X and Y, we denote $X \ll Y$ or $Y \gg X$ if $X \le \epsilon Y$ for some sufficiently small constant $\epsilon > 0$. The needed smallness of the constant ϵ is usually clear from the context. Note that this notation is *different* from the usual Vinogradov asymptotic notation.
- Occasionally we write $X = o(1)$ if $X \ll 1$ and $X = O(1)$ if $X \sim 1$.
- We adopt the following convention for Fourier transform pair:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx,$$
$$f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{ix\cdot\xi} d\xi.$$

- The fractional Laplacian $D^s = (-\Delta)^{s/2}$ is defined via Fourier transform as

$$\widehat{D^s f}(\xi) = |\xi|^s \hat{f}(\xi),$$

whenever it is well-defined. For $\mathbb{R}^2$, we define $\nabla^\perp$ as

$$\nabla^\perp = (-\partial_2, \partial_1).$$

- For $s \in \mathbb{R}$, we define

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi,$$
$$\|f\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^d} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi.$$

- For $0 < \alpha < 1$, we define the Hölder semi-norm $\|\cdot\|_{\dot{C}^\alpha}$ as

$$\|f\|_{\dot{C}^\alpha} = \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

Also

$$\|f\|_{C^{1,\alpha}(\mathbb{R}^d)} = \|f\|_\infty + \|\nabla f\|_\infty + \|\nabla f\|_{\dot{C}^\alpha}.$$

It is well-known that

$$\|f\|_{C^{1,\alpha}} \sim \|f\|_\infty + \|\nabla f\|_{\dot{C}^\alpha}.$$

2. Proof of Theorem 1.1 for $s = 0$

We shall first consider the case dimension $d = 2$ (see Remark 2.4 for $d \geq 3$). By using the scaling symmetry (see Remark 1.2) it suffices for us to prove it for $R_0 = 1$ and $0 < T < \infty$.

We begin with an easy L^2 approximation lemma. Suppose u^a, u are smooth solutions to the following two systems:

$$\begin{cases} \partial_t u^a + u^a \cdot \nabla u^a = -\nabla p^a + E^a, \\ \nabla \cdot u^a = 0, \\ u^a\Big|_{t=0} = f; \end{cases} \qquad \begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p, \\ \nabla \cdot u = 0, \\ u\Big|_{t=0} = f. \end{cases}$$

Lemma 2.1. *We have*

$$\begin{aligned}\max_{0\le t\le T} \|u-u^a\|_2 &\le e^{T\cdot \max_{0\le t\le T}\|\nabla u^a(t)\|_\infty} \cdot \|E^a\|_{L_t^1 L_x^2([0,T]\times\mathbb{R}^d)} \\ &\le e^{T\cdot \max_{0\le t\le T}\|\nabla u^a(t)\|_\infty} \cdot T \cdot \max_{0\le t\le T}\|E^a(t)\|_2.\end{aligned}$$

Proof. Set $\eta = u - u^a$. Note that η satisfies

$$\begin{cases} \partial_t \eta + (u^a\cdot\nabla)\eta + (\eta\cdot\nabla)u^a + (\eta\cdot\nabla)\eta = -\nabla\tilde{p} - E^a, \\ \nabla\cdot\eta = 0, \\ \eta\Big|_{t=0} = 0, \end{cases}$$

where $\tilde{p} = p - p^a$. Obviously

$$\frac{1}{2}\frac{d}{dt}(\|\eta(t)\|_2^2) \le \|\nabla u^a\|_\infty \cdot \|\eta\|_2^2 + \|E^a\|_2 \cdot \|\eta\|_2.$$

The result then follows from integrating in time. □

We shall build two approximate solutions. The first one is a simple wave packet with unit L^2 norm having frequency almost localized around $|\xi| \sim \lambda \gg 1$. Fix $\phi_0 \in C_c^\infty(\mathbb{R}^2)$ such that $\phi_0(x) = 1$ for $|x| \le 1/2$ and $\phi_0(x) = 0$ for $|x| \ge 1$. Let $\lambda \gg 1$, $\mu = \lambda/\sqrt{\log \lambda}$ and define a stream function

$$\Phi^{(0)}(x) = \frac{1}{\mu\lambda}\phi_0(\frac{x}{\mu}) \sin \lambda x_1.$$

Define (recall $\nabla^\perp = (-\partial_2, \partial_1)$)

$$\begin{aligned} u^{(0)}(x) = \nabla^\perp \Phi^{(0)} &= \nabla^\perp(\frac{1}{\mu\lambda}\phi_0(\frac{x}{\mu}) \sin \lambda x_1) \\ &= \frac{1}{\mu^2\lambda}(\nabla^\perp\phi_0)(\frac{x}{\mu}) \sin \lambda x_1 + \frac{1}{\mu}\phi_0(\frac{x}{\mu}) \cdot (0, \cos \lambda x_1). \end{aligned}$$

Note that the second term carries the main bulk of L^2 mass. The following lemma shows that $u^{(0)}$ is an almost steady state solution (with pressure $p = 0$).

Lemma 2.2 (Approximate solution without boost). *We have*

$$(u^{(0)} \cdot \nabla)u^{(0)} = E^{(0)},$$

where $\|E^{(0)}\|_2 \lesssim \lambda^{-1.9}$.

Proof Direct calculation. Write $E^{(0)} = (E_1^{(0)}, E_2^{(0)})$. Note that in terms of the stream function $\Phi^{(0)}$, one has

$$\begin{aligned} E_1^{(0)} &= \partial_2\Phi^{(0)}\partial_{12}\Phi^{(0)} - \partial_1\Phi^{(0)}\partial_{22}\Phi^{(0)}, \\ E_2^{(0)} &= -\partial_2\Phi^{(0)}\partial_{11}\Phi^{(0)} + \partial_1\Phi^{(0)}\partial_{12}\Phi^{(0)}. \end{aligned}$$

Easy to check that

$$\begin{aligned} &\|\partial_2\Phi^{(0)}\|_\infty \lesssim \frac{1}{\lambda\mu^2}, \quad \|\partial_{12}\Phi^{(0)}\|_2 \lesssim \frac{1}{\mu}, \\ &\|\partial_1\Phi^{(0)}\|_\infty \lesssim \frac{1}{\mu}, \quad \|\partial_{22}\Phi^{(0)}\|_2 \lesssim \frac{1}{\lambda\mu^2}, \\ &\|\partial_{11}\Phi^{(0)}\|_2 \lesssim \lambda. \end{aligned}$$

Thus

$$\|E^{(0)}\|_2 \lesssim \frac{1}{\lambda\mu^3} + \frac{1}{\mu^2}.$$

□

Now suppose that we take a constant velocity vector $c = (c_1, c_2)$ and apply the Galilean boost to $u^{(0)}$, i.e. take

$$\tilde{u}(t, x) = u^{(0)}(x - ct) + c.$$

Then clearly

$$\partial_t\tilde{u} + (\tilde{u} \cdot \nabla)\tilde{u} = E^{(0)}(x - ct).$$

This is almost good for us except for the fact that the constant vector c is not in $L^2(\mathbb{R}^2)$. To fix this issue, we shall make use of the finite support of $u^{(0)}$. Assume $|c| \le 1$ and $0 \le t \le T$. Then obviously

$$\mathrm{supp}(u^{(0)}(\cdot - ct)) \subset \{x : \ |x| \le \mu + T\} \subset \{x : \ |x| \le 1.1\mu\},$$

if λ is sufficiently large (recall $\mu = \lambda/\sqrt{\log \lambda}$). Choose $\psi \in C_c^\infty(\mathbb{R}^2)$ such that $\psi(x) = 1$ for $|x| \le 2$. Define

$$u^a(t,x) = \underbrace{u^{(0)}(x - ct)}_{=:v(t,x)} + \underbrace{\nabla^\perp((-c_1 x_2 + c_2 x_1)\psi(\frac{x}{\mu}))}_{=:\beta(x)}.$$

Observe that if $x \in \mathrm{supp}(v(t,\cdot))$ (with $0 \le t \le T$), then $\beta(x) = c$. The following lemma is then evident.

Lemma 2.3 (Boosted approximate solution). *Assume* $|c| \lesssim 1/\lambda$ *and* $0 \le t \le T$. *Then for* $\lambda \gg 1$,

$$\max_{0\le t\le T} \|\partial_t u^a + (u^a \cdot \nabla)u^a\|_2 \lesssim \lambda^{-1.9}.$$

Proof Observe

$$\begin{aligned}
&\partial_t u^a + (u^a \cdot \nabla)u^a \\
&\quad = -c \cdot \nabla v + ((v+\beta)\cdot\nabla)(v+\beta) \\
&\quad = E^{(0)}(x - ct) + (\beta \cdot \nabla)\beta.
\end{aligned}$$

□

We now complete the proof of Theorem 1.1 for $d = 2$ and $s = 0$ case. Later we indicate the modifications needed for $d \ge 3$ and $s = 0$.

Proof of Theorem 1.1 for $d = 2$, $s = 0$. We shall choose $\lambda \gg 1$ and $c = (\frac{1}{\lambda} \cdot \frac{1}{10T}, 0)$.

Step 1. Inflation for the approximate solutions $u^{(0)}$ and u^a. Observe that at $t = 0$, we have

$$\|u^{(0)}(\cdot) - u^a(0,\cdot)\|_2 \le \|\beta\|_2 \lesssim \frac{\mu}{\lambda} = \frac{1}{\sqrt{\log\lambda}} \ll 1.$$

On the other hand for $t = T$, thanks to the $O(1)$ phase shift in x_1, it is not difficult to check that

$$\begin{aligned}
\|u^{(0)}(\cdot) - u^a(T,\cdot)\|_2 &\ge \|u^{(0)}(\cdot) - u^{(0)}(\cdot - cT)\|_2 - \|\beta\|_2 \\
&\gtrsim \sqrt{1 - \cos(\frac{1}{10})} - O(\lambda^{-1}) - O(\frac{1}{\sqrt{\log\lambda}} \cdot \frac{1}{T}) \\
&\ge 2c_0,
\end{aligned}$$

where c_0 is an absolute constant.

Step 2. Inflation for the exact solutions. Let u^1 denote the exact solution corresponding to initial data $u^{(0)}$. Then by Lemma 2.1 and Lemma 2.2, we have

$$\max_{0\le t\le T} \|u^1 - u^{(0)}\|_2 \lesssim e^{T\|\nabla u^{(0)}\|_\infty} \cdot \frac{1}{\lambda^{1.9}} \lesssim \lambda^{-1}.$$

Let u^2 denote the exact solution corresponding to initial data $u^a(0, \cdot)$. Similarly we have

$$\max_{0 \le t \le T} \|u^2 - u^a\|_2 \lesssim \lambda^{-1}.$$

It follows from Step 1 that for $t = 0$,

$$\|u^1(0, \cdot) - u^2(0, \cdot)\|_2 \lesssim \frac{1}{\sqrt{\log \lambda}} \ll 1.$$

And for $t = T$,

$$\|u^1(T, \cdot) - u^2(T, \cdot)\|_2 \ge c_0 > 0.$$

□

Remark 2.4. For dimension $d \ge 3$, one can similarly define a scalar function

$$\Phi^{(0)}(x) = \frac{1}{\mu^{\frac{d}{2}}\lambda} \phi_0(\frac{x}{\mu}) \sin \lambda x_1,$$

and define (below $\nabla^\perp = (-\partial_2, \partial_1, 0, \ldots, 0)$)

$$\begin{aligned} u^{(0)}(x) = \nabla^\perp \Phi^{(0)} &= \nabla^\perp (\frac{1}{\mu^{\frac{d}{2}}\lambda} \phi_0(\frac{x}{\mu}) \sin \lambda x_1) \\ &= \frac{1}{\mu^{1+\frac{d}{2}}\lambda} (\nabla^\perp \phi_0)(\frac{x}{\mu}) \sin \lambda x_1 + \frac{1}{\mu^{\frac{d}{2}}} \phi_0(\frac{x}{\mu}) \cdot (0, \cos \lambda x_1, 0, \ldots, 0). \end{aligned}$$

Recall[3] $\mu = \lambda / \sqrt{\log \lambda}$. Thus for $d \ge 3$ we have

$$\|u^{(0)}\|_{C^{1,\frac{1}{4}}(\mathbb{R}^d)} \lesssim \lambda^{-\frac{1}{5}} \ll 1.$$

By standard $C^{1,\alpha}$ $(0 < \alpha < 1)$ local wellposedness theory, one can guarantee that the exact solution emanating from initial data $u^{(0)}$ will have lifespan covering the time interval $[0, T]$. Similar estimate also holds for the Galilean boosted approximate solution. It follows easily that the preceding estimates hold also for $d \ge 3$. We omit the repetitive details.

3. Modification for H^s, $s > 0$

We shall explain the modification needed for $d = 2$, $s > 0$. It will become clear shortly that the case $d \ge 3$ requires very little changes.

For notational simplicity we shall take $T = 1$. The case for general $T > 0$ is similar and the numerology is essentially the same.

For $d = 2$, we shall take $\mu = \lambda^\delta$, $0 < \delta < 1$, $\delta > 1 - s$ and define

$$\begin{aligned} u_s^{(0)}(x) &= \frac{1}{\lambda^{1+s}} \nabla^\perp (\frac{1}{\mu} \phi_0(\frac{x}{\mu}) \sin \lambda x_1) \\ &= \frac{1}{\lambda^{1+s}\mu^2} (\nabla^\perp \phi_0)(\frac{x}{\mu}) \sin \lambda x_1 + \frac{1}{\lambda^s \mu} \phi_0(\frac{x}{\mu}) \cdot (0, \cos \lambda x_1). \end{aligned}$$

[3] One can even take $\mu = \lambda$ or other scalings. In some sense the problem is simpler in $d \ge 3$.

Note that

$$\|u_s^{(0)}\|_\infty \lesssim \frac{1}{\lambda^s \mu}, \quad \|\nabla u_s^{(0)}\|_\infty \lesssim \lambda^{1-s}\mu^{-1}, \quad \|u_s^{(0)}\|_{H^s} \lesssim 1,$$

and for integer $m = 2 + [s]$

$$\|u_s^{(0)}\|_{H^m} \lesssim \lambda^{m-s}.$$

The Galilean boosted approximate solution is defined as

$$u_s^a(t,x) = u_s^{(0)}(x_1 - \frac{t}{\lambda}, x_2, t) + \underbrace{\frac{1}{\lambda}\nabla^\perp((-x_2)\cdot \psi(\frac{x}{\mu}))}_{=:\beta_s(x)}.$$

Note that[4]

$$\|\beta_s\|_2 \lesssim \frac{\mu}{\lambda}, \quad \|\beta_s\|_{\dot{H}^s} \lesssim \frac{\mu}{\lambda}\cdot\frac{1}{\mu^s},$$
$$\|\beta_s\|_\infty \lesssim \frac{1}{\lambda}, \quad \|\nabla\beta_s\|_\infty \lesssim \frac{1}{\lambda\mu}, \quad \|\partial^2\beta_s\|_\infty \lesssim \frac{1}{\lambda\mu^2},$$

where $\|\partial^2\beta_s\|_\infty = \sum_{i,j}\|\partial_i\partial_j\beta_s\|_\infty$. Take $0 < \alpha < 1$ such that $\alpha < \frac{\delta-(1-s)}{2}$. Then[5]

$$\|u_s^{(0)}\|_{C^{1,\alpha}} + \|u_s^a(0,\cdot)\|_{C^{1,\alpha}} \lesssim \lambda^{-\alpha}.$$

Let u^1, u^2 denote the exact solutions corresponding to the initial data $u_s^{(0)}$ and $u_s^a(0,\cdot)$ respectively. Standard local wellposedness theory in $C^{1,\alpha}$ then yields

$$\max_{0\le t\le 1}(\|\nabla u^1\|_\infty + \|\nabla u^2\|_\infty) \lesssim 1.$$

Energy estimate further gives

$$\max_{0\le t\le 1}(\|u^1\|_{H^m} + \|u^2\|_{H^m}) \lesssim \lambda^{m-s}.$$

Lemma 3.1 (Approximate solution without boost). *We have*

$$(u_s^{(0)}\cdot\nabla)u_s^{(0)} = E_s^{(0)},$$

where $\|E_s^{(0)}\|_2 \lesssim \lambda^{-2s}\mu^{-2}$ *and* $\|E_s^{(0)}\|_{H^s} \lesssim \lambda^{-s}\mu^{-2}$. *Also*

$$\max_{0\le t\le 1}\|u^1 - u_s^{(0)}\|_{H^s} \lesssim \lambda^{-s(1-\frac{s}{m})}\cdot\mu^{-2(1-\frac{s}{m})}.$$

---

[4] It is instructive to investigate why this construction does not work for $s < 0$. To ensure $\|\nabla u_s^{(0)}\|_\infty \lesssim 1$ one needs $\lambda^{1-s} \lesssim \mu$. On the other hand to ensure $\|\beta_s\|_{\dot{H}^s} \lesssim 1$, we need $\mu^{1-s} \lesssim \lambda$. These two clearly cannot be fulfilled simultaneously for $s < 0$. It is also not difficult to check that a similar obstruction happens for the dimension $d \ge 3$.

[5] Thanks to the $C^{1,\alpha}$ estimate, one can easily generalize this construction to higher dimension $d \ge 3$ for which the lifespan of the solution will cover the time interval $[0, T]$.

Proof The first two estimates on $E_s^{(0)}$ are direct calculations. For the third inequality, just observe that by Lemma 2.1,

$$\max_{0\le t\le 1} \|u^1 - u_s^{(0)}\|_2 \lesssim \lambda^{-2s}\mu^{-2}.$$

The result then follows from interpolation with H^m norm. □

Lemma 3.2 (Boosted approximate solution). *We have*

$$\max_{0\le t\le 1} \|\partial_t u_s^a + (u_s^a\cdot\nabla)u_s^a\|_2 \lesssim \lambda^{-2s}\mu^{-2} + \lambda^{-2},$$
$$\max_{0\le t\le 1} \|\partial_t u_s^a + (u_s^a\cdot\nabla)u_s^a\|_{H^s} \lesssim \lambda^{-s}\mu^{-2} + \lambda^{-2},$$
$$\max_{0\le t\le 1} \|u_s^a - u^2\|_{H^s} \lesssim \lambda^{-\alpha_0},$$

where $\alpha_0 > 0$ *depends on* (s, δ).

Proof. For the first two inequalities, one can just use the relation

$$\partial_t u_s^a + (u_s^a\cdot\nabla)u_s^a = E_s^{(0)}(x-ct) + (\beta_s\cdot\nabla)\beta_s.$$

For the third inequality, denote $w = u^2 - u_s^a$. Note that by Lemma 2.1,

$$\max_{0\le t\le 1}\|w\|_2 \lesssim \lambda^{-2s-2\delta} + \lambda^{-2}.$$

We now discuss two cases. In the first case we assume $0 < s < 2$. By using interpolation with H^m norm (recall $m = 2 + [s]$), we have

$$\|w\|_{H^s} \lesssim \|w\|_2^{1-\frac{s}{m}}\cdot\|w\|_{H^m}^{\frac{s}{m}} \lesssim (\lambda^{-2s-2\delta} + \lambda^{-2})^{1-\frac{s}{m}}\cdot\lambda^{(m-s)\cdot\frac{s}{m}} \lesssim \lambda^{-\alpha_0},$$

for some $\alpha_0 > 0$. Note here $(m-s)\cdot\frac{s}{m} = (1-\frac{s}{m})s$ and the condition $s < 2$ is crucially needed.

Now for the case $s \ge 2$ we observe that w satisfies the equation

$$\partial_t w + u_s^a\cdot\nabla w + w\cdot\nabla u_s^a + w\cdot\nabla w = -\nabla p - E^a,$$

where $E^a = \partial_t u_s^a + (u_s^a\cdot\nabla)u_s^a$. The L^2 norm of w was already estimated.

For the $\dot{H}^s$ norm, we compute

$$\frac{1}{2}\frac{d}{dt}(\|D^s w\|_2^2) \le \Big(\|D^s(u_s^a\cdot\nabla w) - u_s^a\cdot\nabla D^s w\|_2 + \|D^s(w\cdot\nabla u_s^a)\|_2 + \|D^s(w\cdot\nabla w) - w\cdot\nabla D^s w\|_2 + \|D^s E^a\|_2\Big)\cdot\|D^s w\|_2.$$

By using a commutator estimate (see Corollary 5.2 in [3]), we have (below we need $s > 1$ in order to control $\|\nabla w\|_2$)

$$\begin{aligned}\|D^s(u_s^a\cdot\nabla w) - u_s^a\cdot\nabla D^s w\|_2 &\lesssim \|D^s u_s^a\|_\infty\cdot\|\nabla w\|_2 + \|D^s w\|_2\cdot\|\nabla u_s^a\|_\infty\\ &\lesssim \lambda^{-\delta}(\|w\|_2 + \|D^s w\|_2) + \|D^s w\|_2\\ &\lesssim \lambda^{-2s-3\delta} + \lambda^{-2-\delta} + \|D^s w\|_2.\end{aligned}$$

On the other hand, by using fractional Leibniz, we have

$$\|D^s(w \cdot \nabla u_s^a)\|_2 \lesssim \|D^s w\|_2 \cdot \|\nabla u_s^a\|_\infty + \|w\|_2 \cdot \|D^s \nabla u_s^a\|_\infty$$
$$\lesssim \|D^s w\|_2 + (\lambda^{-2s-2\delta} + \lambda^{-2}) \cdot \lambda^{1-\delta}.$$

Also

$$\|D^s(w \cdot \nabla w) - w \cdot \nabla D^s w\|_2 \lesssim \|\nabla w\|_\infty \|D^s w\|_2 \lesssim \|D^s w\|_2.$$

Integrating in time then yields the result. □

Remark. Even though our ultimate goal is to show H^s-instability for solutions u^1 and u^2, it is still instructive to understand why one cannot obtain H^s-stability for solutions u^1 and u^2 from the above (standard) energy type arguments. Denote $\tilde{w} = u^2 - u^1$. Observe that $\|\tilde{w}\|_2 = O(\mu/\lambda)$. By using similar estimates on $\|D^s \tilde{w}\|_2$ as above, one sees that we need to estimate

$$\|D^s \nabla u^1\|_\infty \cdot \|\tilde{w}\|_2 \approx \frac{\lambda}{\mu} \cdot \frac{\mu}{\lambda} \sim 1.$$

This shows that $\|\tilde{w}\|_{H^s}$ cannot be bounded from above by λ^{-c} for some $c > 0$ on the unit time interval.

Proof of Theorem 1.1 for $s > 0$. By Lemmas 3.1 and 3.2, we obtain

$$\|u^1(1,\cdot) - u^2(1,\cdot)\|_{\dot{H}^s} \gtrsim \|u_s^{(0)} - u_s^a(1,\cdot)\|_{\dot{H}^s} - o(1) \geq c_0 > 0,$$

where c_0 is some constant depending only on s. □

4. Proof of Theorem 1.3

In this section we give the proof of Theorem 1.3. We first explain the proof in the case $d = 2$, $s = 0$ (i.e. L^2) where the perturbation argument is easy to carry out. In the later subsection we give the proof for general $s \geq 0$.

4.1. Proof of Theorem 1.3 for $d = 2$, $s = 0$. This is the simplest case since in dimension two the lifespan is not an issue for smooth solutions. WLOG we will take $T = 1$. Our analysis below is a slight modification of the corresponding argument in Section 2. Thus we shall adopt similar notation as in Section 2 and only sketch the needed modifications. We shall denote the given solution u^g as u.

The basic idea is as follows. Recall that u solves the equation

$$\partial_t u + (u \cdot \nabla) u = -\nabla p.$$

Now let v be an almost solution (say $v = \text{const} \cdot u^{(0)}$ as in Section 2) which solves

$$\partial_t v + (v \cdot \nabla) v = -\nabla \tilde{p}_1 + \tilde{E}^1.$$

We then define a new approximate solution in the form

$$u_a^{\text{new}}(t,x) = u(t,x) + \underbrace{v(t, x - x_*)}_{=:v_*},$$

where $|x_*|$ will be taken sufficiently large. Clearly then

$$\partial_t u_a^{\text{new}} + (u_a^{\text{new}} \cdot \nabla) u_a^{\text{new}} = -\nabla \tilde{p} + \tilde{E}^1(x - x_*) + (u \cdot \nabla) v_* + (v_* \cdot \nabla) u,$$

where $v_*(t, x) = v(t, x - x_*)$. Now let u^{new} be the exact solution corresponding to the initial data $u_a^{\text{new}}(0, \cdot)$. Then we have

$$\begin{aligned}\max_{0\le t\le 1} \|u_a^{\text{new}}(t,\cdot) - u^{\text{new}}(t,\cdot)\|_2 &\le \exp(\text{const} \cdot \max_{0\le t\le 1} \|\nabla u_a^{\text{new}}(t,\cdot)\|_\infty) \\ &\cdot (\|\tilde{E}^1\|_{L_t^\infty L_x^2} + \|u \cdot \nabla v_*\|_{L_t^\infty L_x^2} + \|v_* \cdot \nabla u\|_{L_t^\infty L_x^2}).\end{aligned}$$

It follows that if $|x_*|$ is sufficiently large, then u^{new} can remain close to u_a^{new} in L^2.

Step 1. Construction of u_n^1. Since by assumption $u \in C_t^0 L^2$ and the time interval $[0, 1]$ is compact, we can find $R_\lambda > 1$ sufficiently large such that

$$\sup_{0\le t\le 1} \|u(t,\cdot)\|_{L_x^2(|x|\ge R_\lambda)} \le \lambda^{-10}.$$

Define the approximate solution u_a^1 as

$$u_a^1(t, x) = u(t, x) + v_*^1(x) = u(t, x) + v^1(x - x_*),$$

where $x_* \in \mathbb{R}^2$ satisfies $|x_*| > R_\lambda + 10^6\mu$, and by choosing $\phi_1 \in C_c^\infty(\{x : 1 < |x| < 2\})$ we introduce

$$\begin{aligned} v^1(x) &= \frac{\eta}{10} u^{(0)}(x) \\ &= \frac{\eta}{10} \nabla^\perp \Big(\frac{1}{\mu\lambda} \phi_1\Big(\frac{x}{\mu}\Big) \sin \lambda x_1\Big), \quad \mu = \lambda/\sqrt{\log \lambda}.\end{aligned}$$

Clearly

$$\partial_t u_a^1 + (u_a^1 \cdot \nabla) u_a^1 = -\nabla p + E_a^1,$$

where

$$E_a^1 = (u \cdot \nabla) v_*^1 + (v_*^1 \cdot \nabla) u + (v_*^1 \cdot \nabla) v_*^1,$$

and $v_*^1 = v^1(\cdot - x_*)$. It is not difficult to check that

$$\begin{aligned}\|E_a^1\|_2 &\lesssim \|u \cdot \nabla v_*^1\|_2 + \|v_*^1 \cdot \nabla u\|_2 + \|v_*^1 \cdot \nabla v_*^1\|_2 \\ &\lesssim \eta \cdot \lambda^{-1} + \eta \cdot \lambda^{-\frac{1}{2}} \lesssim \eta \cdot \lambda^{-\frac{1}{2}}.\end{aligned}$$

Here in bounding the term $\|u \cdot \nabla v_*^1\|_2$, we used the fact that $\text{supp}(v_*^1) \subset B(x_*, 2\mu)$ which induces a spatial localization on u:

$$u \cdot \nabla v_*^1 = u \chi_{|x|\ge R_\lambda + \mu} \cdot \nabla v_*^1,$$

thus yielding

$$\|u \cdot \nabla v_*^1\|_2 \lesssim \lambda^{-10} \cdot \eta \cdot \frac{\lambda}{\mu} \lesssim \eta \cdot \lambda^{-1}.$$

Let u_n^1 be the exact solution corresponding to initial data $u_a^1(0,\cdot)$. It follows that

$$\max_{0\le t\le 1}\|u_n^1(t,\cdot)-u_a^1(t,\cdot)\|_2 \lesssim e^{\mathrm{const}\cdot\sqrt{\log\lambda}}\cdot\eta\cdot\lambda^{-\frac{1}{2}} \lesssim \eta\cdot\lambda^{-\frac{1}{4}}.$$

Step 2. Construction of u_n^2. Define $v^2(t,x)$ as the boosted version of $v^1(x)$, i.e.

$$v^2(t,x) = v^1(x-ct) + \nabla^{\perp}((-c_1x_2+c_2x_1)\psi_1(\frac{x}{\mu})),$$

where $c=(c_1,c_2)=(\frac{1}{\lambda},0)$, and $\psi_1\in C_c^{\infty}(\{x:\ \frac{1}{100}<|x|<100\})$ satisfies $\psi_1(x)\equiv 1$ for $\frac{1}{10}\le|x|\le 10$. One should note that v^2 is supported on the annulus $|x|\sim\mu$ for all $0\le t\le 1$. Then clearly

$$\partial_t v^2 + (v^2\cdot\nabla)v^2 = E^2,$$

where

$$\max_{0\le t\le 1}\|E^2(t,\cdot)\|_2 \lesssim \lambda^{-1.9}.$$

Now define

$$u_a^2(t,x) = u(t,x) + v_*^2(t,x) = u(t,x) + v^2(t,x-x_*).$$

Observe that

$$\partial_t u_a^2 + (u_a^2\cdot\nabla)u_a^2 = -\nabla p + E_a^2,$$

where

$$E_a^2(t,x) = E^2(t,x-x_*) + (u\cdot\nabla)v_*^2 + (v_*^2\cdot\nabla)u.$$

Let u_n^2 be the exact solution corresponding to initial data $u_a^2(0,\cdot)$. Then clearly for λ sufficiently large

$$\max_{0\le t\le 1}\|u_n^2(t,\cdot)-u_a^2(t,\cdot)\|_2 \lesssim \eta\cdot\lambda^{-\frac{1}{4}}.$$

Step 3. Inflation. It follows from previous two steps that

$$\|u_n^2(1,\cdot)-u_n^1(1,\cdot)\|_2 \ge \|v^1(\cdot)-v^1(\cdot-c)\|_2 - \frac{\mathrm{const}}{\sqrt{\log\lambda}} - \frac{\mathrm{const}\cdot\eta}{\lambda^{\frac{1}{4}}} \gtrsim \eta,$$

as $\lambda\to\infty$. The desired result then follows.

Remark 4.1. One can even take $x_*=0$ by a minor change of the argument above. The idea is as follows. First, by a density argument (see Lemma 4.3), one can assume without loss of generality that $u^{(g)}(0,x)$ initially has compact support and $u^{(g)}\in C_t^1H^{10}$. By a computation of moments, this then easily yields that

$$\max_{0\le t\le T}\int_{\mathbb{R}^2}|u^{(g)}(t,x)|^2(1+|x|^2)^{\frac{1}{2}}dx \lesssim 1.$$

This then yields

$$\max_{0\le t\le T} \|u^{(g)}\|_{L_x^2(|x|\sim\mu)} \lesssim \mu^{-\frac{1}{2}}. \tag{4.1}$$

Adopting the same notation as in the previous proof, we then choose

$$u_a^1(t,x) = u^{(g)}(t,x) + v^1(x),$$

such that

$$\partial_t u_a^1 + (u_a^1 \cdot \nabla) u_a^1 = -\nabla p + E_a^1,$$

where

$$E_a^1 = (u^{(g)} \cdot \nabla) v^1 + (v^1 \cdot \nabla) u^{(g)} + (v^1 \cdot \nabla) v^1.$$

In bounding the error term E_a^1, the main troublesome term is the first term. By using the decay estimate (4.1), we have

$$\begin{aligned}\|(u^{(g)} \cdot \nabla) v^1\|_{L_x^2} &\lesssim \|u^{(g)}\|_{L_x^2(|x|\sim\mu)} \cdot \frac{\lambda}{\mu} + \lambda^{-\frac{1}{10}} \\ &\lesssim \lambda^{-\frac{1}{10}},\end{aligned}$$

which is clearly acceptable. The error estimate for the approximate solution u_a^2 is similar.

4.2. Proof of Theorem 1.3 for $d \ge 2$, $s \ge 0$. We first prove two auxiliary perturbation lemmas. These two lemmas are certainly not in the most sharpest forms and are more or less "trivial" from the point of view of standard local wellposedness theory. But these weak formulations suffice for our construction later on. We record them here along with the proofs for the sake of completeness.

Lemma 4.2 (Near solutions generate exact solutions). *Let $d \ge 2$ and $s_0 > \frac{d}{2} + 1$. Let $T > 0$. Suppose $\tilde u \in C_t^0 H^{s_0} \cap C_t^1 H^{s_0-1}([0,T] \times \mathbb{R}^d)$ is a near solution to Euler, i.e.:*

$$\begin{cases} \partial_t \tilde u + (\tilde u \cdot \nabla)\tilde u = -\nabla \tilde p + \tilde f, & (t,x) \in (0,T] \times \mathbb{R}^d; \\ \nabla \cdot \tilde u = 0, \\ \tilde u \Big|_{t=0} = u_0, \end{cases}$$

where $\nabla \tilde p \in C_t^0 H^{s_0-1}$, $\tilde f \in C_t^0 H^{s_0-1}$. Denote

$$A_1 = \max_{0\le t\le T} \|\tilde u(t,\cdot)\|_{H^{s_0}}.$$

There is a constant $\rho_1 = \rho_1(A_1, s_0, d, T) > 0$ sufficiently small such that if

$$\|\tilde f\|_{L_t^1 L_x^2([0,T]\times\mathbb{R}^d)} \le \rho_1,$$

then corresponding to the same initial data u_0, there exists a unique solution $u \in C_t^0 H^{s_0}$ to the incompressible Euler with lifespan covering the interval $[0,T]$. Furthermore, for *any* $0 < s < s_0$ *and any* $\epsilon > 0$, *there exists* $\rho_2 = \rho_2(A_1, s_0, s, d, T, \epsilon) > 0$ *sufficiently small such that if*

$$\|\tilde{f}\|_{L_t^1 L_x^2([0,T]\times\mathbb{R}^d)} \leq \rho_2,$$

then

$$\max_{0\leq t\leq T} \|u - \tilde{u}\|_{H^s} < \epsilon.$$

Proof. Let u be the local solution corresponding to initial data u_0 in $C_t^0 H^{s_0}$. Initially u exists on some time interval $[0, T_0]$ for $T_0 = T_0(u_0) > 0$. We must show that u can be extended to $[0, T]$. For notational convenience, we redefine A_1 as

$$A_1 = \max_{0\leq t\leq T} (\|\tilde{u}(t,\cdot)\|_{H^{s_0}} + \|\nabla\tilde{u}(t,\cdot)\|_\infty).$$

Denote

$$\rho = \|\tilde{f}\|_{L_t^1 L_x^2([0,T]\times\mathbb{R}^d)}.$$

Let $C_1 = C_1(s_0, d) > 0$, $C_2 = C_2(s_0, d) > 0$ be two fixed constants whose value will be specified later in the proof. Denote

$$B = 2\|u_0\|_{H^{s_0}} e^{2C_2 T A_1}, \quad \theta = \frac{s_0 - \frac{d}{2} - 1}{2s_0}.$$

We claim that if ρ is sufficiently small such that

$$C_1 \cdot (e^{TA_1} \cdot \rho)^\theta \cdot (A_1 + B)^{1-\theta} \leq A_1,$$

then $u(t)$ remains trapped in the set

$$X = \{v : \quad \|v\|_{H^{s_0}} \leq B\}.$$

Indeed let t_0 be the first time that u leaves the set X. Since $u(0) = u_0$ is in the interior of X one must have $t_0 > 0$. Assume $t_0 \leq T$ and we shall arrive at a contradiction. Clearly at t_0 we have $\|u(t_0,\cdot)\|_{H^{s_0}} = B$. Now we aim to show that $\|u(t_0,\cdot)\|_{H^{s_0}} < B$ by using a different computation.

To this end, denote $\eta = u - \tilde{u}$. By Lemma 2.1, we have

$$\max_{0\leq\tau\leq t_0} \|\eta(\tau)\|_2 \leq e^{t_0 A_1} \cdot \rho \leq e^{TA_1} \cdot \rho.$$

Since $\max_{0\leq\tau\leq t_0} \|\eta(\tau)\|_{H^{s_0}} \leq A_1 + B$, Sobolev embedding and interpolation then gives

$$\begin{aligned}\max_{0\leq\tau\leq t_0} \|\nabla\eta(\tau)\|_\infty &\leq C_1 \max_{0\leq\tau\leq t_0} \|\eta(\tau)\|_{H^{\frac{s_0+\frac{d}{2}+1}{2}}} \\ &\leq C_1 (e^{TA_1} \cdot \rho)^\theta \cdot (A_1 + B)^{1-\theta} \leq A_1.\end{aligned}$$

Note here C_1 is the Sobolev embedding constant corresponding to $H^{\frac{s_0+\frac{d}{2}+1}{2}-1} \hookrightarrow L^\infty$. Then

$$\max_{0\leq\tau\leq t_0} \|\nabla u(\tau)\|_\infty \leq 2A_1.$$

By a standard energy estimate, we then obtain

$$\|u(t_0)\|_{H^{s_0}} \leq \|u_0\|_{H^{s_0}} e^{C_2 T \cdot 2A_1} = \frac{B}{2},$$

where $C_2 > 0$ depends only on (s_0, d). This then gives the desired contradiction. Hence, the solution u has lifespan covering the interval $[0, T]$. By further taking ρ sufficiently small such that

$$(e^{TA_1} \cdot \rho)^{\theta_1} \cdot (A_1 + B)^{1-\theta_1} < \epsilon,$$

where $\theta_1 = \frac{s_0 - s}{s_0}$, one can obviously achieve $\|u - \tilde{u}\|_{H^s} < \epsilon$. □

Lemma 4.3 (Small subcritical perturbation preserves lifespan). *Let $d \geq 2$ and $s_0 > \frac{d}{2} + 1$. Let $T > 0$. Suppose $u \in C_t^0 H^{s_0} \cap C_t^1 H^{s_0 - 1}([0, T] \times \mathbb{R}^d)$ is a given solution to incompressible Euler with initial data $u(0) = u_0$. There exists $\delta_0 = \delta_0(u, s_0, d, T) > 0$ sufficiently small such that if*

$$\|v_0 - u_0\|_{H^{s_0}} \leq \delta_0,$$

then the solution v corresponding to initial data v_0 has lifespan covering the interval $[0, T]$.

Remark. One can show that v remain close to u in H^{s_0} norm on the whole time interval $[0, T]$ if δ_0 is sufficiently small. Such a stability result is standard (this is just continuous dependence on initial data) and can be proved by pursuing a Lagrangian reformulation [cf. (1.2)]. But we shall not need this fact in the later construction.

Proof. The proof is similar to the previous lemma and we shall sketch the "trapping" argument. Denote

$$A_1 = \max_{0 \leq t \leq T} (\|u(t)\|_{H^{s_0}} + \|\nabla u(t)\|_\infty).$$

Let $C_1 = C_1(s_0, d) > 0$, $C_2 = C_2(s_0, d) > 0$ be two constants whose value will be specified later. Denote

$$B = 2A_1 e^{C_2 A_1 T}, \quad \theta = \frac{s_0 - \frac{d}{2} - 1}{2s_0}.$$

We show that if $\delta_0 < \frac{A_1}{2}$ satisfies

$$C_1 (\delta_0 e^{A_1 T})^\theta (A_1 + B)^{1-\theta} \leq A_1,$$

then v remains trapped in the set

$$X = \{\tilde{v} : \ \|\tilde{v}\|_{H^{s_0}} \leq B\}.$$

Indeed consider the first moment $t_0 > 0$ that v escapes the set X. Assume $t_0 \leq T$ and we shall deduce a contradiction. Clearly $\|v(t_0)\|_{H^{s_0}} = B$. Set $\eta = v - u$. By a simple L^2 estimate, we have

$$\|\eta\|_2 \leq \delta_0 e^{A_1 T}.$$

Interpolating with the estimate $\|\eta\|_{H^{s_0}} \le A_1 + B$ then gives (uniformly in $t \le t_0$)

$$\|\nabla\eta\|_\infty \le C_1\|\eta\|_{H^{\frac{s_0+\frac{d}{2}+1}{2}}} \le C_1 \cdot (\delta_0 e^{A_1 T})^\theta (A_1 + B)^{1-\theta} \le A_1.$$

This then yields $\|\nabla v\|_\infty \le 2A_1$. A standard H^{s_0}-energy estimate then gives

$$\|v(t_0)\|_{H^{s_0}} \le \|v_0\|_{H^{s_0}} e^{C_2 A_1 T} \le \frac{3}{2} A_1 e^{C_2 A_1 T} < B,$$

hence a contradiction. □

We now complete the proof of Theorem 1.3 for $d \ge 2$, $s \ge 0$.

Proof of Theorem 1.3. Step 1. By Theorem 1.1, we can find two sequences of solutions $\{v_n^1\}_{n=1}^\infty$, $\{v_n^2\}_{n=1}^\infty$, with $v_n^1,\ v_n^2 \in C_t^0 H^m([0, T] \times \mathbb{R}^d)$, for all $m \ge 0$, such that $\|v_n^1(0, \cdot)\|_{H^s} = \frac{\eta}{6}$, $\|v_n^2(0, \cdot)\|_{H^s} = \frac{\eta}{6}$,

$$\lim_{n\to\infty} \|v_n^1(0, \cdot) - v_n^2(0, \cdot)\|_{H^s} = 0,$$

and

$$\liminf_{n\to\infty} \|v_n^1(T, \cdot) - v_n^2(T, \cdot)\|_{\dot{H}^s} \ge \tilde{c}_0\eta,$$

where $\tilde{c}_0 > 0$ depends only on (s, d).

Step 2. Mollification of $u^{(g)}$. Even though $u^{(g)}$ has regularity $C_t^0 H^s \cap C_t^0 H^{s_0}$, we need to work with mollified initial data which will have high enough regularity for the perturbation argument in later steps. Denote $\tilde{s}_0 = \max\{s,\ s_0\}$. For each $n \ge 1$, we find $u_{0,n}^{(g)} \in C_c^\infty(\mathbb{R}^d)$ with

$$\|u_{0,n}^{(g)}(\cdot) - u^{(g)}(0, \cdot)\|_{H^{\tilde{s}_0}} < \min\{\delta_0(u^{(g)}, \tilde{s}_0, d, T),\ \frac{\eta}{6n}\},$$

where δ_0 is the same as in Lemma 4.3. Denote by $u_n^{(g)}$ the solution to Euler corresponding to the initial data $u_{0,n}^{(g)}$. Then by Lemma 4.3, $u_n^{(g)}$ still has the life span covering $[0, T]$. Note that we do not need $u_n^{(g)}$ to remain close to $u^{(g)}$ at time $t = T$.

Step 3. Approximate solutions. Define

$$\tilde{u}_n^1(t, x) = u_n^{(g)}(t, x) + v_n^1(t, x - x_n^1),$$
$$\tilde{u}_n^2(t, x) = u_n^{(g)}(t, x) + v_n^2(t, x - x_n^2),$$

where the centers x_n^1, x_n^2 will be taken sufficiently far away from the origin. We shall explain the choice of x_n^1 since the choice of x_n^2 will be similar. Now note that

$$\partial_t\tilde{u}_n^1 + (\tilde{u}_n^1 \cdot \nabla)\tilde{u}_n^1 = -\nabla\tilde{p}_n^1 + \tilde{f}_n^1,$$

where

$$\tilde{f}_n^1(t, x) = (u_n^{(g)}(t, x) \cdot \nabla)v_n^1(t, x - x_n^1) + (v_n^1(t, x - x_n^1) \cdot \nabla)u_n^{(g)}(t, x).$$

Note that $v_n^1 \in C_t^0 H^m$ for all $m \geq 0$. For any fixed (t, x), thanks to the smoothness and decay, we have

$$\lim_{|y|\to\infty} (u_n^{(g)}(t,x)\cdot\nabla)v_n^1(t,x-y) + (v_n^1(t,x-y)\cdot\nabla)u_n^{(g)}(t,x) = 0.$$

By Lebesgue Dominated Convergence, it is clear that

$$\lim_{|y|\to\infty} \|(u_n^{(g)}(t,x)\cdot\nabla)v_n^1(t,x-y) + (v_n^1(t,x-y)\cdot\nabla)u_n^{(g)}(t,x)\|_{L_t^1 L_x^2([0,T]\times\mathbb{R}^d)} = 0.$$

Note that for all $m \geq 0$,

$$\max_{0\le t\le T} \|\tilde{u}_n^1\|_{H^m} \le \max_{0\le t\le T} (\|u_n^{(g)}\|_{H^m} + \|v_n^1\|_{H^m})$$

which is independent of x_n^1. Denote by u_n^1 the exact solutions corresponding to the initial data $\tilde{u}_n^1(0, x)$. By Lemma 4.2, one can then find x_n^1 with $|x_n^1|$ sufficiently large such that

$$\|u_n^1 - \tilde{u}_n^1\|_{H^s\cap H^{s_0}} \le \min\{\frac{\eta}{6}, \frac{\tilde{c}_0\eta}{6}\}, \quad \forall\, 0 \le t \le T.$$

Similarly, one can choose x_n^2 for u_n^2. Clearly

$$\liminf_{n\to\infty} \|u_n^1(T,\cdot) - u_n^2(T,\cdot)\|_{\dot{H}^s} \ge c_0\eta,$$

where $c_0 = \tilde{c}_0/10$. □

Acknowledgments. J. Bourgain was supported in part by NSF No. DMS-1301619. D. Li was supported in part by Hong Kong RGC Grant GRF 16307317 and 16309518. We thank the anonymous referees for very helpful remarks and suggestions. The second author would like to thank Hyunju Kwon for some helpful comments.

References

1. Himonas, A., Misiołek, G.: Non-uniform dependence on initial data of solutions to the Euler equations of hydrodynamics. Commun. Math. Phys. **296**(1), 285–301 (2010)
2. Inci, H.: On the Well-Posedness of the Incompressible Euler Equation. Dissertation, University of Zurich (2012). arXiv:1301.5997
3. Li, D.: On Kato-Ponce and fractional Leibniz. Rev. Mat. Iberoam. **35**(1), 23–100 (2019)
4. Vishik, M.: Instability and non-uniqueness in the Cauchy problem for the Euler equations of an ideal incompressible fluid. Part I. arXiv:1805.09426
5. Vishik, M.: Instability and non-uniqueness in the Cauchy problem for the Euler equations of an ideal incompressible fluid. Part II. arXiv:1805.09440
6. Wiedemann, E.: Weak-strong uniqueness in fluid dynamics. Preprint (2017). arXiv:1705.04220

Communicated by W. Schlag