In[1]:=
```
If[! TrueQ[Private`$ev], Block[{Private`$ev = True},
  Print["Total time: ", First@AbsoluteTiming[ClearAll["Global`*"];
     NotebookEvaluate[EvaluationNotebook[], InsertResults → True]], " s"]]]
```

```
Total time: 2.88996 s
```

In[2]:=
```
ClearAll["Global`*"];
(* functions in (A.2)*)
q1[t_] := 21 539 / 93 423 + 127 023 / 185 578 * t^2 - 1 / 8 885 055 * t^3 +
   54 169 / 401 949 * t^4 - 3 / 44 981 * t^5 + 202 / 73 305 * t^6 - 113 / 80 657 * t^7 +
   293 / 59 051 * t^8 - 604 / 151 861 * t^9 + 127 / 76 892 * t^10 - 28 / 94 431 * t^11;
q2[t_] := 18 176 / 78 783 - 21 / 15 850 t + 295 367 / 428 350 t^2 - 1415 / 123 249 t^3 +
   31 027 / 204 823 t^4 - 5162 / 329 873 t^5 + 3025 / 287 391 t^6 - 17 / 36 388 t^7 +
   2 / 74 523 t^8 + 5 / 86 563 t^9 - 1 / 120 831 t^10 + 1 / 1 183 575 t^11;
f1[t_] := Exp[-t] / t;
p2[t_] := q2[t] + q1[9 / 10] - q2[9 / 10] + (q1'[9 / 10] - q2'[9 / 10]) (t - 9 / 10);
ggg[t_] := (2 Exp[t] ExpIntegralEi[-4 t] - Exp[-t] ExpIntegralEi[-2 t]) / t;
AAAAAA = (p2[5 / 2] × ggg'[5 / 2] + p2'[5 / 2] × ggg[5 / 2]) /
    (f1[5 / 2] × ggg'[5 / 2] - f1'[5 / 2] × ggg[5 / 2]) 1 / (p2[5 / 2])^2;
BBBBBB = -(p2[5 / 2] × f1'[5 / 2] + p2'[5 / 2] × f1[5 / 2]) /
    (f1[5 / 2] × ggg'[5 / 2] - f1'[5 / 2] × ggg[5 / 2]) 1 / (p2[5 / 2])^2;
Q[t_] :=
  If[t ≤ 9 / 10, 1 / q1[t], If[t ≤ 5 / 2, 1 / p2[t], AAAAAA Exp[-t] / t + BBBBBB ggg[t]]];
(*Q_*^2 in (A.3)*)
Qanew[t_] := -32 389 / 844 * t^14 + 692 573 / 1409 * t^13 -
   14 647 839 / 5182 * t^12 + 13 634 891 / 1418 * t^11 - 31 307 644 / 1461 * t^10 +
   22 954 676 / 703 * t^9 - 1 268 792 / 37 * t^8 + 34 165 996 / 1415 * t^7 -
   12 679 168 / 1223 * t^6 + 2 124 876 / 1117 * t^5 + 78 128 / 353 * t^4 +
   15 317 / 829 * t^3 - 138 192 / 1231 * t^2 + 15 295 / 813;
(* W_1 in (A.4)*)
W1[t_] :=
  -126 751 t^22 / 80 621 266 + 998 813 t^21 / 24 102 701 - 44 784 703 t^20 / 90 944 085 +
   532 787 760 t^19 / 154 914 809 - 1 478 230 681 t^18 / 97 178 472 +
   1 104 148 971 t^17 / 27 114 439 - 6 386 998 649 t^16 / 165 889 120 -
   4 563 349 408 t^15 / 24 620 175 + 12 295 664 789 t^14 / 11 848 354 -
   25 742 069 395 t^13 / 8 929 817 + 22 711 315 187 t^12 / 4 190 364 -
   21 505 639 879 t^11 / 2 916 474 + 32 713 853 029 t^10 / 4 462 850 -
   12 946 708 519 t^9 / 2 496 198 + 3 403 022 733 t^8 / 1 411 424 -
   6 729 001 962 t^7 / 11 651 297 - 626 483 074 t^5 / 65 670 795 +
   752 248 559 t^4 / 20 846 520 - 89 936 485 t^2 / 6 984 383 + 93 423 / 21 539;
(* W_2 in (A.5)*)
W2[t_] := 157 * t^8 / 1 255 706 - 5215 * t^7 / 1 535 379 + 25 008 * t^6 / 610 441 -
   1 118 777 * t^5 / 3 886 821 + 1 230 788 * t^4 / 949 047 - 784 873 * t^3 / 202 656 +
   10 232 110 * t^2 / 1 348 117 - 2 084 185 * t / 228 972 + 43 251 / 8233;
(* P_1, P_2 in (B.2)*)
P1[t_] := 2 (W1[t] - 75 × 10^(-6))^2;
P2[t_] := (W1[t] + 75 × 10^(-6)) (W1'[t] - 42 × 10^(-5));
```

```
(* P3, P4 in (B.3)*)
P3[t_] := 2 (W1[t] + 75 × 10^(-6))^2;
P4[t_] := (W1[t] + 75 × 10^(-6))^4;
(* for example, Dig5[312]=31200; Dig5[3123423]=31234*)
Dig5[x_] := IntegerPart[x * 10^5 / 10^(IntegerLength[x])];
(* for example,
FU[401256/309977]=40126/30997 gives an upper bound of 401256/309977*)
FU[s_] := If[s > 0, (Dig5[Numerator[s]] + 1) / Dig5[Denominator[s]] *
    10^(IntegerLength[Numerator[s]] - IntegerLength[Denominator[s]]),
  Print["Error: Input must be a positive number."];
  $Failed]
(* for example, NL[401256/3077]=6;  DL[401256/3077]=4*)
NL[x_] := IntegerLength[Numerator[x]]; DL[x_] := IntegerLength[Denominator[x]];
(*The following function  polyMax corresponds to formula (1.24) in Lemma 1.4. *)
polyMax[p_, a_, b_, dx_] := Module[{d, pValues, M2, Lam, Nsub, coeffList},
  If[! PolynomialQ[p, x], Return["Error: p must be a polynomial in x."];];
  coeffList = CoefficientList[p, x];
  d = Exponent[p, x];
  Nsub = Ceiling[(b - a) / dx];
  If[6 Nsub^2 - d^2 (d^2 - 1) ≤ 0, Return["Error: lambda must be positive."];];
  pValues =
   ParallelTable[FromDigits[Reverse[coeffList], a + k * dx], {k, 0, Nsub}];
  M2 = Max[Abs[pValues]];
  Lam = 6 Nsub^2 / (6 Nsub^2 - d^2 (d^2 - 1));
  Lam * M2]
CAD[poly_, var_, a_, b_] :=
 Module[{cond}, cond = Reduce[ForAll[var, a ≤ var ≤ b, poly > 0], Reals];
  Return[cond];]
(*The following functions are used in Lemma 4.4*)
H[x1_, x2_, x3_, x4_, y1_, y2_, y3_, y4_, p1_, p2_] :=
  4 * p1 * x3^2 * y1^2 + 4 * p1 * p2 * x2 * x4 * y1^2 - 4 * p1 * p2 * x1 * x4 * y1 * y2 -
   p2^2 * x4^2 * y2^2 - 8 * p1 * x1 * x3 * y1 * y3 - 4 * p2 * x3 * x4 * y2 * y3 +
   4 * p1 * x1^2 * y3^2 + 4 * p2 * x2 * x4 * y3^2 - 4 * p1 * p2 * x1 * x2 * y1 * y4 +
   4 * p1 * p2 * x1^2 * y2 * y4 + 4 * p2 * x3^2 * y2 * y4 +
   2 * p2^2 * x2 * x4 * y2 * y4 - 4 * p2 * x2 * x3 * y3 * y4 - p2^2 * x2^2 * y4^2;
HM[x1_, x2_, x3_, x4_, y1_, y2_, y3_, y4_, p1_, p2_] :=
  4 * p1 * x3^2 * y1^2 + 4 * p1 * p2 * x2 * x4 * y1^2 + 4 * p1 * p2 * x1 * x4 * y1 * y2 +
   p2^2 * x4^2 * y2^2 + 8 * p1 * x1 * x3 * y1 * y3 + 4 * p2 * x3 * x4 * y2 * y3 +
   4 * p1 * x1^2 * y3^2 + 4 * p2 * x2 * x4 * y3^2 + 4 * p1 * p2 * x1 * x2 * y1 * y4 +
   4 * p1 * p2 * x1^2 * y2 * y4 + 4 * p2 * x3^2 * y2 * y4 +
   2 * p2^2 * x2 * x4 * y2 * y4 + 4 * p2 * x2 * x3 * y3 * y4 + p2^2 * x2^2 * y4^2;
```

Toy Examples

toy example 1

In[23]:=
```
CAD[3 x P2[x] + 3 (W1[x] - 75 × 10^(-6))^2 - 1, x, 0, 1/2] > 0
```

Out[23]=
```
True > 0
```

toy example 2

In[24]:=
```
Clear[U, Ud, U2d, p, R];
T = 3 / 10;
U[x_, t_] :=
 t - 31 / 5 t^3 + 1 / 3 t^4 + 472 / 21 t^5 - 25 t^6 + (-1 / 6 t^3 + 21 / 20 t^5 - 8 / 7 t^6) x;
Ud[x_, t_] := Derivative[0, 1][U][x, t];
U2d[x_, t_] := Derivative[0, 2][U][x, t];
R[x_, t_] := U2d[x, t] + (x + 2 Qanew[t] - 1) U[x, t];
p = Integrate[R[x, t]^2, {t, 0, T}];
Ud[x, T]
2 × 75 / 10^5 T + T^(1 / 2) (polyMax[p, 144 / 10, 192 / 10, 1 / 10])^(1 / 2) < 7 / 10^3
```

Out[28]=
$$-\frac{1291}{14\,000}-\frac{26\,793\,x}{1\,400\,000}$$

Out[29]=
```
True
```

Case l=0: Theorem 5.1

Proposition 5.2: $\tau \in [12.4, 19.2]$

In[30]:=
```
Clear[Uaapp, Vaapp, Ubapp, Vbapp, Uad, Vad, Ubd, Vbd, Ua2d, Va2d, Ub2d, Vb2d];
Uaapp[τ_, t_] :=
  (4550 * t^7) / 131 - (5787 * t^6) / 91 + (3986 * t^5) / 105 - (334 * t^4) / 211 -
   (727 * t^3) / 120 + (t^2) / 242 + t + ((58 * t^7) / 49 - (305 * t^6) / 129 +
      (90 * t^5) / 59 - (6 * t^4) / 61 - (10 * t^3) / 63 - (t^2) / 3170) * τ;
Vaapp[τ_, t_] :=
  - (2251 * t^7) / 81 + (4841 * t^6) / 99 - (11 060 * t^5) / 403 + (89 * t^4) / 102 +
   (429 * t^3) / 136 - (t^2) / 140 + ((-9 * t^7) / 92 + (17 * t^6) / 162 + (5 * t^5) / 111 -
      (3 * t^4) / 133 + (t^3) / 260 - (t^2) / 3409) * τ;
Ubapp[τ_, t_] :=
  - (3423 * t^7) / 124 + (4105 * t^6) / 84 - (3577 * t^5) / 130 + (72 * t^4) / 79 +
   (129 * t^3) / 41 - (t^2) / 155 + ((5 * t^7) / 83 - (t^6) / 52 - (4 * t^5) / 95 +
      (t^4) / 61 - (t^3) / 382 + (t^2) / 5233) * τ;
Vbapp[τ_, t_] :=
  (5468 * t^7) / 155 - (8018 * t^6) / 121 + (3078 * t^5) / 79 - (406 * t^4) / 225 -
   (1417 * t^3) / 235 + (t^2) / 461 + t + ((-241 * t^7) / 443 + (288 * t^6) / 253 -
      (95 * t^5) / 148 + (15 * t^4) / 374 + (61 * t^3) / 372 + (t^2) / 15 381) * τ;
Uad[x_, t_] := Derivative[0, 1][Uaapp][x, t];
Vad[x_, t_] := Derivative[0, 1][Vaapp][x, t];
Ubd[x_, t_] := Derivative[0, 1][Ubapp][x, t];
Vbd[x_, t_] := Derivative[0, 1][Vbapp][x, t];
Ua2d[x_, t_] := Derivative[0, 2][Uaapp][x, t];
Va2d[x_, t_] := Derivative[0, 2][Vaapp][x, t];
Ub2d[x_, t_] := Derivative[0, 2][Ubapp][x, t];
Vb2d[x_, t_] := Derivative[0, 2][Vbapp][x, t];
```

(5.2)

In[43]:=
```
Clear[pa, pb, x, es, du, tauL, tauR, Ya1, Ya2, Ya3, Ya4, Yb1, Yb2, Yb3, Yb4, T];
tauL = 124 / 10;
tauR = 192 / 10;
pa = 1 / 10;
pb = 427 / 100;
es = 45 / 10^3;
du = 2 * 10^-2;
T = 475 / 1000;
Ya1[x_] := Uaapp[x, T];
Ya2[x_] := Vaapp[x, T]; Ya3[x_] := Uad[x, T];
Ya4[x_] := Vad[x, T];
Yb1[x_] := Ubapp[x, T]; Yb2[x_] := Vbapp[x, T]; Yb3[x_] := Ubd[x, T];
Yb4[x_] := Vbd[x, T];
Hepsilon[x_] := Simplify[
   H[Ya1[x], Ya2[x], Ya3[x], Ya4[x], Yb1[x], Yb2[x], Yb3[x], Yb4[x], pa, pb] -
    (HM[Abs[Ya1[x]] + es, Abs[Ya2[x]] + es, Abs[Ya3[x]] + es, Abs[Ya4[x]] + es,
       Abs[Yb1[x]] + es, Abs[Yb2[x]] + es, Abs[Yb3[x]] + es, Abs[Yb4[x]] + es,
       pa, pb] - HM[Abs[Ya1[x]], Abs[Ya2[x]], Abs[Ya3[x]], Abs[Ya4[x]],
       Abs[Yb1[x]], Abs[Yb2[x]], Abs[Yb3[x]], Abs[Yb4[x]], pa, pb])];
Ya2[x] - es
Ya4[x] - es
HepsilonExpanded = Expand[Hepsilon[x]];
CAD[HepsilonExpanded, x, tauL, tauR]
```

Out[48]=
$$\frac{576\,038\,932\,877\,554\,217}{7\,000\,831\,918\,080\,000\,000}+\frac{1\,919\,017\,510\,830\,489\,821\,x}{2\,002\,003\,641\,630\,720\,000\,000}$$

Out[49]=
$$\frac{58\,118\,833\,252\,035\,821}{175\,020\,797\,952\,000\,000}+\frac{191\,526\,365\,774\,652\,211\,x}{16\,683\,363\,680\,256\,000\,000}$$

Out[51]= True

(5.3)

In[52]:=
```
Clear[tauL, tauR, dx, T, Za, Zb, x, Ra, Rb, A1, A2, B1, B2];
tauL = 124 / 10;
tauR = 192 / 10;
dx = 2 * 10^-2;
T = 475 / 1000;
A1[x_, t_] := Ua2d[x, t] - (1 - x - 2 Qanew[t]) Uaapp[x, t] - Qanew[t] × Vaapp[x, t];
A2[x_, t_] := Va2d[x, t] - (1 + x - 2 Qanew[t]) Vaapp[x, t] - Qanew[t] × Uaapp[x, t];
B1[x_, t_] := Ub2d[x, t] - (1 - x - 2 Qanew[t]) Ubapp[x, t] - Qanew[t] × Vbapp[x, t];
B2[x_, t_] := Vb2d[x, t] - (1 + x - 2 Qanew[t]) Vbapp[x, t] - Qanew[t] × Ubapp[x, t];
Za = Integrate[ExpandAll[A1[x, t]^2 + A2[x, t]^2], {t, 0, T}];
Zb = Integrate[ExpandAll[B1[x, t]^2 + B2[x, t]^2], {t, 0, T}];
Ra = polyMax[Za, tauL, tauR, dx];
Rb = polyMax[Zb, tauL, tauR, dx];
Max[FU[Ra], FU[Rb]]
```

Out[61]= $\frac{14\,039}{18\,384\,000}$

(5.4)

In[62]:=
```
k = √(tauR + 1);
{1/2 √((2 k T + Sinh[2 k T])/k) < 1493/1000,
 (3 T Sinh[k T])/(2 k) < 6690/10^4, 1493/1000 √(14 039/18 384 000) + 6690/10^4 × 75/10^5 ≤ 418/10^4}
```

Out[62]= {True, True, True}

(5.5)

In[63]:=
```
dx/2 Max[Abs[Ya1'[tauL]], Abs[Ya1'[tauR]], Abs[Ya2'[tauL]], Abs[Ya2'[tauR]],
  Abs[Ya3'[tauL]], Abs[Ya3'[tauR]], Abs[Ya4'[tauL]], Abs[Ya4'[tauR]],
  Abs[Yb1'[tauL]], Abs[Yb1'[tauR]], Abs[Yb2'[tauL]], Abs[Yb2'[tauR]],
  Abs[Yb3'[tauL]], Abs[Yb3'[tauR]], Abs[Yb4'[tauL]], Abs[Yb4'[tauR]]] < 1/1000
```

Out[63]= True

(5.1)

In[64]:=
$$k = \sqrt{tauR + 1}\ ;$$
$$\frac{1}{2}\sqrt{\frac{2\,k\,T + Sinh[2\,k\,T]}{k}}\ \sqrt{Max[FU[Ra], FU[Rb]]} + \frac{3\,T\,Sinh[k\,T]}{2\,k}\ \frac{75}{10^5} + \frac{1}{20}\ dx < es$$

Out[65]= True

Proposition C.1: $\tau \in [5.4, 12.4]$

In[66]:=
```
Clear[Uaapp, Vaapp, Ubapp, Vbapp, Uad, Vad, Ubd, Vbd, Ua2d, Va2d, Ub2d, Vb2d];
Uaapp[τ_, t_] := (250 858 * t^10) / 4155 - (42 245 * t^9) / 226 + (7428 * t^8) / 35 -
  (8522 * t^7) / 109 - (9443 * t^6) / 240 + (11 239 * t^5) / 283 - (698 * t^4) / 265 -
  (1186 * t^3) / 203 - (10 * t^2) / 749 + t + (5 * t^10 / 402 - 14 * t^9 / 341 + 7 * t^8 / 135 -
     5 * t^7 / 214 - t^6 / 96 + t^5 / 81 - t^4 / 1295 + t^3 / 12 292) * τ^2 +
  (303 * t^10 / 272 - 642 * t^9 / 179 + 843 * t^8 / 199 - 111 * t^7 / 68 -
     207 * t^6 / 244 + 584 * t^5 / 629 - 26 * t^4 / 435 - 48 * t^3 / 299 - t^2 / 3075) * τ;
Vaapp[τ_, t_] := -(13 012 * t^10) / 297 + (50 423 * t^9) / 387 - (35 597 * t^8) / 261 +
   (11 461 * t^7) / 333 + (13 129 * t^6) / 321 - (20 636 * t^5) / 669 + (431 * t^4) / 177 +
   (255 * t^3) / 88 + (2 * t^2) / 167 + (t^10 / 5317 - t^9 / 336 + 4 * t^8 / 435 -
      t^7 / 90 + 3 * t^6 / 389 - t^5 / 459 + t^4 / 2762 - t^3 / 28 556) * τ^2 +
   (15 * t^10 / 256 - 99 * t^9 / 232 + 493 * t^8 / 450 - 534 * t^7 / 389 +
      248 * t^6 / 285 - 25 * t^5 / 102 + 7 * t^4 / 172 - t^3 / 255 + t^2 / 4980) * τ;
Ubapp[τ_, t_] :=
  -(3355 * t^10) / 78 + (51 257 * t^9) / 401 - (55 000 * t^8) / 413 + (7825 * t^7) / 244 +
   (12 499 * t^6) / 298 - (11 800 * t^5) / 379 + (236 * t^4) / 95 + (1727 * t^3) / 597 +
   (4 * t^2) / 327 + (-t^10 / 167 + 5 * t^9 / 288 - 5 * t^8 / 249 + 5 * t^7 / 466 -
      t^6 / 503 + t^5 / 2013 - t^4 / 10 812 + t^3 / 94 132) * τ^2 + ((103 * t^10) / 334 -
      (122 * t^9) / 165 + (95 * t^8) / 201 + (35 * t^7) / 172 - (115 * t^6) / 337 +
      (23 * t^5) / 240 - (6 * t^4) / 409 + (t^3) / 805 - (t^2) / 18 830) * τ;
Vbapp[τ_, t_] :=
  (18 556 * t^10) / 313 - (107 695 * t^9) / 587 + (66 232 * t^8) / 319 - (23 047 * t^7) / 308 -
   (13 337 * t^6) / 328 + (7012 * t^5) / 175 - (167 * t^4) / 62 - (1249 * t^3) / 214 -
   (4 * t^2) / 293 + t + ((14 * t^10) / 741 - (18 * t^9) / 301 + (13 * t^8) / 171 -
      (7 * t^7) / 166 + (2 * t^6) / 375 + (t^5) / 132 + (t^4) / 12 270) * τ^2 +
   ((-376 * t^10) / 245 + (634 * t^9) / 129 - (2809 * t^8) / 465 +
      (1457 * t^7) / 491 + (21 * t^6) / 85 - (268 * t^5) / 353 +
      (t^4) / 33 + (35 * t^3) / 214 + (t^2) / 6317) * τ;
Uad[x_, t_] := Derivative[0, 1][Uaapp][x, t];
Vad[x_, t_] := Derivative[0, 1][Vaapp][x, t];
Ubd[x_, t_] := Derivative[0, 1][Ubapp][x, t];
Vbd[x_, t_] := Derivative[0, 1][Vbapp][x, t];
Ua2d[x_, t_] := Derivative[0, 2][Uaapp][x, t];
Va2d[x_, t_] := Derivative[0, 2][Vaapp][x, t];
Ub2d[x_, t_] := Derivative[0, 2][Ubapp][x, t];
Vb2d[x_, t_] := Derivative[0, 2][Vbapp][x, t];
```

(C.2)

In[78]:=
```
Clear[pa, pb, x, es, du, tauL, tauR, Ya1, Ya2, Ya3, Ya4, Yb1, Yb2, Yb3, Yb4, T];
tauL = 54 / 10;
tauR = 124 / 10;
pa = 96 / 10;
pb = 343 / 100;
es = 26 / 10^3;
du = 10^-2;
T = 2 / 3;
Ya1[x_] := Uaapp[x, T];
Ya2[x_] := Vaapp[x, T]; Ya3[x_] := Uad[x, T];
Ya4[x_] := Vad[x, T];
Yb1[x_] := Ubapp[x, T]; Yb2[x_] := Vbapp[x, T]; Yb3[x_] := Ubd[x, T];
Yb4[x_] := Vbd[x, T];
Hepsilon[x_] :=
  H[Ya1[x], Ya2[x], Ya3[x], Ya4[x], Yb1[x], Yb2[x], Yb3[x], Yb4[x], pa, pb] -
   (HM[Abs[Ya1[x]] + es, Abs[Ya2[x]] + es, Abs[Ya3[x]] + es, Abs[Ya4[x]] + es,
      Abs[Yb1[x]] + es, Abs[Yb2[x]] + es, Abs[Yb3[x]] + es, Abs[Yb4[x]] + es,
      pa, pb] - HM[Abs[Ya1[x]], Abs[Ya2[x]], Abs[Ya3[x]], Abs[Ya4[x]],
      Abs[Yb1[x]], Abs[Yb2[x]], Abs[Yb3[x]], Abs[Yb4[x]], pa, pb]);
Ya2[x] - es
Ya4[x] - es
HepsilonExpanded = Expand[Hepsilon[x]];
CAD[HepsilonExpanded, x, tauL, tauR]
```

Out[83]=
$$\frac{16\,148\,372\,893\,594\,385\,391\,157}{95\,118\,348\,656\,680\,395\,205\,500} + \frac{65\,331\,293\,804\,303\,x}{19\,197\,701\,089\,755\,075} + \frac{1\,778\,959\,623\,075\,792\,502\,x^2}{20\,776\,616\,952\,736\,362\,382\,665}$$

Out[84]=
$$\frac{7\,609\,339\,494\,027\,734\,729\,969}{31\,706\,116\,218\,893\,465\,068\,500} + \frac{150\,313\,914\,413\,207\,x}{6\,399\,233\,696\,585\,025} + \frac{1\,115\,287\,103\,240\,575\,649\,x^2}{1\,385\,107\,796\,849\,090\,825\,511}$$

Out[86]= True

(C.3)

In[87]:=
```
Max[Abs[Ya1'[tauL]], Abs[Ya1'[tauR]], Abs[Ya2'[tauL]], Abs[Ya2'[tauR]],
  Abs[Ya3'[tauL]], Abs[Ya3'[tauR]], Abs[Ya4'[tauL]], Abs[Ya4'[tauR]],
  Abs[Yb1'[tauL]], Abs[Yb1'[tauR]], Abs[Yb2'[tauL]], Abs[Yb2'[tauR]],
  Abs[Yb3'[tauL]], Abs[Yb3'[tauR]], Abs[Yb4'[tauL]], Abs[Yb4'[tauR]]] < 2/5
```

Out[87]= True

(C.4)

In[88]:=
```
Clear[tauL, tauR, dx, T, Za, Zb, x, Ra, Rb, A1, A2, B1, B2];
tauL = 54 / 10;
tauR = 124 / 10;
T = 2 / 3;
dx = 10^-2;
A1[x_, t_] := Ua2d[x, t] - (1 - x - 2 Qanew[t]) Uaapp[x, t] - Qanew[t] × Vaapp[x, t];
A2[x_, t_] := Va2d[x, t] - (1 + x - 2 Qanew[t]) Vaapp[x, t] - Qanew[t] × Uaapp[x, t];
B1[x_, t_] := Ub2d[x, t] - (1 - x - 2 Qanew[t]) Ubapp[x, t] - Qanew[t] × Vbapp[x, t];
B2[x_, t_] := Vb2d[x, t] - (1 + x - 2 Qanew[t]) Vbapp[x, t] - Qanew[t] × Ubapp[x, t];
Za = Integrate[ExpandAll[A1[x, t]^2 + A2[x, t]^2], {t, 0, T}];
Zb = Integrate[ExpandAll[B1[x, t]^2 + B2[x, t]^2], {t, 0, T}];
Ra = polyMax[Za, tauL, tauR, dx];
Rb = polyMax[Zb, tauL, tauR, dx];
k = √(tauR + 1 );
1/2 √((2 k T + Sinh[2 k T])/k) √(Max[FU[Ra], FU[Rb]]) + (3 T Sinh[k T])/(2 k) 75/10^5 + 1/5 dx < es
```

Out[98]= True

Proposition C.1: $\tau \in [2.1, 5.4]$

In[99]:=
```
Clear[Uaapp, Vaapp, Ubapp, Vbapp, Uad, Vad, Ubd, Vbd, Ua2d, Va2d, Ub2d, Vb2d];
Uaapp[τ_, t_] := (42673 * t^12) / 692 - (1055462 * t^11) / 3005 + (424795 * t^10) / 488 -
  (435706 * t^9) / 359 + (419355 * t^8) / 413 - (130882 * t^7) / 269 +
  (20712 * t^6) / 211 + (3913 * t^5) / 423 + (405 * t^4) / 241 - (4633 * t^3) / 746 +
  t^2 / 278 + t + ((89 * t^12) / 67 - (1981 * t^11) / 262 + (4426 * t^10) / 235 -
     (21136 * t^9) / 795 + (5435 * t^8) / 239 - (6325 * t^7) / 559 + (437 * t^6) / 176 +
     (23 * t^5) / 135 + (15 * t^4) / 296 - (95 * t^3) / 558 + t^2 / 7198) * τ +
  ((-2 * t^12) / 231 + (13 * t^11) / 401 - t^10 / 23 + (5 * t^9) / 276 + (4 * t^8) / 451 -
     t^7 / 383 - (5 * t^6) / 328 + (3 * t^5) / 227 - t^4 / 1153 + t^3 / 11554) * τ^2;
Vaapp[τ_, t_] := -(26307 * t^12) / 491 +
   (81288 * t^11) / 271 - (257262 * t^10) / 353 + (290879 * t^9) / 293 -
   (214088 * t^8) / 265 + (91013 * t^7) / 243 - (19248 * t^6) / 263 -
   (1193 * t^5) / 211 - (267 * t^4) / 238 + (1981 * t^3) / 619 - t^2 / 519 +
   ((-233 * t^12) / 354 + (1250 * t^11) / 367 - (1958 * t^10) / 259 + (3649 * t^9) / 394 -
      (6277 * t^8) / 944 + (888 * t^7) / 329 - (228 * t^6) / 409 + (21 * t^5) / 244 -
      (3 * t^4) / 347 + t^3 / 1861 - t^2 / 53354) * τ +
   ((3 * t^12) / 152 - (20 * t^11) / 207 + (58 * t^10) / 279 - (20 * t^9) / 77 + (101 * t^8) /
       481 - (23 * t^7) / 200 + (9 * t^6) / 206 - t^5 / 96 + t^4 / 636 - t^3 / 7055) * τ^2;
Ubapp[τ_, t_] := -(23768 * t^12) / 441 +
   (30452 * t^11) / 101 - (91512 * t^10) / 125 + (987850 * t^9) / 991 -
   (236014 * t^8) / 291 + (129403 * t^7) / 344 - (160111 * t^6) / 2171 -
   (4746 * t^5) / 859 - (397 * t^4) / 348 + (1284 * t^3) / 401 - t^2 / 498 +
   ((179 * t^12) / 278 - (2323 * t^11) / 697 + (2951 * t^10) / 399 - (1077 * t^9) / 119 +
      (1509 * t^8) / 233 - (521 * t^7) / 200 + (34 * t^6) / 65 - (15 * t^5) / 193 +
      (3 * t^4) / 409 - (t^3) / 2412 + (t^2) / 79649) * τ +
   ((-4 * t^12) / 423 + (16 * t^11) / 343 - (39 * t^10) / 392 + (38 * t^9) / 319 -
      (19 * t^8) / 219 + (27 * t^7) / 703 - (2 * t^6) / 215 +
      t^5 / 592 - (t^4) / 4961 + (t^3) / 67049) * τ^2;
Vbapp[τ_, t_] := (27683 * t^12) / 447 - (214333 * t^11) / 608 + (289907 * t^10) / 332 -
   (320070 * t^9) / 263 + (845926 * t^8) / 831 - (18539 * t^7) / 38 +
   (25544 * t^6) / 259 + (1335 * t^5) / 146 + (765 * t^4) / 451 - (733 * t^3) / 118 +
   t^2 / 273 + t + ((-369 * t^12) / 280 + (4729 * t^11) / 630 - (4883 * t^10) / 261 +
      (7160 * t^9) / 271 - (3865 * t^8) / 171 + (3944 * t^7) / 351 - (547 * t^6) / 223 -
      (148 * t^5) / 831 - (4 * t^4) / 81 + (74 * t^3) / 435 - t^2 / 7602) * τ +
   ((t^12) / 72 - (74 * t^11) / 961 + (79 * t^10) / 420 - (107 * t^9) / 408 +
      (74 * t^8) / 331 - (22 * t^7) / 197 + (8 * t^6) / 323 +
      (t^5) / 255 + (t^4) / 1959 - (t^3) / 27802) * τ^2;
Uad[x_, t_] := Derivative[0, 1][Uaapp][x, t];
Vad[x_, t_] := Derivative[0, 1][Vaapp][x, t];
Ubd[x_, t_] := Derivative[0, 1][Ubapp][x, t];
Vbd[x_, t_] := Derivative[0, 1][Vbapp][x, t];
Ua2d[x_, t_] := Derivative[0, 2][Uaapp][x, t];
Va2d[x_, t_] := Derivative[0, 2][Vaapp][x, t];
Ub2d[x_, t_] := Derivative[0, 2][Ubapp][x, t];
Vb2d[x_, t_] := Derivative[0, 2][Vbapp][x, t];
```

(C.2)

In[111]:=
```
Clear[pa, pb, x, es, du, tauL, tauR, Ya1, Ya2, Ya3, Ya4, Yb1, Yb2, Yb3, Yb4, T];
tauL = 21 / 10;
tauR = 54 / 10;
pa = 4;
pb = 182 / 100;
es = 2 / 100;
du = 10^-2;
T = 856 / 1000;
Ya1[x_] := Uaapp[x, T];
Ya2[x_] := Vaapp[x, T]; Ya3[x_] := Uad[x, T];
Ya4[x_] := Vad[x, T];
Yb1[x_] := Ubapp[x, T]; Yb2[x_] := Vbapp[x, T]; Yb3[x_] := Ubd[x, T];
Yb4[x_] := Vbd[x, T];
Hepsilon[x_] :=
  H[Ya1[x], Ya2[x], Ya3[x], Ya4[x], Yb1[x], Yb2[x], Yb3[x], Yb4[x], pa, pb] -
   (HM[Abs[Ya1[x]] + es, Abs[Ya2[x]] + es, Abs[Ya3[x]] + es, Abs[Ya4[x]] + es,
      Abs[Yb1[x]] + es, Abs[Yb2[x]] + es, Abs[Yb3[x]] + es, Abs[Yb4[x]] + es,
      pa, pb] - HM[Abs[Ya1[x]], Abs[Ya2[x]], Abs[Ya3[x]], Abs[Ya4[x]],
      Abs[Yb1[x]], Abs[Yb2[x]], Abs[Yb3[x]], Abs[Yb4[x]], pa, pb]);
Ya2[x] - es
Ya4[x] - es
HepsilonExpanded = Expand[Hepsilon[x]];
CAD[HepsilonExpanded, x, tauL, tauR]
```

Out[116]=
$$\frac{388\,004\,538\,452\,058\,137\,925\,231\,684\,439\,910\,672\,125\,848\,431\,312\,467}{1\,823\,968\,464\,669\,914\,729\,574\,724\,027\,770\,571\,410\,655\,975\,341\,796\,875} + \frac{634\,300\,122\,037\,847\,394\,619\,437\,192\,697\,618\,587\,669\,892\,669\,303\,x}{60\,187\,947\,459\,956\,398\,825\,480\,369\,850\,993\,156\,433\,105\,468\,750\,000} + \frac{1\,951\,295\,907\,924\,724\,951\,569\,378\,317\,240\,701\,952\,759\,x^2}{5\,398\,963\,974\,127\,235\,356\,718\,301\,773\,071\,289\,062\,500\,000}$$

Out[117]=
$$\frac{3\,591\,219\,254\,307\,886\,508\,964\,726\,317\,789\,110\,225\,309\,291\,294\,669}{29\,183\,495\,434\,718\,635\,673\,195\,584\,444\,329\,142\,570\,495\,605\,468\,750} + \frac{294\,473\,063\,498\,780\,853\,920\,597\,919\,226\,384\,486\,105\,103\,447\,x}{5\,732\,185\,472\,376\,799\,888\,140\,987\,604\,856\,491\,088\,867\,187\,500} + \frac{334\,816\,193\,089\,149\,066\,307\,615\,506\,196\,634\,565\,607\,x^2}{129\,575\,135\,379\,053\,648\,561\,239\,242\,553\,710\,937\,500\,000}$$

Out[119]= True

(C.3)

In[120]:=
```
Max[Abs[Ya1'[tauL]], Abs[Ya1'[tauR]], Abs[Ya2'[tauL]], Abs[Ya2'[tauR]],
  Abs[Ya3'[tauL]], Abs[Ya3'[tauR]], Abs[Ya4'[tauL]], Abs[Ya4'[tauR]],
  Abs[Yb1'[tauL]], Abs[Yb1'[tauR]], Abs[Yb2'[tauL]], Abs[Yb2'[tauR]],
  Abs[Yb3'[tauL]], Abs[Yb3'[tauR]], Abs[Yb4'[tauL]], Abs[Yb4'[tauR]]] < 2/5
```

Out[120]= True

(C.4)

In[121]:=
```
Clear[tauL, tauR, dx, T, Za, Zb, x, Ra, Rb, A1, A2, B1, B2];
tauL = 21 / 10;
tauR = 54 / 10;
T = 856 / 1000;
dx = 10^-2;
A1[x_, t_] := Ua2d[x, t] - (1 - x - 2 Qanew[t]) Uaapp[x, t] - Qanew[t] × Vaapp[x, t];
A2[x_, t_] := Va2d[x, t] - (1 + x - 2 Qanew[t]) Vaapp[x, t] - Qanew[t] × Uaapp[x, t];
B1[x_, t_] := Ub2d[x, t] - (1 - x - 2 Qanew[t]) Ubapp[x, t] - Qanew[t] × Vbapp[x, t];
B2[x_, t_] := Vb2d[x, t] - (1 + x - 2 Qanew[t]) Vbapp[x, t] - Qanew[t] × Ubapp[x, t];
Za = Integrate[ExpandAll[A1[x, t]^2 + A2[x, t]^2], {t, 0, T}];
Zb = Integrate[ExpandAll[B1[x, t]^2 + B2[x, t]^2], {t, 0, T}];
Ra = polyMax[Za, tauL, tauR, dx];
Rb = polyMax[Zb, tauL, tauR, dx];
k = √(tauR + 1);
1/2 √((2 k T + Sinh[2 k T])/k) √(Max[FU[Ra], FU[Rb]]) + (3 T Sinh[k T])/(2 k) 75/10^5 + 1/5 dx < es
```

Out[131]= True

Proposition C.1: $\tau \in [1, 1.2]$

In[132]:=
```
Clear[Uaapp, Vaapp, Ubapp, Vbapp, Uad, Vad, Ubd, Vbd, Ua2d, Va2d, Ub2d, Vb2d];
Uaapp[τ_, t_] := t + (112 * t^2) / 7071 - (84 069 * t^3) / 12 941 + (19 306 * t^4) / 3625 -
  (156 439 * t^5) / 8122 + (4 529 108 * t^6) / 18 517 - (12 407 469 * t^7) / 12 391 +
  (13 802 505 * t^8) / 6041 - (19 938 272 * t^9) / 5805 + (53 183 415 * t^10) / 14 657 -
  (42 924 858 * t^11) / 15 571 + (8 404 358 * t^12) / 5591 - (3 526 473 * t^13) / 6148 +
  (1 591 789 * t^14) / 11 029 - (278 264 * t^15) / 13 315 + (3326 * t^16) / 3039 +
  (2680 * t^17) / 49 349 + (-(249 * t^17) / 5389 + (4374 * t^16) / 6893 - (15 398 * t^15) /
      3823 + (219 958 * t^14) / 14 015 - (212 295 * t^13) / 5051 + (688 472 * t^12) / 8401 -
    (1 117 303 * t^11) / 9315 + (1 588 706 * t^10) / 11 909 - (4 482 105 * t^9) / 39 941 +
    (501 013 * t^8) / 7201 - (359 491 * t^7) / 12 021 + (30 645 * t^6) / 3956 -
    (5614 * t^5) / 6493 + (1923 * t^4) / 10 346 - (1271 * t^3) / 7015 + (3 * t^2) / 4790) *
   τ + ((54 * t^17) / 7553 - (1012 * t^16) / 11 019 + (6161 * t^15) / 11 484 -
```

```
    (39 423 * t^14) / 20 906 + (260 558 * t^13) / 58 587 - (121 265 * t^12) / 16 324 +
    (57 598 * t^11) / 6371 - (64 982 * t^10) / 7999 + (68 601 * t^9) / 12 683 -
    (17 790 * t^8) / 6677 + (7547 * t^7) / 7746 - (2944 * t^6) / 10 785 +
    (2152 * t^5) / 35 229 - (33 * t^4) / 4823 + (7 * t^3) / 12 667 - (t^2) / 40 643) * τ^2;
Vaapp[τ_, t_] := (3681 * t^17) / 15 061 - (57 636 * t^16) / 11 473 +
  (825 829 * t^15) / 18 764 - (57 112 156 * t^14) / 255 125 +
  (11 597 999 * t^13) / 15 502 - (20 590 648 * t^12) / 11 777 +
  (25 621 273 * t^11) / 8685 - (51 191 939 * t^10) / 14 066 + (39 140 407 * t^9) / 11 979 -
  (5 210 419 * t^8) / 2505 + (13 541 181 * t^7) / 15 350 - (352 048 * t^6) / 1631 +
  (183 459 * t^5) / 8386 - (43 947 * t^4) / 9515 + (43 703 * t^3) / 12 583 -
  (205 * t^2) / 15 119 + ((271 * t^17) / 5454 - (6838 * t^16) / 10 085 +
     (52 723 * t^15) / 12 435 - (43 681 * t^14) / 2714 + (243 056 * t^13) / 5871 -
     (2 308 060 * t^12) / 30 281 + (771 033 * t^11) / 7463 -
     (949 681 * t^10) / 9110 + (1 067 194 * t^9) / 13 673 - (126 229 * t^8) / 2964 +
     (127 476 * t^7) / 7783 - (33 624 * t^6) / 7837 + (12 196 * t^5) / 15 311 -
     (531 * t^4) / 5368 + (89 * t^3) / 11 587 - (4 * t^2) / 12 041) * τ +
  ((-80 * t^17) / 14 649 + (945 * t^16) / 13 466 - (221 639 * t^15) / 540 245 +
     (12 380 * t^14) / 8571 - (232 353 * t^13) / 67 993 + (74 064 * t^12) / 12 901 -
     (58 461 * t^11) / 8284 + (65 299 * t^10) / 10 126 - (41 500 * t^9) / 9413 +
     (4898 * t^8) / 2173 - (13 581 * t^7) / 15 874 + (2569 * t^6) / 10 804 -
     (305 * t^5) / 6601 + (56 * t^4) / 9319 - (2 * t^3) / 4113 + t^2 / 46 045) * τ^2;
Ubapp[τ_, t_] := (2463 * t^17) / 9715 - (36 251 * t^16) / 7047 +
  (285 916 * t^15) / 6391 - (2 011 509 * t^14) / 8881 + (8 515 714 * t^13) / 11 285 -
  (23 298 569 * t^12) / 13 241 + (119 068 063 * t^11) / 40 167 -
  (44 459 792 * t^10) / 12 171 + (21 775 652 * t^9) / 6645 - (17 028 738 * t^8) / 8167 +
  (8 508 863 * t^7) / 9624 - (1 251 442 * t^6) / 5783 + (792 161 * t^5) / 36 032 -
  (54 532 * t^4) / 11 771 + (34 531 * t^3) / 9939 - (156 * t^2) / 11 465 +
  ((68 * t^17) / 6929 - (745 * t^16) / 7468 + (6177 * t^15) / 15 824 -
     (6535 * t^14) / 12 444 - (4174 * t^13) / 3315 + (66 375 * t^12) / 9109 -
     (146 303 * t^11) / 8824 + (212 006 * t^10) / 9193 - (201 460 * t^9) / 9491 +
     (43 291 * t^8) / 3347 - (88 537 * t^7) / 17 825 + (19 981 * t^6) / 17 933 -
     (1831 * t^5) / 10 383 + (147 * t^4) / 8275 - (35 * t^3) / 33 337 + (t^2) / 32 146) * τ +
  ((-28 * t^17) / 9343 + (365 * t^16) / 9599 - (2944 * t^15) / 13 441 +
     (12 991 * t^14) / 17 140 - (32 497 * t^13) / 18 494 + (24 623 * t^12) / 8543 -
     (16 321 * t^11) / 4738 + (17 317 * t^10) / 5685 - (8489 * t^9) / 4231 +
     (90 568 * t^8) / 91 895 - (2930 * t^7) / 8137 + (1761 * t^6) / 18 083 -
     (143 * t^5) / 7848 + (162 * t^4) / 71 441 - (t^3) / 5726 + (t^2) / 135 052) * τ^2;
Vbapp[τ_, t_] := (281 * t^17) / 6789 + (29 119 * t^16) / 23 022 -
  (134 735 * t^15) / 6146 + (1 565 941 * t^14) / 10 578 - (6 744 679 * t^13) / 11 576 +
  (9 325 931 * t^12) / 6140 - (21 688 406 * t^11) / 7811 + (11 901 372 * t^10) / 3263 -
  (36 827 911 * t^9) / 10 681 + (28 814 971 * t^8) / 12 573 -
  (11 522 451 * t^7) / 11 476 + (609 949 * t^6) / 2486 - (47 380 * t^5) / 2441 +
  (53 649 * t^4) / 10 037 - (70 099 * t^3) / 10 788 + (243 * t^2) / 15 278 + t +
  (-(431 * t^17) / 12 307 + (1891 * t^16) / 4423 - (17 301 * t^15) / 7532 +
     (79 657 * t^14) / 11 355 - (75 969 * t^13) / 5918 + (128 952 * t^12) / 10 477 +
     (15 653 * t^11) / 11 987 - (186 559 * t^10) / 8361 + (394 344 * t^9) / 11 453 -
     (395 025 * t^8) / 13 636 + (260 073 * t^7) / 18 221 - (29 944 * t^6) / 8821 +
```

```
      (271 * t^5) / 18 282 - (807 * t^4) / 10 802 + (1133 * t^3) / 6583 - (2 * t^2) / 9289) *
    τ + ((27 * t^17) / 6227 - (1706 * t^16) / 30 739 + (4549 * t^15) / 14 101 -
      (14 721 * t^14) / 13 070 + (52 013 * t^13) / 19 762 - (52 482 * t^12) / 12 091 +
      (30 786 * t^11) / 5933 - (64 433 * t^10) / 14 173 + (24 681 * t^9) / 8450 -
      (10 701 * t^8) / 7799 + (2498 * t^7) / 5227 - (973 * t^6) / 7295 +
      (267 * t^5) / 7960 - (22 * t^4) / 6887 + (3 * t^3) / 11 911 - (t^2) / 91 156) * τ^2;
Uad[x_, t_] := Derivative[0, 1][Uaapp][x, t];
Vad[x_, t_] := Derivative[0, 1][Vaapp][x, t];
Ubd[x_, t_] := Derivative[0, 1][Ubapp][x, t];
Vbd[x_, t_] := Derivative[0, 1][Vbapp][x, t];
Ua2d[x_, t_] := Derivative[0, 2][Uaapp][x, t];
Va2d[x_, t_] := Derivative[0, 2][Vaapp][x, t];
Ub2d[x_, t_] := Derivative[0, 2][Ubapp][x, t];
Vb2d[x_, t_] := Derivative[0, 2][Vbapp][x, t];
```

(C.2)

In[144]:=
```
Clear[pa, pb, x, es, du, tauL, tauR, Ya1, Ya2, Ya3, Ya4, Yb1, Yb2, Yb3, Yb4, T];
tauL = 1;
tauR = 12 / 10;
pa = 2 / 10;
pb = 1 / 10;
es = 26 / 10^3;
du = 2 * 10^-3;
T = 16 / 10;
Ya1[x_] := Uaapp[x, T];
Ya2[x_] := Vaapp[x, T]; Ya3[x_] := Uad[x, T];
Ya4[x_] := Vad[x, T];
Yb1[x_] := Ubapp[x, T]; Yb2[x_] := Vbapp[x, T]; Yb3[x_] := Ubd[x, T];
Yb4[x_] := Vbd[x, T];
Hepsilon[x_] :=
  H[Ya1[x], Ya2[x], Ya3[x], Ya4[x], Yb1[x], Yb2[x], Yb3[x], Yb4[x], pa, pb] -
   (HM[Abs[Ya1[x]] + es, Abs[Ya2[x]] + es, Abs[Ya3[x]] + es, Abs[Ya4[x]] + es,
      Abs[Yb1[x]] + es, Abs[Yb2[x]] + es, Abs[Yb3[x]] + es, Abs[Yb4[x]] + es,
      pa, pb] - HM[Abs[Ya1[x]], Abs[Ya2[x]], Abs[Ya3[x]], Abs[Ya4[x]],
      Abs[Yb1[x]], Abs[Yb2[x]], Abs[Yb3[x]], Abs[Yb4[x]], pa, pb]);
Ya2[x] - es
Ya4[x] - es
HepsilonExpanded = Expand[Hepsilon[x]];
CAD[HepsilonExpanded, x, tauL, tauR]
```

Out[149]=
$$\frac{37\,389\,545\,769\,645\,188\,674\,446\,457\,332\,298\,893\,599\,825\,742\,634\,640\,074\,118\,988\,011}{135\,616\,875\,730\,386\,557\,401\,158\,096\,210\,574\,201\,584\,799\,282\,583\,010\,864\,257\,812\,500} + \frac{286\,204\,174\,242\,596\,089\,786\,963\,739\,818\,741\,905\,069\,601\,949\,948\,402\,920\,821\,473\,024\,x}{2\,591\,197\,925\,650\,411\,899\,040\,947\,550\,886\,665\,341\,473\,746\,454\,793\,665\,313\,720\,703\,125} + \frac{2\,077\,197\,586\,997\,254\,312\,491\,594\,821\,611\,639\,677\,503\,650\,972\,545\,310\,024\,273\,183\,589\,056\,x^2}{140\,459\,747\,475\,274\,958\,091\,636\,208\,362\,930\,928\,722\,114\,856\,630\,794\,324\,534\,759\,521\,484\,375}$$

Out[150]=
$$\frac{3\,937\,443\,944\,856\,404\,806\,116\,904\,737\,870\,809\,558\,377\,184\,175\,148\,176\,171\,744\,821}{37\,972\,725\,204\,508\,236\,072\,324\,266\,938\,960\,776\,443\,743\,799\,123\,243\,041\,992\,187\,500} + \frac{23\,837\,863\,680\,267\,123\,897\,780\,207\,246\,925\,155\,847\,676\,054\,994\,764\,657\,142\,435\,328\,x}{103\,647\,917\,026\,016\,475\,961\,637\,902\,035\,466\,613\,658\,949\,858\,191\,746\,612\,548\,828\,125} + \frac{179\,556\,008\,848\,835\,130\,967\,686\,964\,733\,392\,257\,908\,813\,625\,508\,505\,477\,302\,451\,255\,728\,x^2}{3\,121\,327\,721\,672\,776\,846\,480\,804\,630\,287\,353\,971\,602\,552\,369\,573\,207\,211\,883\,544\,921\,875}$$

Out[152]= True

(C.3)

In[153]:=
```
Max[Abs[Ya1'[tauL]], Abs[Ya1'[tauR]], Abs[Ya2'[tauL]], Abs[Ya2'[tauR]],
  Abs[Ya3'[tauL]], Abs[Ya3'[tauR]], Abs[Ya4'[tauL]], Abs[Ya4'[tauR]],
  Abs[Yb1'[tauL]], Abs[Yb1'[tauR]], Abs[Yb2'[tauL]], Abs[Yb2'[tauR]],
  Abs[Yb3'[tauL]], Abs[Yb3'[tauR]], Abs[Yb4'[tauL]], Abs[Yb4'[tauR]]] < 2/5
```

Out[153]= True

(C.4)

In[154]:=
```
Clear[tauL, tauR, dx, T, Za, Zb, x, Ra, Rb, A1, A2, B1, B2];
tauL = 1;
tauR = 12 / 10;
T = 16 / 10;
dx = 2 * 10^-3;
A1[x_, t_] := Ua2d[x, t] - (1 - x - 2 Qanew[t]) Uaapp[x, t] - Qanew[t] × Vaapp[x, t];
A2[x_, t_] := Va2d[x, t] - (1 + x - 2 Qanew[t]) Vaapp[x, t] - Qanew[t] × Uaapp[x, t];
B1[x_, t_] := Ub2d[x, t] - (1 - x - 2 Qanew[t]) Ubapp[x, t] - Qanew[t] × Vbapp[x, t];
B2[x_, t_] := Vb2d[x, t] - (1 + x - 2 Qanew[t]) Vbapp[x, t] - Qanew[t] × Ubapp[x, t];
Za = Integrate[ExpandAll[A1[x, t]^2 + A2[x, t]^2], {t, 0, T}];
Zb = Integrate[ExpandAll[B1[x, t]^2 + B2[x, t]^2], {t, 0, T}];
Ra = polyMax[Za, tauL, tauR, dx];
Rb = polyMax[Zb, tauL, tauR, dx];
k = √(tauR + 1);
1/2 √((2 k T + Sinh[2 k T])/k) √(Max[FU[Ra], FU[Rb]]) + (3 T Sinh[k T])/(2 k) 75/10^5 + 1/5 dx < es
```

Out[164]= True

Proposition C.1: $\tau \in [1.2, 2.1]$

(C.2)

In[165]:=
```
Clear[pa, pb, x, es, du, tauL, tauR, Ya1, Ya2, Ya3, Ya4, Yb1, Yb2, Yb3, Yb4, T];
tauL = 12 / 10;
tauR = 21 / 10;
pa = 1;
pb = 1 / 2;
es = 23 / 10^3;
du = 10^-2;
T = 11 / 10;
Ya1[x_] := Uaapp[x, T];
Ya2[x_] := Vaapp[x, T]; Ya3[x_] := Uad[x, T];
Ya4[x_] := Vad[x, T];
Yb1[x_] := Ubapp[x, T]; Yb2[x_] := Vbapp[x, T]; Yb3[x_] := Ubd[x, T];
Yb4[x_] := Vbd[x, T];
Hepsilon[x_] :=
  H[Ya1[x], Ya2[x], Ya3[x], Ya4[x], Yb1[x], Yb2[x], Yb3[x], Yb4[x], pa, pb] -
   (HM[Abs[Ya1[x]] + es, Abs[Ya2[x]] + es, Abs[Ya3[x]] + es, Abs[Ya4[x]] + es,
      Abs[Yb1[x]] + es, Abs[Yb2[x]] + es, Abs[Yb3[x]] + es, Abs[Yb4[x]] + es,
      pa, pb] - HM[Abs[Ya1[x]], Abs[Ya2[x]], Abs[Ya3[x]], Abs[Ya4[x]],
      Abs[Yb1[x]], Abs[Yb2[x]], Abs[Yb3[x]], Abs[Yb4[x]], pa, pb]);
Ya2[x] - es
Ya4[x] - es
HepsilonExpanded = Expand[Hepsilon[x]];
CAD[HepsilonExpanded, x, tauL, tauR]
```

Out[170]=
$$\frac{1\,368\,358\,052\,173\,157\,750\,784\,980\,624\,005\,615\,999\,838\,604\,825\,302\,259\,045\,863\,775\,739}{5\,842\,835\,553\,631\,319\,870\,482\,351\,516\,979\,088\,666\,926\,168\,339\,925\,000\,000\,000\,000\,000} + \frac{163\,410\,003\,294\,769\,333\,102\,434\,735\,655\,859\,004\,318\,360\,771\,458\,304\,708\,840\,433\,800\,211\,x}{5\,613\,776\,768\,774\,393\,197\,208\,183\,097\,352\,347\,101\,448\,709\,013\,598\,600\,000\,000\,000\,000\,000} + \left(3\,985\,331\,314\,462\,767\,651\,047\,384\,677\,836\,349\,008\,462\,896\,331\,667\,229\,785\,864\,787\,038\,273\,921\,x^2\right) / \left(2\,630\,048\,574\,439\,891\,329\,569\,563\,014\,649\,440\,384\,209\,291\,212\,615\,924\,815\,060\,000\,000\,000\,000\,000\,000\right)$$

Out[171]=
$$\frac{12\,580\,749\,316\,047\,438\,413\,369\,276\,419\,853\,848\,666\,648\,834\,170\,655\,715\,923\,401\,671}{233\,713\,422\,145\,252\,794\,819\,294\,060\,679\,163\,546\,677\,046\,733\,597\,000\,000\,000\,000\,000} + \frac{11\,305\,754\,343\,616\,303\,782\,502\,875\,718\,739\,463\,369\,907\,852\,699\,271\,822\,451\,768\,667\,733\,x}{112\,275\,535\,375\,487\,863\,944\,163\,661\,947\,046\,942\,028\,974\,180\,271\,972\,000\,000\,000\,000\,000} + \left(180\,484\,421\,136\,231\,189\,769\,838\,350\,553\,022\,421\,215\,491\,022\,865\,640\,143\,777\,829\,928\,834\,293\,x^2\right) / \left(21\,917\,071\,453\,665\,761\,079\,746\,358\,455\,412\,003\,201\,744\,093\,438\,466\,040\,125\,500\,000\,000\,000\,000\right)$$

Out[173]= True

(C.3)

In[174]:=
```
Max[Abs[Ya1'[tauL]], Abs[Ya1'[tauR]], Abs[Ya2'[tauL]], Abs[Ya2'[tauR]],
  Abs[Ya3'[tauL]], Abs[Ya3'[tauR]], Abs[Ya4'[tauL]], Abs[Ya4'[tauR]],
  Abs[Yb1'[tauL]], Abs[Yb1'[tauR]], Abs[Yb2'[tauL]], Abs[Yb2'[tauR]],
  Abs[Yb3'[tauL]], Abs[Yb3'[tauR]], Abs[Yb4'[tauL]], Abs[Yb4'[tauR]]] < 2/5
```

Out[174]= True

(C.4)

In[175]:=
```
Clear[tauL, tauR, dx, T, Za, Zb, x, Ra, Rb, A1, A2, B1, B2];
tauL = 12 / 10;
tauR = 21 / 10;
T = 11 / 10;
dx = 10^-2;
A1[x_, t_] := Ua2d[x, t] - (1 - x - 2 Qanew[t]) Uaapp[x, t] - Qanew[t] × Vaapp[x, t];
A2[x_, t_] := Va2d[x, t] - (1 + x - 2 Qanew[t]) Vaapp[x, t] - Qanew[t] × Uaapp[x, t];
B1[x_, t_] := Ub2d[x, t] - (1 - x - 2 Qanew[t]) Ubapp[x, t] - Qanew[t] × Vbapp[x, t];
B2[x_, t_] := Vb2d[x, t] - (1 + x - 2 Qanew[t]) Vbapp[x, t] - Qanew[t] × Ubapp[x, t];
Za = Integrate[ExpandAll[A1[x, t]^2 + A2[x, t]^2], {t, 0, T}];
Zb = Integrate[ExpandAll[B1[x, t]^2 + B2[x, t]^2], {t, 0, T}];
Ra = polyMax[Za, tauL, tauR, dx];
Rb = polyMax[Zb, tauL, tauR, dx];
k = √(tauR + 1);
1/2 √((2 k T + Sinh[2 k T])/k) √(Max[FU[Ra], FU[Rb]]) + (3 T Sinh[k T])/(2 k) 75/10^5 + 1/5 dx < es
```

Out[185]= True

Case l=1: Section 7

(7.15), please check toy example 1

(7.18)

In[186]:=
```
CAD[2 + 867/100 x^2 + 3 x^3 P2[x], x, 1/2, 1]
```

Out[186]= True

(7.21)

In[187]:=
```
CAD[2 + 3 x^2 + 3 x^3 P2[x], x, 1, 17/10]
```

Out[187]= True

(7.34)

In[188]:=
```
Clear[y1, y];
y1[t_] := -3 t^8/341 + 46 t^7/525 - 487 t^6/1404 + 1801 t^5/2640 - 2669 t^4/4312 + 81 t^2/185;
y[t_] := t^2 y1[t];
CAD[ExpandAll[(y''[x] - 2 y1[x] - x^2 (W1[x] + 75 / 10^6)) / x^2], x, 0, 17 / 10]
```

Out[189]= True

(7.35)

In[190]:=
```
CAD[ExpandAll[(2 / x^2 + 3 x P2[x]) y1[x] + W1[x] - 75 / 10^6], x, 0, 17 / 10]
```

Out[190]= True

l=1: Section 8

Check $l_-(t)>0$, $t\in[0, 1.8]$

In[191]:=
```
gtilde[t_] := 29 / 5 + (5324 t^2) / 11 721 - (5242 t^3) / 6305 - (453 401 t^4) / 8690 -
    (1 372 577 t^5) / 5985 + (11 561 100 t^6) / 5101 - (104 060 861 t^7) / 8909 +
    (269 766 735 t^8) / 5476 - (1 847 962 994 t^9) / 11 735 + (3 127 972 553 t^10) / 8468 -
    (117 464 741 599 t^11) / 181 890 + (272 712 958 583 t^12) / 316 809 -
    (188 840 245 805 t^13) / 212 143 + (59 891 505 057 t^14) / 83 009 -
    (82 501 815 384 t^15) / 179 359 + (88 378 047 540 t^16) / 384 383 -
    (13 934 342 915 t^17) / 156 279 + (22 540 646 527 t^18) / 857 154 -
    (4 889 665 718 t^19) / 857 587 + (1 193 343 423 t^20) / 1 392 527 -
    (61 031 641 t^21) / 765 161 + (2 655 007 t^22) / 766 661;
Iminus[t_] := t^2 (gtilde[t] - t / 1600);
Iplus[t_] := t^2 (gtilde[t] + t / 1600);
Itilde[t_] := t^2 gtilde[t];
CAD[gtilde[x] - x / 1600, x, 0, 18 / 10]
```

Out[195]= True

(8.3)

In[196]:=
```
Clear[p, x];
P2plus[t_] := (W1[t] - 75 × 10^(-6)) (W1'[t] + 42 × 10^(-5));
p = Simplify[ExpandAll[(Iplus''[x] * x^2 * Iminus[x] - (2 + 3 x^3 P2plus[x]) *
        Iplus[x] * Iminus[x] - x^4 (W1[x] + 75 / 10^6) * Iplus[x]) / x^5]];
CAD[p, x, 0, 12 / 10]
Clear[p, x];
p = Simplify[ExpandAll[(-Iminus''[x] * x^2 * Iplus[x] + (2 + 3 x^3 P2[x]) *
        Iplus[x] * Iminus[x] + x^4 (W1[x] - 75 / 10^6) * Iminus[x]) / x^5]];
CAD[p, x, 0, 12 / 10]
```

Out[197]= True

Out[199]= True

(8.4)

In[200]:=
```
Max[Abs[Itilde'[6 / 5] - Iplus'[6 / 5] / Iplus[6 / 5] * Iminus[6 / 5]],
  Abs[Itilde'[6 / 5] - Iminus'[6 / 5] / Iminus[6 / 5] * Iplus[6 / 5]]] < 62 / 10^4
```

Out[200]= True

(8.5)

In[201]:=
```
(* Check 1st inequality in (8.5) *)
Fminus[t_] :=
 Itilde''[t] - 2 gtilde[t] - 3 t * P2plus[t] * Itilde[t] - t^2 (W1[t] + 75 / 10^6)
Fplus[t_] :=
  Itilde''[t] - 2 gtilde[t] - 3 t * P2[t] * Itilde[t] - t^2 (W1[t] - 75 / 10^6);
CAD[Fminus[x], x, 12 / 10, 18 / 10]
{CAD[39 / 10^4 - Fminus[x], x, 12 / 10, 18 / 10],
 CAD[39 / 10^4 + Fminus[x], x, 12 / 10, 18 / 10],
 CAD[39 / 10^4 - Fplus[x], x, 12 / 10, 18 / 10],
 CAD[39 / 10^4 + Fplus[x], x, 12 / 10, 18 / 10]}
```

Out[203]= True

Out[204]= {True, True, True, True}

In[205]:=
```
(* Check 2nd inequality in (8.5) *)
Clear[p]; p = 7 / 5 * x^2 - 2 - 3 x^3 P2plus[x]; CAD[p, x, 12 / 10, 18 / 10]
Clear[p];
p = 7 / 5 * x^2 + 2 + 3 x^3 P2[x];  CAD[p, x, 12 / 10, 18 / 10]
```

Out[205]= True

Out[206]= True

(8.7)

In[207]:=
```
Clear[t, y];
DSolve[{y''[t] == 7 / 5 y[t] + 39 / 10^4 ,
  y[6 / 5] == (12 / 10)^3 / 1600, y'[6 / 5] == 62 / 10^4}, y[t], t]
```

Out[207]=
$$\left\{\left\{y[t] \to \frac{1}{700\,000}\, e^{-\frac{6\sqrt{\frac{7}{5}}}{5}-\sqrt{\frac{7}{5}}\,t}\left(1353\, e^{\frac{12\sqrt{\frac{7}{5}}}{5}} - 310\sqrt{35}\, e^{\frac{12\sqrt{\frac{7}{5}}}{5}} + 1353\, e^{2\sqrt{\frac{7}{5}}\,t} + 310\sqrt{35}\, e^{2\sqrt{\frac{7}{5}}\,t} - 1950\, e^{\frac{6\sqrt{\frac{7}{5}}}{5}+\sqrt{\frac{7}{5}}\,t}\right)\right\}\right\}$$

In[208]:=
```
Clear[yz];
yz[t_] := 1 / 700 000 E^(-6 Sqrt[7/5] / 5 - Sqrt[7/5] t)
  (1353 E^(12 Sqrt[7/5] / 5) - 310 Sqrt[35] E^(12 Sqrt[7/5] / 5) + 1353 E^(2 Sqrt[7/5] t) + 310 Sqrt[35] E^(2 Sqrt[7/5] t) - 1950 E^(6 Sqrt[7/5] / 5 + Sqrt[7/5] t));
{yz[9 / 5] < 62 / 10^4, yz'[9 / 5] < 114 / 10^4}
```

Out[209]= {True, True}

(8.8)

In[210]:=
```
CAD[-2 - 3 x^3 P2plus[x], x, 12 / 10, 16 / 10]
Integrate[-2 / t^2 - 3 t P2[t], {t, 12 / 10, 16 / 10}] < 244 / 10^3
{CAD[3 / 20 x^2 + 2 + 3 x^3 P2[x], x, 16 / 10, 18 / 10],
 CAD[3 / 20 x^2 - 2 - 3 x^3 P2[x], x, 16 / 10, 18 / 10]}
{CAD[3 / 20 x^2 + 2 + 3 x^3 P2plus[x], x, 16 / 10, 18 / 10],
 CAD[3 / 20 x^2 - 2 - 3 x^3 P2plus[x], x, 16 / 10, 18 / 10]}
Integrate[Fplus[t], {t, 12 / 10, 18 / 10}] < 165 / 10^5
62 / 10^4 + (244 / 10^3 + 2 / 10 * 15 / 100) * 62 / 10^4 + 165 / 10^5 < 96 / 10^4
```

Out[210]= True

Out[211]= True

Out[212]= {True, True}

Out[213]= {True, True}

Out[214]= True

Out[215]= True

(8.9)

In[216]:=
```
Integrate[t^2 (Itilde[t] - 62 / 10^4) (W1[t] - 75 / 10^6), {t, 0, 18 / 10}] > 107 / 100
```

Out[216]= True

$$J_+\left(\frac{9}{5}\right) \geq I\left(\frac{9}{5}\right) \geq J_-\left(\frac{9}{5}\right),\ J'_+\left(\frac{9}{5}\right) > I'\left(\frac{9}{5}\right) \geq J'_-\left(\frac{9}{5}\right).$$

In[217]:=
```
Clear[Jp, Jm];
Jp[x_] := -46 355/4182 + 162 617 x/2916 - 215 987 x^2/2017 + 428 192 x^3/3543 - 38 323 x^4/412 + 104 356 x^5/2011 -
  37 198 x^6/1725 + 20 181 x^7/2999 - 5962 x^8/3809 + 923 x^9/3491 - 17 x^10/556 + 11 x^11/5279 - x^12/27 017 - x^13/189 311 + x^14/3 302 419;
Jm[x_] :=
  -12 413/1105 + 171 747 x/3038 - 174 823 x^2/1612 + 47 397 503 x^3/388 000 - 307 187 x^4/3277 + 540 785 x^5/10 386 - 28 489 x^6/1325 +
   20 281 x^7/3051 - 5023 x^8/3296 + 1218 x^9/4849 - 187 x^10/6760 + 4 x^11/2441 + x^12/121 108 - x^13/124 156 + x^14/2 629 395;
{Jp[9/5] > Iplus[9/5] > Iminus[9/5] > Jm[9/5],
 Jp'[9/5] > Iplus'[9/5] > Iminus'[9/5] > Jm'[9/5]}
```

Out[219]= {True, True}

(8.11) and note that $J_m(t)<J_p(t)$, then $-J_p(t)<-J_m(t)$

In[220]:=
```
{83/100 < -Jp[35/10] < -Jm[35/10] < 1051/1000, 16/100 < -Jp'[35/10] < -Jm'[35/10] < 57/100}
```

Out[220]= {True, True}

(8.13)

In[221]:=
```
Clear[x, Bt1, Bt2, pt1, pt2, pt3, pt4];
Bt1 = (W1'[x] + 5 / 10^5) (W1[x] - 6 / 10^6);
Bt2 = (W1'[x] - 5 / 10^5) (W1[x] + 6 / 10^6);
pt1 = x^2 Jp''[x] - 2 Jp[x] - 3 x^3 Bt1 * Jp[x] - x^4 (W1[x] + 6 / 10^6);
pt2 = x^2 Jp''[x] - 2 Jp[x] - 3 x^3 Bt2 * Jp[x] - x^4 (W1[x] + 6 / 10^6);
pt3 = x^2 Jm''[x] - 2 Jm[x] - 3 x^3 Bt1 * Jm[x] - x^4 (W1[x] - 6 / 10^6);
pt4 = x^2 Jm''[x] - 2 Jm[x] - 3 x^3 Bt2 * Jm[x] - x^4 (W1[x] - 6 / 10^6);
{CAD[pt1, x, 18 / 10, 25 / 10], CAD[pt2, x, 18 / 10, 25 / 10],
 CAD[-pt3, x, 18 / 10, 25 / 10], CAD[-pt4, x, 18 / 10, 25 / 10]}
```

Out[226]= {True, True, True, True}

Check $J_{\pm}$<0 on [2.5, 3.5]

In[227]:=
```
{CAD[-Jp[x], x, 25 / 10, 35 / 10], CAD[-Jm[x], x, 25 / 10, 35 / 10]}
```
Out[227]= {True, True}

(8.14) and (8.15)

In[228]:=
```
Clear[x, Ct1, Ct2, pt1, pt2];
Ct1 = (-W2'[x] + 5 / 10^5) (W2[x] + 6 / 10^6);
Ct2 = (-W2'[x] - 5 / 10^5) (W2[x] - 6 / 10^6);
pt1 = x^2 Jp''[x] - 2 Jp[x] + 3 x^3 Ct1 * Jp[x] - x^4 (W2[x] + 6 / 10^6);
pt2 = x^2 Jm''[x] - 2 Jm[x] + 3 x^3 Ct2 * Jm[x] - x^4 (W2[x] - 6 / 10^6);
{CAD[pt1, x, 25 / 10, 35 / 10], CAD[-pt2, x, 25 / 10, 35 / 10]}
```
Out[232]= {True, True}

(8.16)

In[233]:=
```
{CAD[106/100 - Jm[x], x, 9 / 5, 7 / 2], CAD[106/100 + Jm[x], x, 9 / 5, 7 / 2]}

Integrate[Jm[t] t^2 W1[t], {t, 9/5, 5/2}] + Integrate[Jm[t] t^2 W2[t], {t, 5/2, 7/2}] -
  6/10^6 * 106/100 Integrate[t^2, {t, 9/5, 7/2}] > -401/1000
```
Out[233]= {True, True}

Out[234]= True

$H_{-}(t)$ in Lemma 8.3

In[235]:=
```
Clear[t];
DSolve[{Hm''[t] == 18525 / 10^4 * 1 / t^2 Hm[t] - 28 / 10 t Exp[-t],
  Hm[7 / 2] == 83 / 100, Hm'[7 / 2] == 16 / 100}, Hm[t], t]
```
Out[235]=
$$\left\{\left\{Hm[t] \to \frac{1}{568\,400\, t^{19/20}}\left(103\,733 \times 2^{1/20} \times 7^{19/20} + 10\,788 \times 2^{19/20} \times 7^{1/20}\, t^{29/10} - 548\,800\, t^{29/10}\, \text{Gamma}\left[\frac{21}{20}, \frac{7}{2}\right] + 548\,800\, t^{29/10}\, \text{Gamma}\left[\frac{21}{20}, t\right] + 548\,800\, \text{Gamma}\left[\frac{79}{20}, \frac{7}{2}\right] - 548\,800\, \text{Gamma}\left[\frac{79}{20}, t\right]\right)\right\}\right\}$$

(8.19)

In[236]:=
$$10\,788 \times 2^{19/20} \times 7^{1/20} - 548\,800\ \text{Gamma}\left[\frac{21}{20}, \frac{7}{2}\right] > 0$$

Out[236]= True

$H_+(t)$ in Lemma 8.3

In[237]:=
```
Clear[t];
DSolve[{Hp''[t] == 2 / t^2 Hp[t] - 27 / 10 t Exp[-t],
  Hp[7 / 2] == 1051 / 1000, Hp'[7 / 2] == 57 / 100}, Hp[t], t]
```

Out[237]=
$$\left\{\left\{\text{Hp}[t] \to \frac{1}{294\,000\,t} e^{-\frac{7}{2}-t}\left(-1\,587\,600\,e^{7/2} + 28\,212\,975\,e^{t} + 36\,701\,e^{\frac{7}{2}+t} - 1\,587\,600\,e^{7/2}\,t - 793\,800\,e^{7/2}\,t^2 - 264\,600\,e^{t}\,t^3 + 24\,368\,e^{\frac{7}{2}+t}\,t^3\right)\right\}\right\}$$

In[238]:=
$$\text{Integrate}\left[\text{ExpandAll}\left[\frac{1}{294\,000\,t} e^{-\frac{7}{2}-t}\left(-1\,587\,600\,e^{7/2} + 28\,212\,975\,e^{t} + 36\,701\,e^{\frac{7}{2}+t} - 1\,587\,600\,e^{7/2}\,t - 793\,800\,e^{7/2}\,t^2 - 264\,600\,e^{t}\,t^3 + 24\,368\,e^{\frac{7}{2}+t}\,t^3\right) t\ 19 / 7\ \text{Exp}[-t]\right], \{t, 7/2, \text{Infinity}\}\right] < \frac{65}{100}$$

Out[238]= True

Proposition 8.4

In[239]:=
$$\frac{107}{100} - \frac{401}{1000} + \frac{65}{100} > 0$$

Out[239]= True

Appendix A

(A .6)

In[240]:=
```
Clear[p]; p = Qanew[x];
CAD[p, x, 0, 8 / 5]
Clear[p]; p = -Simplify[Qanew'[x] / x];
CAD[p, x, 0, 8 / 5]
```

Out[241]= True

Out[243]= True

(A .8)

In[244]:=
```
(*Check the first inequality in (A.8) via Corollary 1.5*)
Clear[p];
p = 75 * 10^(-5) q1[x]^2 + (q1[x]^2 × Qanew[x] - (1 - 7 * 10^(-5) q1[x])^2);
CAD[p, x, 0, 9 / 10]
Clear[p];
p = 75 * 10^(-5) q1[x]^2 - ( q1[x]^2 × Qanew[x] - (1 - 7 * 10^(-5) q1[x])^2);
CAD[p, x, 0, 9 / 10]
```

Out[245]= True

Out[247]= True

In[248]:=
```
(*Check the second inequality in (A.8) via Corollary 1.5*)
Clear[p];
p = 75 * 10^(-5) p2[x]^2 + ( p2[x]^2 × Qanew[x] - (1 - 7 * 10^(-5) p2[x])^2);
CAD[p, x, 9 / 10, 8 / 5]
Clear[p];
p = 75 * 10^(-5) p2[x]^2 - ( p2[x]^2 × Qanew[x] - (1 - 7 * 10^(-5) p2[x])^2);
CAD[p, x, 9 / 10, 8 / 5]
```

Out[249]= True

Out[251]= True

(A .9)

In[252]:=
```
(*Check the first inequality in (A.9) via Corollary 1.5*)
Clear[p];
p = 75 * 10^(-5) q1[x]^2 + (q1[x]^2 × Qanew[x] - (1 + 7 * 10^(-5) q1[x])^2);
CAD[p, x, 0, 9 / 10]
Clear[p];
p = 75 * 10^(-5) q1[x]^2 - (q1[x]^2 × Qanew[x] - (1 + 7 * 10^(-5) q1[x])^2);
CAD[p, x, 0, 9 / 10]
```

Out[253]= True

Out[255]= True

In[256]:=
```
(*Check the second inequality in (A.9) via Corollary 1.5*)
Clear[p];
p = 75 * 10^(-5) p2[x]^2 + (p2[x]^2 × Qanew[x] - (1 + 7 * 10^(-5) p2[x])^2);
CAD[p, x, 9 / 10, 8 / 5]
Clear[p];
p = 75 * 10^(-5) p2[x]^2 - (p2[x]^2 × Qanew[x] - (1 + 7 * 10^(-5) p2[x])^2);
CAD[p, x, 9 / 10, 8 / 5]
```

Out[257]= True

Out[259]= True

(A .10)

In[260]:=
```
(*Check the first inequality in (A.10) via Corollary 1.5*)
Clear[p]; p = 32 * 10^(-7) q1[x] + (1 - q1[x] × W1[x]);
CAD[p, x, 0, 9 / 10]
Clear[p]; p = 32 * 10^(-7) q1[x] - (1 - q1[x] × W1[x]);
CAD[p, x, 0, 9 / 10]
```

Out[261]= True

Out[263]= True

In[264]:=
```
(*Check the second inequality in (A .10) via Corollary 1.5*)
Clear[p]; p = 32 * 10^(-7) p2[x] + (1 - p2[x] × W1[x]);
CAD[p, x, 9 / 10, 5 / 2]
Clear[p]; p = 32 * 10^(-7) p2[x] - (1 - p2[x] × W1[x]);
CAD[p, x, 9 / 10, 5 / 2]
```

Out[265]= True

Out[267]= True

(A .11)

In[268]:=
```
(*Check the first inequality in (A.11) via Corollary 1.5*)
Clear[p]; p = 57 * 10^(-6) q1[x]^2 + (q1'[x] + q1[x]^2 × W1'[x]);
CAD[p, x, 0, 9 / 10]
Clear[p]; p = 57 * 10^(-6) q1[x]^2 - (q1'[x] + q1[x]^2 × W1'[x]);
CAD[p, x, 0, 9 / 10]
```

Out[269]= True

Out[271]= True

In[272]:=
```
(*Check the second inequality in (A.11) via Corollary 1.5*)
Clear[p]; p = 57 * 10^(-6) p2[x]^2 + (p2'[x] + p2[x]^2 × W1'[x]);
CAD[p, x, 9 / 10, 5 / 2]
Clear[p]; p = 57 * 10^(-6) p2[x]^2 - (p2'[x] + p2[x]^2 × W1'[x]);
CAD[p, x, 9 / 10, 5 / 2]
```

Out[273]= True

Out[275]= True

(A .13)

In[276]:=
```
(*Check the first inequality in (A.13) via Corollary 1.5*)
Clear[p]; p = 18 * 10^(-7) * p2[x] + (1 - p2[x] × W1[x]);
CAD[p, x, 18 / 10, 5 / 2]
Clear[p]; p = 18 * 10^(-7) * p2[x] - (1 - p2[x] × W1[x]);
CAD[p, x, 18 / 10, 5 / 2]
```

Out[277]= True

Out[279]= True

In[280]:=
```
(*Check the second inequality in (A.13) via Corollary 1.5*)
Clear[p]; p = 29 * 10^(-6) * p2[x]^2 + (p2'[x] + p2[x]^2 × W1'[x]);
CAD[p, x, 18 / 10, 5 / 2]
Clear[p]; p = 29 * 10^(-6) * p2[x]^2 - (p2'[x] + p2[x]^2 × W1'[x]);
CAD[p, x, 18 / 10, 5 / 2]
```

Out[281]= True

Out[283]= True

(A .14)

In[284]:=
```
Clear[p];
p = 48 * 10^(-7) * x + (x W2''[x] + 2 W2'[x] - x W2[x] + x W2[x]^3);
CAD[p, x, 5 / 2, 7 / 2]
Clear[p];
p = 48 * 10^(-7) * x - (x W2''[x] + 2 W2'[x] - x W2[x] + x W2[x]^3);
CAD[p, x, 5 / 2, 7 / 2]
```

Out[285]= True

Out[287]= True

(A .15)

In[288]:=
```
{CAD[W2[x], x, 5 / 2, 7 / 2], CAD[9 / 100 - W2[x], x, 5 / 2, 7 / 2]}
```

Out[288]= {True, True}

(A.18)

In[289]:=
$$\frac{48 \times 10^{-7}}{\left(1 - 2\,\frac{243}{10\,000} - \frac{81}{10\,000}\right)} < \frac{51}{10^7}$$

Out[289]= True

$$\max_{\frac{5}{2} \le t \le \frac{7}{2}} \left| \eta(t) \right| < \frac{5}{10^6}$$

In[290]:=
$$\frac{3 \times \frac{9}{100}\left(\frac{51}{10^7}\right)^2 + \left(\frac{51}{10^7}\right)^3 + 48 \times 10^{-7}}{1 - 3\left(\frac{243}{10\,000}\right)^2} < \frac{5}{10^6}$$

Out[290]= True

(A.19)

In[291]:=
$$\frac{48 \times 10^{-7}}{1 - \frac{243}{10\,000} - \frac{81}{10\,000}} < \frac{5}{10^6}$$

Out[291]= True

(A.20)

In[292]:=
$$\frac{83}{10^7}+\frac{109}{147}\left(\frac{5}{10^6}+\frac{48}{10^7}\right)\leq\frac{156}{10^7}$$

Out[292]= True

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In[293]:=
$$\left(\frac{7}{2}\right)^3\left(\text{W2}'\left[\frac{7}{2}\right]-\frac{156}{10^7}\right)-\left(\frac{7}{2}\right)^2\left(\text{W2}\left[\frac{7}{2}\right]+\frac{5}{10^6}\right)>-2$$

Out[293]= True

(A.21)

In[294]:=
$$\left\{\left(\frac{17}{10}\right)^3\left(\text{W1}'\left[\frac{17}{10}\right]-\frac{42}{10^5}\right)-\left(\frac{17}{10}\right)^2\left(\text{W1}\left[\frac{17}{10}\right]+\frac{75}{10^6}\right)>-\frac{31}{10},\ \left(\frac{17}{10}\right)^2\left(1-\left(\text{W1}\left[\frac{17}{10}\right]+\frac{75}{10^6}\right)^2\right)-2>0\right\}$$

$$\text{W1}\left[\frac{17}{10}\right]+\frac{75}{10^6}<\frac{\frac{93}{10}-\sqrt{\left(-\frac{93}{10}\right)^2-4\times\frac{867}{100}\times2}}{2\times\frac{867}{100}}$$

Out[294]= {True, True}

Out[295]= True

(A.22)

In[296]:=
$$2-6\left(\text{W2}\left[\frac{7}{2}\right]+156\times10\text{^}(-7)\right)>\frac{18\,525}{10\,000}$$

Out[296]= True

(A .23)

In[297]:=
```
Clear[p]; p = 2 + x^3 (W1[x] + 75 * 10^(-6)) (W1'[x] - 42 * 10^(-5));
CAD[p, x, 0, 17 / 10]
```

Out[298]= True

Appendix B

Case 1≤τ≤1.2

(B.4)

In[299]:=
```
Clear[A]; A = 3/10^4; 1 + 10/13 (-12872/10000) > 98/10000
(31/100 - 2 (W2[7/2] + 5 × 10^(-6))^2 - 1/(2 A (26/10)^2) (W2[7/2] + 5 × 10^(-6))^4) -
 A/4 (W2[7/2] + 5 × 10^(-6))^2 > 0
```

Out[299]= True

Out[300]= True

(B.5)

In[301]:=
```
Clear[A];
A = 3/10^4;
31/100 - 2 (W1[5/2] + 75 × 10^(-6))^2 - 1/(2 A (26/10)^2) (W1[5/2] + 75 × 10^(-6))^4 > 27/100
```

Out[301]= True

(B.6)

In[302]:=
```
Clear[af, p];
af = 1 / 2441 Sum[1 / k! (-13 / 5 (x - 3))^k, {k, 0, 3}] ;
p =
  W2[x] - 5 × 10^(-6) + 2 (10 / 26 - 630 / 26 af) (W2'[x] - 156 × 10^(-7)) - 2 × 10^(-7);
CAD[p, x, 5 / 2, 7 / 2]
```

Out[305]= True

$\frac{1}{2}AQ\left(\frac{8}{5}\right)^4 < \frac{3}{10^6}$ before (B.9)

In[306]:=
```
Clear[A]; A = 3/10^4; 1/2 A (W1[8/5] + 75 × 10^(-6))^4 < 3/10^6
```

Out[306]= True

(B.9)

In[307]:=
```
Clear[a1f, p, pnew];
a1f = 73 / 15 070 Sum[1 / k! (-13 / 5 (x - 41 / 20))^k, {k, 0, 5}];
p = P1[x] + (10 / 26 - 630 / 26 a1f) 4 P2[x];
A = 3 × 10^(-4);
pnew = 31 / 100 - P3[x] - 1 / (2 A) (10 / 26 - 630 / 26 a1f)^2 P4[x];
{CAD[p - 5 / 2 × 10^(-4), x, 8 / 5, 5 / 2], CAD[pnew - 2 × 10^(-2), x, 8 / 5, 5 / 2]}
```

Out[312]= {True, True}

Case $1.2 \leq \tau \leq 2.1$

(B.10)

In[313]:=
$$\text{Clear[A]; A} = \frac{2}{10^3};$$
$$\left\{\frac{1}{2}\,A\left(W1\left[\frac{5}{2}\right] + 75 \times 10^{(-6)}\right)^4 < \frac{1}{10^4},\right.$$
$$\frac{2}{10} + \left(2 - \frac{40}{14} \times \frac{380}{271}\right)\left(W1\left[\frac{5}{2}\right] + 75 \times 10^{(-6)}\right)^2 > \frac{1}{10},$$
$$\left.\frac{171}{100} - 2\left(W1\left[\frac{5}{2}\right] + 75 \times 10^{(-6)}\right)^2 - \frac{1}{2\,A}\left(\frac{10}{14}\right)^2\left(W1\left[\frac{5}{2}\right] + 75 \times 10^{(-6)}\right)^4 > 1\right\}$$

Out[314]= {True, True, True}

$\frac{1}{2} A Q \left(\frac{11}{10}\right)^4 < \frac{1}{10^3}$ before (B.11)

In[315]:=
$$\text{Clear[A]; A} = \frac{2}{10^3};\ \frac{1}{2}\,A\left(W1\left[\frac{11}{10}\right] + 75 \times 10^{(-6)}\right)^4 < \frac{1}{10^3}$$

Out[315]= True

(B.11)

In[316]:=
```
Clear[p, pnew, a2f, x, A];
a2f = 8 / 100 Sum[1 / k! (-7 / 5 (x - 9 / 5))^k , {k, 0, 3}]; A = 2 * 10^(-3);
p = 1 / 5 + P1[x] + (10 / 14 - 461 / 140 a2f) 4 P2[x];
pnew = 171 / 100 - P3[x] - 1 / (2 A) (10 / 14 - 461 / 140 a2f)^2 P4[x];
{CAD[p - 1 / 10, x, 11 / 10, 5 / 2], CAD[pnew - 2 / 100, x, 11 / 10, 5 / 2]}
```

Out[319]= {True, True}

Case $2.1 \leq \tau \leq 5.4$

(B.12)

In[320]:=
```
Clear[A]; A = 1/100;
{1/2 A (W1[5/2] + 75 × 10^(-6))^4 < 1/10^4,
 11/10 + (2 - 4 × 380/271 × 5/2) (W1[5/2] + 75 × 10^(-6))^2 > 1,
 31/10 - 2 (W1[5/2] + 75 × 10^(-6))^2 - 1/(2 A) (5/2)^2 (W1[5/2] + 75 × 10^(-6))^4 > 1}
```

Out[321]= {True, True, True}

$\frac{1}{2}AQ\left(\frac{91}{100}\right)^4 < \frac{9}{10^3}$ and $\frac{1}{2}AQ(T)^4 < \frac{12}{10^3}$ before (B.14)

In[322]:=
```
Clear[A, T];
A = 1/100;
T = 856/1000;
{1/2 A (W1[91/100] + 75 × 10^(-6))^4 < 9/10^3, 1/2 A (W1[T] + 75 × 10^(-6))^4 < 12/10^3}
```

Out[322]= {True, True}

(B.14) and (B.15)

In[323]:=
```
Clear[p, pnew];
p = 11/10 + P1[x] + 4 (x - 845/1000) P2[x];
A = 10^(-2);
pnew = 31/10 - P3[x] - 1/(2 A) (x - 845/1000)^2 P4[x];
{CAD[p - 7/10, x, 91/100, 5/2], CAD[pnew - 16/100, x, 91/100, 5/2]}
{CAD[p - 3, x, 856/1000, 91/100], CAD[pnew - 41 * 10^(-4), x, 856/1000, 91/100]}
```

Out[325]= {True, True}

Out[326]= {True, True}

Case $5.4 \le \tau \le 12.4$

$$\frac{1}{2} A Q \left(\frac{5}{2}\right)^4 < \frac{1}{10^4}$$

$$f_1 > \frac{44}{10} + 2\,Q\left(\frac{5}{2}\right)^2 + 4 \cdot \left(-\frac{380}{271}\right) \cdot \frac{5}{2} \cdot Q\left(\frac{5}{2}\right)^2 > 1, \quad \forall\, t \ge \frac{5}{2};$$

$$f_2 > \frac{64}{10} - 2\,Q\left(\frac{5}{2}\right)^2 - \frac{1}{2A} \cdot \left(\frac{5}{2}\right)^2 \cdot Q\left(\frac{5}{2}\right)^4 > 1, \quad \forall\, t \ge \frac{5}{2}.$$

In[327]:=
```
Clear[A]; A = 2/100;
{1/2 A (W1[5/2] + 75 × 10^(-6))^4 < 1/10^4,
 44/10 + (2 - 4 × 380/271 × 5/2) (W1[5/2] + 75 × 10^(-6))^2 > 1,
 64/10 - 2 (W1[5/2] + 75 × 10^(-6))^2 - 1 / (2 A) (5 / 2)^2 (W1[5/2] + 75 × 10^(-6))^4 > 1}
```

Out[328]=
```
{True, True, True}
```

$\frac{1}{2} A Q\left(\frac{68}{100}\right)^4 < \frac{1}{10}$ and $\frac{1}{2} A Q(T)^4 < \frac{12}{100}$ before (B.16)

In[329]:=
```
Clear[A, T];
A = 2/100;
T = 2/3;
{1/2 A (W1[68/100] + 75 × 10^(-6))^4 < 1/10, 1/2 A (W1[T] + 75 × 10^(-6))^4 < 12/100}
```

Out[329]=
```
{True, True}
```

(B.16) and (B.17)

In[330]:=
```
Clear[p, pnew, A];
A = 2 * 10^(-2);
p = 44 / 10 + P1[x] + 4 (x - 655 / 1000) P2[x];
pnew = 64 / 10 - P3[x] - 1 / (2 A) (x - 655 / 1000)^2 P4[x];
{CAD[p - 3, x, 68 / 100, 5 / 2], CAD[pnew - 8 / 100, x, 68 / 100, 5 / 2]}
{CAD[p - 98 / 10, x, 2 / 3, 68 / 100], CAD[pnew - 14 * 10^(-3), x, 2 / 3, 68 / 100]}
```

Out[332]=
```
{True, True}
```

Out[333]=
```
{True, True}
```

Case $12.4 \le \tau \le 19.2$

$$\frac{1}{2} A Q \left(\frac{5}{2}\right)^4 < \frac{1}{10^4}$$

$$f_1 > \frac{114}{10} + 2\,Q\left(\frac{5}{2}\right)^2 + 4 \cdot \left(-\frac{380}{271}\right) \cdot \frac{5}{2} \cdot Q\left(\frac{5}{2}\right)^2 > 1, \quad \forall\, t \geq \frac{5}{2};$$

$$f_2 > \frac{134}{10} - 2\,Q\left(\frac{5}{2}\right)^2 - \frac{1}{2A} \cdot \left(\frac{5}{2}\right)^2 \cdot Q\left(\frac{5}{2}\right)^4 > 1, \quad \forall\, t \geq \frac{5}{2}.$$

In[334]:=
```
Clear[A]; A = 32/1000;
{1/2 A (W1[5/2] + 75 × 10^(-6))^4 < 1/10^4,
 114/10 + (2 - 4 × 380/271 × 5/2) (W1[5/2] + 75 × 10^(-6))^2 > 1,
 134/10 - 2 (W1[5/2] + 75 × 10^(-6))^2 - 1/(2 A) (5/2)^2 (W1[5/2] + 75 × 10^(-6))^4 > 1}
```

Out[335]= {True, True, True}

$\frac{1}{2} A Q\left(\frac{68}{100}\right)^4 < \frac{2}{10}$ and $\frac{1}{2} A Q(T)^4 < \frac{68}{100}$ before (B.18)

In[336]:=
```
Clear[A, T];
A = 32/1000;
T = 475/1000;
{1/2 A (W1[68/100] + 75 × 10^(-6))^4 < 2/10, 1/2 A (W1[T] + 75 × 10^(-6))^4 < 68/100}
```

Out[336]= {True, True}

(B.18) and (B.19)

In[337]:=
```
Clear[p, pnew, A];
A = 32 / 10^3;
p = 114 / 10 + P1[x] + 4 (x - 46 / 100) P2[x];
pnew = 134 / 10 - P3[x] - 1 / (2 A) (x - 46 / 100)^2 P4[x];
{CAD[p - 95 / 10, x, 68 / 100, 5 / 2], CAD[pnew - 5 / 100, x, 68 / 100, 5 / 2]}
{CAD[p - 123 / 10, x, 475 / 1000, 68 / 100], CAD[pnew - 59 / 1000, x, 475 / 1000, 68 / 100]}
```

Out[339]= {True, True}

Out[340]= {True, True}