

```
In[1]:= If[! TrueQ[Private`$ev], Block[{Private`$ev = True},
  Print["Total time: ", First@AbsoluteTiming[ClearAll["Global`*"];
  NotebookEvaluate[EvaluationNotebook[], InsertResults -> True]], " s"]]]
```

Total time: 17.6 s

```
In[2]:= ClearAll["Global`*"];
(* Q* in (A.3) *)
Qanew[t_] := -32 389 / 844 * t^14 + 692 573 / 1409 * t^13 -
  14 647 839 / 5182 * t^12 + 13 634 891 / 1418 * t^11 - 31 307 644 / 1461 * t^10 +
  22 954 676 / 703 * t^9 - 1 268 792 / 37 * t^8 + 34 165 996 / 1415 * t^7 -
  12 679 168 / 1223 * t^6 + 2 124 876 / 1117 * t^5 + 78 128 / 353 * t^4 +
  15 317 / 829 * t^3 - 138 192 / 1231 * t^2 + 15 295 / 813;
(* W1 in (A.4) *)
W1[t_] :=
-126 751 t^22 / 80 621 266 + 998 813 t^21 / 24 102 701 - 44 784 703 t^20 / 90 944 085 +
  532 787 760 t^19 / 154 914 809 - 1 478 230 681 t^18 / 97 178 472 +
  1 104 148 971 t^17 / 27 114 439 - 6 386 998 649 t^16 / 165 889 120 -
  4 563 349 408 t^15 / 24 620 175 + 12 295 664 789 t^14 / 11 848 354 -
  25 742 069 395 t^13 / 8 929 817 + 22 711 315 187 t^12 / 4 190 364 -
  21 505 639 879 t^11 / 2 916 474 + 32 713 853 029 t^10 / 4 462 850 -
  12 946 708 519 t^9 / 2 496 198 + 3 403 022 733 t^8 / 1 411 424 -
  6 729 001 962 t^7 / 11 651 297 - 626 483 074 t^5 / 65 670 795 +
  752 248 559 t^4 / 20 846 520 - 89 936 485 t^2 / 6 984 383 + 93 423 / 21 539;
(* W2 in (A.5) *)
W2[t_] := 157 * t^8 / 1 255 706 - 5215 * t^7 / 1 535 379 + 25 008 * t^6 / 610 441 -
  1 118 777 * t^5 / 3 886 821 + 1 230 788 * t^4 / 949 047 - 784 873 * t^3 / 202 656 +
  10 232 110 * t^2 / 1 348 117 - 2 084 185 * t / 228 972 + 43 251 / 8233;
(* P1, P2 in (B.2) *)
P1[t_] := 2 (W1[t] - 75 * 10^(-6))^2;
P2[t_] := (W1[t] + 75 * 10^(-6)) (W1'[t] - 42 * 10^(-5));
(* P3, P4 in (B.3) *)
P3[t_] := 2 (W1[t] + 75 * 10^(-6))^2;
P4[t_] := (W1[t] + 75 * 10^(-6))^4;
(* for example, Dig5[312]=31200; Dig5[3123423]=31234*)
Dig5[x_] := IntegerPart[x * 10^5 / 10^(IntegerLength[x])];
(* for example,
FU[401256/309977]= $\frac{40126}{30997}$  gives an upper bound of 401256/309977*)
FU[s_] := If[s > 0, (Dig5[Numerator[s]] + 1) / Dig5[Denominator[s]] *
  10^(IntegerLength[Numerator[s]] - IntegerLength[Denominator[s]]),
  Print["Error: Input must be a positive number."];
  $Failed]
(* for example, NL[401256/3077]=6; DL[401256/3077]=4*)
NL[x_] := IntegerLength[Numerator[x]]; DL[x_] := IntegerLength[Denominator[x]];
(*The following function polyMax corresponds to formula (1.24) in Lemma 1.4. *)
polyMax[p_, a_, b_, dx_] := Module[{d, pValues, M2, Lam, Nsub, coeffList},
  If[! PolynomialQ[p, x], Return["Error: p must be a polynomial in x."];];
```

```

coeffList = CoefficientList[p, x];
d = Exponent[p, x];
Nsub = Ceiling[(b - a) / dx];
If[6 Nsub^2 - d^2 (d^2 - 1) ≤ 0, Return["Error: lambda must be positive."]];
pValues =
  ParallelTable[FromDigits[Reverse[coeffList], a + k * dx], {k, 0, Nsub}];
M2 = Max[Abs[pValues]];
Lam =  $\frac{6 Nsub^2}{6 Nsub^2 - d^2 (d^2 - 1)}$ ;
Lam * M2]
(*The following function polynomialMin corresponds to formula (1.25) in Lemma 1.4.*)
polynomialMin[p_, a_, b_, dx_] := Module[{d, pValues, m1, m2, lam, Nsub, coeffList},
  If[! PolynomialQ[p, x], Return["Error: p must be a polynomial in x."]];
  coeffList = CoefficientList[p, x];
  d = Exponent[p, x];
  Nsub = Ceiling[(b - a) / dx];
  If[6 Nsub^2 - d^2 (d^2 - 1) ≤ 0, Return["Error: lambda must be positive."]];
  pValues =
    ParallelTable[FromDigits[Reverse[coeffList], a + k * dx], {k, 0, Nsub}];
  m1 = Min[pValues];
  m2 = Max[Abs[pValues]];
  lam = (d^2 (d^2 - 1)) / (6 Nsub^2 - d^2 (d^2 - 1));
  m1 - lam * m2]
(*The following functions are used in Lemma 4.4*)
H[x1_, x2_, x3_, x4_, y1_, y2_, y3_, y4_, p1_, p2_] :=
  4 * p1 * x3^2 * y1^2 + 4 * p1 * p2 * x2 * x4 * y1^2 - 4 * p1 * p2 * x1 * x4 * y1 * y2 -
  p2^2 * x4^2 * y2^2 - 8 * p1 * x1 * x3 * y1 * y3 - 4 * p2 * x3 * x4 * y2 * y3 +
  4 * p1 * x1^2 * y3^2 + 4 * p2 * x2 * x4 * y3^2 - 4 * p1 * p2 * x1 * x2 * y1 * y4 +
  4 * p1 * p2 * x1^2 * y2 * y4 + 4 * p2 * x3^2 * y2 * y4 +
  2 * p2^2 * x2 * x4 * y2 * y4 - 4 * p2 * x2 * x3 * y3 * y4 - p2^2 * x2^2 * y4^2;
HM[x1_, x2_, x3_, x4_, y1_, y2_, y3_, y4_, p1_, p2_] :=
  4 * p1 * x3^2 * y1^2 + 4 * p1 * p2 * x2 * x4 * y1^2 + 4 * p1 * p2 * x1 * x4 * y1 * y2 +
  p2^2 * x4^2 * y2^2 + 8 * p1 * x1 * x3 * y1 * y3 + 4 * p2 * x3 * x4 * y2 * y3 +
  4 * p1 * x1^2 * y3^2 + 4 * p2 * x2 * x4 * y3^2 + 4 * p1 * p2 * x1 * x2 * y1 * y4 +
  4 * p1 * p2 * x1^2 * y2 * y4 + 4 * p2 * x3^2 * y2 * y4 +
  2 * p2^2 * x2 * x4 * y2 * y4 + 4 * p2 * x2 * x3 * y3 * y4 + p2^2 * x2^2 * y4^2;

```

Toy Examples

toy example 1

$$\text{In[15]:= polynomialMin}\left[3 x P2[x] + 3 (W1[x] - 75 \times 10^{\wedge}(-6))^{\wedge}2 - 1, 0, \frac{1}{2}, \frac{5}{10^{\wedge}5}\right] > 0$$

Out[15]= True

toy example 2

```
In[16]:= Clear[U, Ud, U2d, p, R];
T = 3 / 10;
U[x_, t_] :=
  t - 31 / 5 t^3 + 1 / 3 t^4 + 472 / 21 t^5 - 25 t^6 + (-1 / 6 t^3 + 21 / 20 t^5 - 8 / 7 t^6) x;
Ud[x_, t_] := Derivative[0, 1][U][x, t];
U2d[x_, t_] := Derivative[0, 2][U][x, t];
R[x_, t_] := U2d[x, t] + (x + 2 Qanew[t] - 1) U[x, t];
p = Integrate[R[x, t]^2, {t, 0, T}];
Ud[x, T]
2 * 75 / 10^5 T + T^(1 / 2) (polyMax[p, 144 / 10, 192 / 10, 1 / 10])^(1 / 2) < 7 / 10^3
```

Out[20]=
$$-\frac{1291}{14000} - \frac{26793x}{1400000}$$

Out[21]= True

Case $l=0$: Theorem 5.1

Proposition 5.2: $\tau \in [12.4, 19.2]$

```

In[22]:= Clear[Uaapp, Vaapp, Ubapp, Vbapp, Uad, Vad, Ubd, Vbd, Ua2d, Va2d, Ub2d, Vb2d];
Uaapp[τ_, t_] :=
  (4550 * t^7) / 131 - (5787 * t^6) / 91 + (3986 * t^5) / 105 - (334 * t^4) / 211 -
  (727 * t^3) / 120 + (t^2) / 242 + t + ((58 * t^7) / 49 - (305 * t^6) / 129 +
  (90 * t^5) / 59 - (6 * t^4) / 61 - (10 * t^3) / 63 - (t^2) / 3170) * τ;
Vaapp[τ_, t_] :=
  - (2251 * t^7) / 81 + (4841 * t^6) / 99 - (11060 * t^5) / 403 + (89 * t^4) / 102 +
  (429 * t^3) / 136 - (t^2) / 140 + ((-9 * t^7) / 92 + (17 * t^6) / 162 + (5 * t^5) / 111 -
  (3 * t^4) / 133 + (t^3) / 260 - (t^2) / 3409) * τ;
Ubapp[τ_, t_] :=
  - (3423 * t^7) / 124 + (4105 * t^6) / 84 - (3577 * t^5) / 130 + (72 * t^4) / 79 +
  (129 * t^3) / 41 - (t^2) / 155 + ((5 * t^7) / 83 - (t^6) / 52 - (4 * t^5) / 95 +
  (t^4) / 61 - (t^3) / 382 + (t^2) / 5233) * τ;
Vbapp[τ_, t_] :=
  (5468 * t^7) / 155 - (8018 * t^6) / 121 + (3078 * t^5) / 79 - (406 * t^4) / 225 -
  (1417 * t^3) / 235 + (t^2) / 461 + t + ((-241 * t^7) / 443 + (288 * t^6) / 253 -
  (95 * t^5) / 148 + (15 * t^4) / 374 + (61 * t^3) / 372 + (t^2) / 15381) * τ;
Uad[x_, t_] := Derivative[0, 1][Uaapp][x, t];
Vad[x_, t_] := Derivative[0, 1][Vaapp][x, t];
Ubd[x_, t_] := Derivative[0, 1][Ubapp][x, t];
Vbd[x_, t_] := Derivative[0, 1][Vbapp][x, t];
Ua2d[x_, t_] := Derivative[0, 2][Uaapp][x, t];
Va2d[x_, t_] := Derivative[0, 2][Vaapp][x, t];
Ub2d[x_, t_] := Derivative[0, 2][Ubapp][x, t];
Vb2d[x_, t_] := Derivative[0, 2][Vbapp][x, t];

```

(5.2)

```

In[35]= Clear[pa, pb, x, es, du, tauL, tauR, Ya1, Ya2, Ya3, Ya4, Yb1, Yb2, Yb3, Yb4, T];
tauL = 124 / 10;
tauR = 192 / 10;
pa = 1 / 10;
pb = 427 / 100;
es = 45 / 10^3;
du = 2 * 10^-2;
T = 475 / 1000;
Ya1[x_] := Uaapp[x, T];
Ya2[x_] := Vaapp[x, T]; Ya3[x_] := Uad[x, T];
Ya4[x_] := Vad[x, T];
Yb1[x_] := Ubapp[x, T]; Yb2[x_] := Vbapp[x, T]; Yb3[x_] := Ubd[x, T];
Yb4[x_] := Vbd[x, T];
Hepsilon[x_] := Simplify[
  H[Ya1[x], Ya2[x], Ya3[x], Ya4[x], Yb1[x], Yb2[x], Yb3[x], Yb4[x], pa, pb] -
  (HM[Abs[Ya1[x]] + es, Abs[Ya2[x]] + es, Abs[Ya3[x]] + es, Abs[Ya4[x]] + es,
    Abs[Yb1[x]] + es, Abs[Yb2[x]] + es, Abs[Yb3[x]] + es, Abs[Yb4[x]] + es,
    pa, pb] - HM[Abs[Ya1[x]], Abs[Ya2[x]], Abs[Ya3[x]], Abs[Ya4[x]],
    Abs[Yb1[x]], Abs[Yb2[x]], Abs[Yb3[x]], Abs[Yb4[x]], pa, pb]);
  Ya2[x] - es
  Ya4[x] - es
HepsilonExpanded = Expand[Hepsilon[x]];
xvals = Range[tauL, tauR, du];
minValue = Min[HepsilonExpanded /. x -> xvals];
minValue > 0

```

Out[40]= $\frac{576\,038\,932\,877\,554\,217}{7\,000\,831\,918\,080\,000\,000} + \frac{1\,919\,017\,510\,830\,489\,821\,x}{2\,002\,003\,641\,630\,720\,000\,000}$

Out[41]= $\frac{58\,118\,833\,252\,035\,821}{175\,020\,797\,952\,000\,000} + \frac{191\,526\,365\,774\,652\,211\,x}{16\,683\,363\,680\,256\,000\,000}$

Out[45]= True

(5.3)

```
In[46]:= Clear[tauL, tauR, dx, T, Za, Zb, x, Ra, Rb, A1, A2, B1, B2];
tauL = 124 / 10;
tauR = 192 / 10;
dx = 2 * 10^-2;
T = 475 / 1000;
A1[x_, t_] := Ua2d[x, t] - (1 - x - 2 Qanew[t]) Uaapp[x, t] - Qanew[t] * Vaapp[x, t];
A2[x_, t_] := Va2d[x, t] - (1 + x - 2 Qanew[t]) Vaapp[x, t] - Qanew[t] * Uaapp[x, t];
B1[x_, t_] := Ub2d[x, t] - (1 - x - 2 Qanew[t]) Ubapp[x, t] - Qanew[t] * Vbapp[x, t];
B2[x_, t_] := Vb2d[x, t] - (1 + x - 2 Qanew[t]) Vbapp[x, t] - Qanew[t] * Ubapp[x, t];
Za = Integrate[ExpandAll[A1[x, t]^2 + A2[x, t]^2], {t, 0, T}];
Zb = Integrate[ExpandAll[B1[x, t]^2 + B2[x, t]^2], {t, 0, T}];
Ra = polyMax[Za, tauL, tauR, dx];
Rb = polyMax[Zb, tauL, tauR, dx];
Max[FU[Ra], FU[Rb]]
```

```
Out[55]=  $\frac{14\,039}{18\,384\,000}$ 
```

(5.4)

```
In[56]:= k = Sqrt[tauR + 1];
{  $\frac{1}{2} \sqrt{\frac{2 k T + \text{Sinh}[2 k T]}{k}} < \frac{1493}{1000}$ ,
 $\frac{3 T \text{Sinh}[k T]}{2 k} < \frac{6690}{10^4}$ ,  $\frac{1493}{1000} \sqrt{\frac{14\,039}{18\,384\,000}} + \frac{6690}{10^4} \times \frac{75}{10^5} \leq \frac{418}{10^4}$  }
```

```
Out[56]= {True, True, True}
```

(5.5)

```
In[57]:=  $\frac{dx}{2} \text{Max}[\text{Abs}[Ya1'[\text{tauL}]], \text{Abs}[Ya1'[\text{tauR}]], \text{Abs}[Ya2'[\text{tauL}]], \text{Abs}[Ya2'[\text{tauR}]],$ 
 $\text{Abs}[Ya3'[\text{tauL}]], \text{Abs}[Ya3'[\text{tauR}]], \text{Abs}[Ya4'[\text{tauL}]], \text{Abs}[Ya4'[\text{tauR}]],$ 
 $\text{Abs}[Yb1'[\text{tauL}]], \text{Abs}[Yb1'[\text{tauR}]], \text{Abs}[Yb2'[\text{tauL}]], \text{Abs}[Yb2'[\text{tauR}]],$ 
 $\text{Abs}[Yb3'[\text{tauL}]], \text{Abs}[Yb3'[\text{tauR}]], \text{Abs}[Yb4'[\text{tauL}]], \text{Abs}[Yb4'[\text{tauR}]]] < \frac{1}{1000}$ 
```

```
Out[57]= True
```

(5.1)

In[58]:= $k = \sqrt{\text{tauR} + 1}$;

$$\frac{1}{2} \sqrt{\frac{2 k T + \text{Sinh}[2 k T]}{k}} \sqrt{\text{Max}[FU[\text{Ra}], FU[\text{Rb}]]} + \frac{3 T \text{Sinh}[k T]}{2 k} \frac{75}{10^5} + \frac{1}{20} dx < es$$

Out[59]:= True

Proposition C.1: $\tau \in [5.4, 12.4]$

```

In[60]:= Clear[Uaapp, Vaapp, Ubapp, Vbapp, Uad, Vad, Ubd, Vbd, Ua2d, Va2d, Ub2d, Vb2d];
Uaapp[τ_, t_] := (250 858 * t^10) / 4155 - (42 245 * t^9) / 226 + (7428 * t^8) / 35 -
  (8522 * t^7) / 109 - (9443 * t^6) / 240 + (11 239 * t^5) / 283 - (698 * t^4) / 265 -
  (1186 * t^3) / 203 - (10 * t^2) / 749 + t + (5 * t^10 / 402 - 14 * t^9 / 341 + 7 * t^8 / 135 -
    5 * t^7 / 214 - t^6 / 96 + t^5 / 81 - t^4 / 1295 + t^3 / 12 292) * τ^2 +
  (303 * t^10 / 272 - 642 * t^9 / 179 + 843 * t^8 / 199 - 111 * t^7 / 68 -
    207 * t^6 / 244 + 584 * t^5 / 629 - 26 * t^4 / 435 - 48 * t^3 / 299 - t^2 / 3075) * τ;
Vaapp[τ_, t_] := -(13 012 * t^10) / 297 + (50 423 * t^9) / 387 - (35 597 * t^8) / 261 +
  (11 461 * t^7) / 333 + (13 129 * t^6) / 321 - (20 636 * t^5) / 669 + (431 * t^4) / 177 +
  (255 * t^3) / 88 + (2 * t^2) / 167 + (t^10 / 5317 - t^9 / 336 + 4 * t^8 / 435 -
    t^7 / 90 + 3 * t^6 / 389 - t^5 / 459 + t^4 / 2762 - t^3 / 28 556) * τ^2 +
  (15 * t^10 / 256 - 99 * t^9 / 232 + 493 * t^8 / 450 - 534 * t^7 / 389 +
    248 * t^6 / 285 - 25 * t^5 / 102 + 7 * t^4 / 172 - t^3 / 255 + t^2 / 4980) * τ;
Ubapp[τ_, t_] :=
  - (3355 * t^10) / 78 + (51 257 * t^9) / 401 - (55 000 * t^8) / 413 + (7825 * t^7) / 244 +
  (12 499 * t^6) / 298 - (11 800 * t^5) / 379 + (236 * t^4) / 95 + (1727 * t^3) / 597 +
  (4 * t^2) / 327 + (-t^10 / 167 + 5 * t^9 / 288 - 5 * t^8 / 249 + 5 * t^7 / 466 -
    t^6 / 503 + t^5 / 2013 - t^4 / 10 812 + t^3 / 94 132) * τ^2 + ((103 * t^10) / 334 -
    (122 * t^9) / 165 + (95 * t^8) / 201 + (35 * t^7) / 172 - (115 * t^6) / 337 +
    (23 * t^5) / 240 - (6 * t^4) / 409 + (t^3) / 805 - (t^2) / 18 830) * τ;
Vbapp[τ_, t_] :=
  (18 556 * t^10) / 313 - (107 695 * t^9) / 587 + (66 232 * t^8) / 319 - (23 047 * t^7) / 308 -
  (13 337 * t^6) / 328 + (7012 * t^5) / 175 - (167 * t^4) / 62 - (1249 * t^3) / 214 -
  (4 * t^2) / 293 + t + ((14 * t^10) / 741 - (18 * t^9) / 301 + (13 * t^8) / 171 -
    (7 * t^7) / 166 + (2 * t^6) / 375 + (t^5) / 132 + (t^4) / 12 270) * τ^2 +
  ((-376 * t^10) / 245 + (634 * t^9) / 129 - (2809 * t^8) / 465 +
    (1457 * t^7) / 491 + (21 * t^6) / 85 - (268 * t^5) / 353 +
    (t^4) / 33 + (35 * t^3) / 214 + (t^2) / 6317) * τ;
Uad[x_, t_] := Derivative[0, 1][Uaapp][x, t];
Vad[x_, t_] := Derivative[0, 1][Vaapp][x, t];
Ubd[x_, t_] := Derivative[0, 1][Ubapp][x, t];
Vbd[x_, t_] := Derivative[0, 1][Vbapp][x, t];
Ua2d[x_, t_] := Derivative[0, 2][Uaapp][x, t];
Va2d[x_, t_] := Derivative[0, 2][Vaapp][x, t];
Ub2d[x_, t_] := Derivative[0, 2][Ubapp][x, t];
Vb2d[x_, t_] := Derivative[0, 2][Vbapp][x, t];

```

(C.2)

```

In[72]= Clear[pa, pb, x, es, du, tauL, tauR, Ya1, Ya2, Ya3, Ya4, Yb1, Yb2, Yb3, Yb4, T];
tauL = 54 / 10;
tauR = 124 / 10;
pa = 96 / 10;
pb = 343 / 100;
es = 26 / 10^3;
du = 10^-2;
T = 2 / 3;
Ya1[x_] := Uaapp[x, T];
Ya2[x_] := Vaapp[x, T]; Ya3[x_] := Uad[x, T];
Ya4[x_] := Vad[x, T];
Yb1[x_] := Ubapp[x, T]; Yb2[x_] := Vbapp[x, T]; Yb3[x_] := Ubd[x, T];
Yb4[x_] := Vbd[x, T];
Hepsilon[x_] :=
  H[Ya1[x], Ya2[x], Ya3[x], Ya4[x], Yb1[x], Yb2[x], Yb3[x], Yb4[x], pa, pb] -
  (HM[Abs[Ya1[x]] + es, Abs[Ya2[x]] + es, Abs[Ya3[x]] + es, Abs[Ya4[x]] + es,
    Abs[Yb1[x]] + es, Abs[Yb2[x]] + es, Abs[Yb3[x]] + es, Abs[Yb4[x]] + es,
    pa, pb] - HM[Abs[Ya1[x]], Abs[Ya2[x]], Abs[Ya3[x]], Abs[Ya4[x]],
    Abs[Yb1[x]], Abs[Yb2[x]], Abs[Yb3[x]], Abs[Yb4[x]], pa, pb]);
Ya2[x] - es
Ya4[x] - es
HepsilonExpanded = Expand[Hepsilon[x]];
xvals = Range[tauL, tauR, du];
minValue = Min[HepsilonExpanded /. x -> xvals];
minValue > 0

```

$$\text{Out[77]} = \frac{16\,148\,372\,893\,594\,385\,391\,157}{95\,118\,348\,656\,680\,395\,205\,500} + \frac{65\,331\,293\,804\,303\,x}{19\,197\,701\,089\,755\,075} + \frac{1\,778\,959\,623\,075\,792\,502\,x^2}{20\,776\,616\,952\,736\,362\,382\,665}$$

$$\text{Out[78]} = \frac{7\,609\,339\,494\,027\,734\,729\,969}{31\,706\,116\,218\,893\,465\,068\,500} + \frac{150\,313\,914\,413\,207\,x}{6\,399\,233\,696\,585\,025} + \frac{1\,115\,287\,103\,240\,575\,649\,x^2}{1\,385\,107\,796\,849\,090\,825\,511}$$

```

Out[82]= True

```

(C.3)

```

In[83]= Max[Abs[Ya1'[tauL]], Abs[Ya1'[tauR]], Abs[Ya2'[tauL]], Abs[Ya2'[tauR]],
  Abs[Ya3'[tauL]], Abs[Ya3'[tauR]], Abs[Ya4'[tauL]], Abs[Ya4'[tauR]],
  Abs[Yb1'[tauL]], Abs[Yb1'[tauR]], Abs[Yb2'[tauL]], Abs[Yb2'[tauR]],
  Abs[Yb3'[tauL]], Abs[Yb3'[tauR]], Abs[Yb4'[tauL]], Abs[Yb4'[tauR]]] <  $\frac{2}{5}$ 

```

```

Out[83]= True

```

(C.4)

```

In[84]= Clear[tauL, tauR, dx, T, Za, Zb, x, Ra, Rb, A1, A2, B1, B2];
tauL = 54 / 10;
tauR = 124 / 10;
T = 2 / 3;
dx = 10-2;
A1[x_, t_] := Ua2d[x, t] - (1 - x - 2 Qanew[t]) Uaapp[x, t] - Qanew[t] × Vaapp[x, t];
A2[x_, t_] := Va2d[x, t] - (1 + x - 2 Qanew[t]) Vaapp[x, t] - Qanew[t] × Uaapp[x, t];
B1[x_, t_] := Ub2d[x, t] - (1 - x - 2 Qanew[t]) Ubapp[x, t] - Qanew[t] × Vbapp[x, t];
B2[x_, t_] := Vb2d[x, t] - (1 + x - 2 Qanew[t]) Vbapp[x, t] - Qanew[t] × Ubapp[x, t];
Za = Integrate[ExpandAll[A1[x, t]^2 + A2[x, t]^2], {t, 0, T}];
Zb = Integrate[ExpandAll[B1[x, t]^2 + B2[x, t]^2], {t, 0, T}];
Ra = polyMax[Za, tauL, tauR, dx];
Rb = polyMax[Zb, tauL, tauR, dx];
k = Sqrt[tauR + 1];

$$\frac{1}{2} \sqrt{\frac{2 k T + \text{Sinh}[2 k T]}{k}} \sqrt{\text{Max}[FU[Ra], FU[Rb]]} + \frac{3 T \text{Sinh}[k T]}{2 k} \frac{75}{10^5} + \frac{1}{5} dx < es$$


```

Out[94]= True

Proposition C.1: $\tau \in [2.1, 5.4]$

```

In[95]:= Clear[Uaapp, Vaapp, Ubapp, Vbapp, Uad, Vad, Ubd, Vbd, Ua2d, Va2d, Ub2d, Vb2d];
Uaapp[τ_, t_] := (42 673 * t^12) / 692 - (1 055 462 * t^11) / 3005 + (424 795 * t^10) / 488 -
  (435 706 * t^9) / 359 + (419 355 * t^8) / 413 - (130 882 * t^7) / 269 +
  (20 712 * t^6) / 211 + (3913 * t^5) / 423 + (405 * t^4) / 241 - (4633 * t^3) / 746 +
  t^2 / 278 + t + ((89 * t^12) / 67 - (1981 * t^11) / 262 + (4426 * t^10) / 235 -
  (21 136 * t^9) / 795 + (5435 * t^8) / 239 - (6325 * t^7) / 559 + (437 * t^6) / 176 +
  (23 * t^5) / 135 + (15 * t^4) / 296 - (95 * t^3) / 558 + t^2 / 7198) * τ +
  ((-2 * t^12) / 231 + (13 * t^11) / 401 - t^10 / 23 + (5 * t^9) / 276 + (4 * t^8) / 451 -
  t^7 / 383 - (5 * t^6) / 328 + (3 * t^5) / 227 - t^4 / 1153 + t^3 / 11 554) * τ^2;
Vaapp[τ_, t_] := -(26 307 * t^12) / 491 +
  (81 288 * t^11) / 271 - (257 262 * t^10) / 353 + (290 879 * t^9) / 293 -
  (214 088 * t^8) / 265 + (91 013 * t^7) / 243 - (19 248 * t^6) / 263 -
  (1193 * t^5) / 211 - (267 * t^4) / 238 + (1981 * t^3) / 619 - t^2 / 519 +
  ((-233 * t^12) / 354 + (1250 * t^11) / 367 - (1958 * t^10) / 259 + (3649 * t^9) / 394 -
  (6277 * t^8) / 944 + (888 * t^7) / 329 - (228 * t^6) / 409 + (21 * t^5) / 244 -
  (3 * t^4) / 347 + t^3 / 1861 - t^2 / 53 354) * τ +
  ((3 * t^12) / 152 - (20 * t^11) / 207 + (58 * t^10) / 279 - (20 * t^9) / 77 + (101 * t^8) /
  481 - (23 * t^7) / 200 + (9 * t^6) / 206 - t^5 / 96 + t^4 / 636 - t^3 / 7055) * τ^2;
Ubapp[τ_, t_] := -(23 768 * t^12) / 441 +
  (30 452 * t^11) / 101 - (91 512 * t^10) / 125 + (987 850 * t^9) / 991 -
  (236 014 * t^8) / 291 + (129 403 * t^7) / 344 - (160 111 * t^6) / 2171 -
  (4746 * t^5) / 859 - (397 * t^4) / 348 + (1284 * t^3) / 401 - t^2 / 498 +
  ((179 * t^12) / 278 - (2323 * t^11) / 697 + (2951 * t^10) / 399 - (1077 * t^9) / 119 +
  (1509 * t^8) / 233 - (521 * t^7) / 200 + (34 * t^6) / 65 - (15 * t^5) / 193 +
  (3 * t^4) / 409 - (t^3) / 2412 + (t^2) / 79 649) * τ +
  ((-4 * t^12) / 423 + (16 * t^11) / 343 - (39 * t^10) / 392 + (38 * t^9) / 319 -
  (19 * t^8) / 219 + (27 * t^7) / 703 - (2 * t^6) / 215 +
  t^5 / 592 - (t^4) / 4961 + (t^3) / 67 049) * τ^2;
Vbapp[τ_, t_] := (27 683 * t^12) / 447 - (214 333 * t^11) / 608 + (289 907 * t^10) / 332 -
  (320 070 * t^9) / 263 + (845 926 * t^8) / 831 - (18 539 * t^7) / 38 +
  (25 544 * t^6) / 259 + (1335 * t^5) / 146 + (765 * t^4) / 451 - (733 * t^3) / 118 +
  t^2 / 273 + t + ((-369 * t^12) / 280 + (4729 * t^11) / 630 - (4883 * t^10) / 261 +
  (7160 * t^9) / 271 - (3865 * t^8) / 171 + (3944 * t^7) / 351 - (547 * t^6) / 223 -
  (148 * t^5) / 831 - (4 * t^4) / 81 + (74 * t^3) / 435 - t^2 / 7602) * τ +
  ((t^12) / 72 - (74 * t^11) / 961 + (79 * t^10) / 420 - (107 * t^9) / 408 +
  (74 * t^8) / 331 - (22 * t^7) / 197 + (8 * t^6) / 323 +
  (t^5) / 255 + (t^4) / 1959 - (t^3) / 27 802) * τ^2;
Uad[x_, t_] := Derivative[0, 1][Uaapp][x, t];
Vad[x_, t_] := Derivative[0, 1][Vaapp][x, t];
Ubd[x_, t_] := Derivative[0, 1][Ubapp][x, t];
Vbd[x_, t_] := Derivative[0, 1][Vbapp][x, t];
Ua2d[x_, t_] := Derivative[0, 2][Uaapp][x, t];
Va2d[x_, t_] := Derivative[0, 2][Vaapp][x, t];
Ub2d[x_, t_] := Derivative[0, 2][Ubapp][x, t];
Vb2d[x_, t_] := Derivative[0, 2][Vbapp][x, t];

```

(C.2)

```

In[107]:= Clear[pa, pb, x, es, du, tauL, tauR, Ya1, Ya2, Ya3, Ya4, Yb1, Yb2, Yb3, Yb4, T];
tauL = 21 / 10;
tauR = 54 / 10;
pa = 4;
pb = 182 / 100;
es = 2 / 100;
du = 10-2;
T = 856 / 1000;
Ya1[x_] := Uaapp[x, T];
Ya2[x_] := Vaapp[x, T]; Ya3[x_] := Uad[x, T];
Ya4[x_] := Vad[x, T];
Yb1[x_] := Ubapp[x, T]; Yb2[x_] := Vbapp[x, T]; Yb3[x_] := Ubd[x, T];
Yb4[x_] := Vbd[x, T];
Hepsilon[x_] :=
  H[Ya1[x], Ya2[x], Ya3[x], Ya4[x], Yb1[x], Yb2[x], Yb3[x], Yb4[x], pa, pb] -
  (HM[Abs[Ya1[x]] + es, Abs[Ya2[x]] + es, Abs[Ya3[x]] + es, Abs[Ya4[x]] + es,
    Abs[Yb1[x]] + es, Abs[Yb2[x]] + es, Abs[Yb3[x]] + es, Abs[Yb4[x]] + es,
    pa, pb] - HM[Abs[Ya1[x]], Abs[Ya2[x]], Abs[Ya3[x]], Abs[Ya4[x]],
    Abs[Yb1[x]], Abs[Yb2[x]], Abs[Yb3[x]], Abs[Yb4[x]], pa, pb]);
Ya2[x] - es
Ya4[x] - es
HepsilonExpanded = Expand[Hepsilon[x]];
xvals = Range[tauL, tauR, du];
minValue = Min[HepsilonExpanded /. x -> xvals];
minValue > 0

```

Out[112]=

$$\frac{1823968464669914729574724027770571410655975341796875634300122037847394619437192697618587669892669303x^3 + 601879474599563988254803698509931564331054687500001951295907924724951569378317240701952759x^2 + 5398963974127235356718301773071289062500000359121925430788650896472631778911022530929129466929183495434718635673195584444329142570495605468750294473063498780853920597919226384486105103447x^2 + 5732185472376799888140987604856491088867187500334816193089149066307615506196634565607x^2 + 129575135379053648561239242553710937500000}{388004538452058137925231684439910672125848431312467}$$

Out[113]=

$$\frac{29183495434718635673195584444329142570495605468750294473063498780853920597919226384486105103447x^3 + 5732185472376799888140987604856491088867187500334816193089149066307615506196634565607x^2 + 129575135379053648561239242553710937500000}{3591219254307886508964726317789110225309291294669}$$

Out[117]= True

(C.3)

```
In[118]= Max[Abs[Ya1'[tauL]], Abs[Ya1'[tauR]], Abs[Ya2'[tauL]], Abs[Ya2'[tauR]],
Abs[Ya3'[tauL]], Abs[Ya3'[tauR]], Abs[Ya4'[tauL]], Abs[Ya4'[tauR]],
Abs[Yb1'[tauL]], Abs[Yb1'[tauR]], Abs[Yb2'[tauL]], Abs[Yb2'[tauR]],
Abs[Yb3'[tauL]], Abs[Yb3'[tauR]], Abs[Yb4'[tauL]], Abs[Yb4'[tauR]]] <  $\frac{2}{5}$ 
```

```
Out[118]= True
```

(C.4)

```
In[119]= Clear[tauL, tauR, dx, T, Za, Zb, x, Ra, Rb, A1, A2, B1, B2];
tauL = 21 / 10;
tauR = 54 / 10;
T = 856 / 1000;
dx = 10^-2;
A1[x_, t_] := Ua2d[x, t] - (1 - x - 2 Qanew[t]) Uaapp[x, t] - Qanew[t] x Vaapp[x, t];
A2[x_, t_] := Va2d[x, t] - (1 + x - 2 Qanew[t]) Vaapp[x, t] - Qanew[t] x Uaapp[x, t];
B1[x_, t_] := Ub2d[x, t] - (1 - x - 2 Qanew[t]) Ubapp[x, t] - Qanew[t] x Vbapp[x, t];
B2[x_, t_] := Vb2d[x, t] - (1 + x - 2 Qanew[t]) Vbapp[x, t] - Qanew[t] x Ubapp[x, t];
Za = Integrate[ExpandAll[A1[x, t]^2 + A2[x, t]^2], {t, 0, T}];
Zb = Integrate[ExpandAll[B1[x, t]^2 + B2[x, t]^2], {t, 0, T}];
Ra = polyMax[Za, tauL, tauR, dx];
Rb = polyMax[Zb, tauL, tauR, dx];
k = Sqrt[tauR + 1];
 $\frac{1}{2} \sqrt{\frac{2 k T + \text{Sinh}[2 k T]}{k}} \sqrt{\text{Max}[FU[Ra], FU[Rb]]} + \frac{3 T \text{Sinh}[k T]}{2 k} \frac{75}{10^5} + \frac{1}{5} dx < es$ 
```

```
Out[129]= True
```

Proposition C.1: $\tau \in [1, 1.2]$

```
In[130]= Clear[Uaapp, Vaapp, Ubapp, Vbapp, Uad, Vad, Ubd, Vbd, Ua2d, Va2d, Ub2d, Vb2d];
Uaapp[tau_, t_] := t + (112 * t^2) / 7071 - (84 069 * t^3) / 12 941 + (19 306 * t^4) / 3625 -
(156 439 * t^5) / 8122 + (4 529 108 * t^6) / 18 517 - (12 407 469 * t^7) / 12 391 +
(13 802 505 * t^8) / 6041 - (19 938 272 * t^9) / 5805 + (53 183 415 * t^10) / 14 657 -
(42 924 858 * t^11) / 15 571 + (8 404 358 * t^12) / 5591 - (3 526 473 * t^13) / 6148 +
(1 591 789 * t^14) / 11 029 - (278 264 * t^15) / 13 315 + (3326 * t^16) / 3039 +
(2680 * t^17) / 49 349 + (- (249 * t^17) / 5389 + (4374 * t^16) / 6893 - (15 398 * t^15) /
3823 + (219 958 * t^14) / 14 015 - (212 295 * t^13) / 5051 + (688 472 * t^12) / 8401 -
(1 117 303 * t^11) / 9315 + (1 588 706 * t^10) / 11 909 - (4 482 105 * t^9) / 39 941 +
(501 013 * t^8) / 7201 - (359 491 * t^7) / 12 021 + (30 645 * t^6) / 3956 -
(5614 * t^5) / 6493 + (1923 * t^4) / 10 346 - (1271 * t^3) / 7015 + (3 * t^2) / 4790) *
tau + ((54 * t^17) / 7553 - (1012 * t^16) / 11 019 + (6161 * t^15) / 11 484 -
```

$$\begin{aligned}
& (39\,423 * t^{14}) / 20\,906 + (260\,558 * t^{13}) / 58\,587 - (121\,265 * t^{12}) / 16\,324 + \\
& (57\,598 * t^{11}) / 6\,371 - (64\,982 * t^{10}) / 7\,999 + (68\,601 * t^9) / 12\,683 - \\
& (17\,790 * t^8) / 6\,677 + (7547 * t^7) / 7\,746 - (2944 * t^6) / 10\,785 + \\
& (2152 * t^5) / 35\,229 - (33 * t^4) / 4\,823 + (7 * t^3) / 12\,667 - (t^2) / 40\,643) * \tau^2; \\
\text{Vaapp}[\tau_, t_] := & (3681 * t^{17}) / 15\,061 - (57\,636 * t^{16}) / 11\,473 + \\
& (825\,829 * t^{15}) / 18\,764 - (57\,112\,156 * t^{14}) / 255\,125 + \\
& (11\,597\,999 * t^{13}) / 15\,502 - (20\,590\,648 * t^{12}) / 11\,777 + \\
& (25\,621\,273 * t^{11}) / 8\,685 - (51\,191\,939 * t^{10}) / 14\,066 + (39\,140\,407 * t^9) / 11\,979 - \\
& (5\,210\,419 * t^8) / 2\,505 + (13\,541\,181 * t^7) / 15\,350 - (352\,048 * t^6) / 1\,631 + \\
& (183\,459 * t^5) / 8\,386 - (43\,947 * t^4) / 9\,515 + (43\,703 * t^3) / 12\,583 - \\
& (205 * t^2) / 15\,119 + ((271 * t^{17}) / 5\,454 - (6838 * t^{16}) / 10\,085 + \\
& (52\,723 * t^{15}) / 12\,435 - (43\,681 * t^{14}) / 2\,714 + (243\,056 * t^{13}) / 5\,871 - \\
& (2\,308\,060 * t^{12}) / 30\,281 + (771\,033 * t^{11}) / 7\,463 - \\
& (949\,681 * t^{10}) / 9\,110 + (1\,067\,194 * t^9) / 13\,673 - (126\,229 * t^8) / 2\,964 + \\
& (127\,476 * t^7) / 7\,783 - (33\,624 * t^6) / 7\,837 + (12\,196 * t^5) / 15\,311 - \\
& (531 * t^4) / 5\,368 + (89 * t^3) / 11\,587 - (4 * t^2) / 12\,041) * \tau + \\
& ((-80 * t^{17}) / 14\,649 + (945 * t^{16}) / 13\,466 - (221\,639 * t^{15}) / 540\,245 + \\
& (12\,380 * t^{14}) / 8\,571 - (232\,353 * t^{13}) / 67\,993 + (74\,064 * t^{12}) / 12\,901 - \\
& (58\,461 * t^{11}) / 8\,284 + (65\,299 * t^{10}) / 10\,126 - (41\,500 * t^9) / 9\,413 + \\
& (4898 * t^8) / 2\,173 - (13\,581 * t^7) / 15\,874 + (2569 * t^6) / 10\,804 - \\
& (305 * t^5) / 6\,601 + (56 * t^4) / 9\,319 - (2 * t^3) / 4\,113 + t^2 / 46\,045) * \tau^2; \\
\text{Ubapp}[\tau_, t_] := & (2463 * t^{17}) / 9\,715 - (36\,251 * t^{16}) / 70\,47 + \\
& (285\,916 * t^{15}) / 6\,391 - (2\,011\,509 * t^{14}) / 8\,881 + (8\,515\,714 * t^{13}) / 11\,285 - \\
& (23\,298\,569 * t^{12}) / 13\,241 + (119\,068\,063 * t^{11}) / 40\,167 - \\
& (44\,459\,792 * t^{10}) / 12\,171 + (21\,775\,652 * t^9) / 6\,645 - (17\,028\,738 * t^8) / 8\,167 + \\
& (8\,508\,863 * t^7) / 9\,624 - (1\,251\,442 * t^6) / 5\,783 + (792\,161 * t^5) / 36\,032 - \\
& (54\,532 * t^4) / 11\,771 + (34\,531 * t^3) / 9\,939 - (156 * t^2) / 11\,465 + \\
& ((68 * t^{17}) / 6\,929 - (745 * t^{16}) / 7\,468 + (61\,77 * t^{15}) / 15\,824 - \\
& (6535 * t^{14}) / 12\,444 - (41\,74 * t^{13}) / 3\,315 + (66\,375 * t^{12}) / 9\,109 - \\
& (146\,303 * t^{11}) / 8\,824 + (212\,006 * t^{10}) / 9\,193 - (201\,460 * t^9) / 9\,491 + \\
& (43\,291 * t^8) / 3\,347 - (88\,537 * t^7) / 17\,825 + (19\,981 * t^6) / 17\,933 - \\
& (1831 * t^5) / 10\,383 + (147 * t^4) / 8\,275 - (35 * t^3) / 33\,337 + (t^2) / 32\,146) * \tau + \\
& ((-28 * t^{17}) / 9\,343 + (365 * t^{16}) / 9\,599 - (2944 * t^{15}) / 13\,441 + \\
& (12\,991 * t^{14}) / 17\,140 - (32\,497 * t^{13}) / 18\,494 + (24\,623 * t^{12}) / 8\,543 - \\
& (16\,321 * t^{11}) / 4\,738 + (17\,317 * t^{10}) / 5\,685 - (8\,489 * t^9) / 4\,231 + \\
& (90\,568 * t^8) / 91\,895 - (2930 * t^7) / 8\,137 + (1\,761 * t^6) / 18\,083 - \\
& (143 * t^5) / 7\,848 + (162 * t^4) / 71\,441 - (t^3) / 5\,726 + (t^2) / 135\,052) * \tau^2; \\
\text{Vbapp}[\tau_, t_] := & (281 * t^{17}) / 6\,789 + (29\,119 * t^{16}) / 23\,022 - \\
& (134\,735 * t^{15}) / 6\,146 + (1\,565\,941 * t^{14}) / 10\,578 - (6\,744\,679 * t^{13}) / 11\,576 + \\
& (9\,325\,931 * t^{12}) / 6\,140 - (21\,688\,406 * t^{11}) / 7\,811 + (11\,901\,372 * t^{10}) / 3\,263 - \\
& (36\,827\,911 * t^9) / 10\,681 + (28\,814\,971 * t^8) / 12\,573 - \\
& (11\,522\,451 * t^7) / 11\,476 + (609\,949 * t^6) / 2\,486 - (47\,380 * t^5) / 2\,441 + \\
& (53\,649 * t^4) / 10\,037 - (70\,099 * t^3) / 10\,788 + (243 * t^2) / 15\,278 + t + \\
& (- (431 * t^{17}) / 12\,307 + (1891 * t^{16}) / 4\,423 - (17\,301 * t^{15}) / 7\,532 + \\
& (79\,657 * t^{14}) / 11\,355 - (75\,969 * t^{13}) / 5\,918 + (128\,952 * t^{12}) / 10\,477 + \\
& (15\,653 * t^{11}) / 11\,987 - (186\,559 * t^{10}) / 8\,361 + (394\,344 * t^9) / 11\,453 - \\
& (395\,025 * t^8) / 13\,636 + (260\,073 * t^7) / 18\,221 - (29\,944 * t^6) / 8\,821 +
\end{aligned}$$

```

      (271 * t^5) / 18282 - (807 * t^4) / 10802 + (1133 * t^3) / 6583 - (2 * t^2) / 9289) *
    tau + ((27 * t^17) / 6227 - (1706 * t^16) / 30739 + (4549 * t^15) / 14101 -
      (14721 * t^14) / 13070 + (52013 * t^13) / 19762 - (52482 * t^12) / 12091 +
      (30786 * t^11) / 5933 - (64433 * t^10) / 14173 + (24681 * t^9) / 8450 -
      (10701 * t^8) / 7799 + (2498 * t^7) / 5227 - (973 * t^6) / 7295 +
      (267 * t^5) / 7960 - (22 * t^4) / 6887 + (3 * t^3) / 11911 - (t^2) / 91156) * tau^2;
    Uad[x_, t_] := Derivative[0, 1][Uaapp][x, t];
    Vad[x_, t_] := Derivative[0, 1][Vaapp][x, t];
    Ubd[x_, t_] := Derivative[0, 1][Ubapp][x, t];
    Vbd[x_, t_] := Derivative[0, 1][Vbapp][x, t];
    Ua2d[x_, t_] := Derivative[0, 2][Uaapp][x, t];
    Va2d[x_, t_] := Derivative[0, 2][Vaapp][x, t];
    Ub2d[x_, t_] := Derivative[0, 2][Ubapp][x, t];
    Vb2d[x_, t_] := Derivative[0, 2][Vbapp][x, t];

```

(C.2)

```

In[142]= Clear[pa, pb, x, es, du, tauL, tauR, Ya1, Ya2, Ya3, Ya4, Yb1, Yb2, Yb3, Yb4, T];
tauL = 1;
tauR = 12 / 10;
pa = 2 / 10;
pb = 1 / 10;
es = 26 / 10^3;
du = 2 * 10^-3;
T = 16 / 10;
Ya1[x_] := Uaapp[x, T];
Ya2[x_] := Vaapp[x, T]; Ya3[x_] := Uad[x, T];
Ya4[x_] := Vad[x, T];
Yb1[x_] := Ubapp[x, T]; Yb2[x_] := Vbapp[x, T]; Yb3[x_] := Ubd[x, T];
Yb4[x_] := Vbd[x, T];
Hepsilon[x_] :=
  H[Ya1[x], Ya2[x], Ya3[x], Ya4[x], Yb1[x], Yb2[x], Yb3[x], Yb4[x], pa, pb] -
  (HM[Abs[Ya1[x]] + es, Abs[Ya2[x]] + es, Abs[Ya3[x]] + es, Abs[Ya4[x]] + es,
    Abs[Yb1[x]] + es, Abs[Yb2[x]] + es, Abs[Yb3[x]] + es, Abs[Yb4[x]] + es,
    pa, pb] - HM[Abs[Ya1[x]], Abs[Ya2[x]], Abs[Ya3[x]], Abs[Ya4[x]],
    Abs[Yb1[x]], Abs[Yb2[x]], Abs[Yb3[x]], Abs[Yb4[x]], pa, pb]);
Ya2[x] - es
Ya4[x] - es
HepsilonExpanded = Expand[Hepsilon[x]];
xvals = Range[tauL, tauR, du];
minValue = Min[HepsilonExpanded /. x -> xvals];
minValue > 0

```

```

Out[147]=
37 389 545 769 645 188 674 446 457 332 298 893 599 825 742 634 640 074 118 988 011
135 616 875 730 386 557 401 158 096 210 574 201 584 799 282 583 010 864 257 812 500
286 204 174 242 596 089 786 963 739 818 741 905 069 601 949 948 402 920 821 473 024 x
2 591 197 925 650 411 899 040 947 550 886 665 341 473 746 454 793 665 313 720 703 125
2 077 197 586 997 254 312 491 594 821 611 639 677 503 650 972 545 310 024 273 183 589 056 x^2
140 459 747 475 274 958 091 636 208 362 930 928 722 114 856 630 794 324 534 759 521 484 375
3 937 443 944 856 404 806 116 904 737 870 809 558 377 184 175 148 176 171 744 821

```

```

Out[148]=
37 972 725 204 508 236 072 324 266 938 960 776 443 743 799 123 243 041 992 187 500
23 837 863 680 267 123 897 780 207 246 925 155 847 676 054 994 764 657 142 435 328 x
103 647 917 026 016 475 961 637 902 035 466 613 658 949 858 191 746 612 548 828 125
179 556 008 848 835 130 967 686 964 733 392 257 908 813 625 508 505 477 302 451 255 728 x^2
3 121 327 721 672 776 846 480 804 630 287 353 971 602 552 369 573 207 211 883 544 921 875

```

```

Out[152]= True

```

(C.3)

```
In[153]= Max[Abs[Ya1'[tauL]], Abs[Ya1'[tauR]], Abs[Ya2'[tauL]], Abs[Ya2'[tauR]],
Abs[Ya3'[tauL]], Abs[Ya3'[tauR]], Abs[Ya4'[tauL]], Abs[Ya4'[tauR]],
Abs[Yb1'[tauL]], Abs[Yb1'[tauR]], Abs[Yb2'[tauL]], Abs[Yb2'[tauR]],
Abs[Yb3'[tauL]], Abs[Yb3'[tauR]], Abs[Yb4'[tauL]], Abs[Yb4'[tauR]]] <  $\frac{2}{5}$ 
```

```
Out[153]= True
```

(C.4)

```
In[154]= Clear[tauL, tauR, dx, T, Za, Zb, x, Ra, Rb, A1, A2, B1, B2];
tauL = 1;
tauR = 12 / 10;
T = 16 / 10;
dx = 2 * 10-3;
A1[x_, t_] := Ua2d[x, t] - (1 - x - 2 Qanew[t]) Uaapp[x, t] - Qanew[t] * Vaapp[x, t];
A2[x_, t_] := Va2d[x, t] - (1 + x - 2 Qanew[t]) Vaapp[x, t] - Qanew[t] * Uaapp[x, t];
B1[x_, t_] := Ub2d[x, t] - (1 - x - 2 Qanew[t]) Ubapp[x, t] - Qanew[t] * Vbapp[x, t];
B2[x_, t_] := Vb2d[x, t] - (1 + x - 2 Qanew[t]) Vbapp[x, t] - Qanew[t] * Ubapp[x, t];
Za = Integrate[ExpandAll[A1[x, t]^2 + A2[x, t]^2], {t, 0, T}];
Zb = Integrate[ExpandAll[B1[x, t]^2 + B2[x, t]^2], {t, 0, T}];
Ra = polyMax[Za, tauL, tauR, dx];
Rb = polyMax[Zb, tauL, tauR, dx];
k = Sqrt[tauR + 1];
 $\frac{1}{2} \sqrt{\frac{2 k T + \text{Sinh}[2 k T]}{k}} \sqrt{\text{Max}[FU[Ra], FU[Rb]]} + \frac{3 T \text{Sinh}[k T]}{2 k} \frac{75}{10^5} + \frac{1}{5} dx < es$ 
```

```
Out[164]= True
```

Proposition C.1: $\tau \in [1.2, 2.1]$

(C.2)

```
In[165]:= Clear[pa, pb, x, es, du, tauL, tauR, Ya1, Ya2, Ya3, Ya4, Yb1, Yb2, Yb3, Yb4, T];
tauL = 12 / 10;
tauR = 21 / 10;
pa = 1;
pb = 1 / 2;
es = 23 / 10^3;
du = 10^-2;
T = 11 / 10;
Ya1[x_] := Uaapp[x, T];
Ya2[x_] := Vaapp[x, T]; Ya3[x_] := Uad[x, T];
Ya4[x_] := Vad[x, T];
Yb1[x_] := Ubapp[x, T]; Yb2[x_] := Vbapp[x, T]; Yb3[x_] := Ubd[x, T];
Yb4[x_] := Vbd[x, T];
Hepsilon[x_] :=
  H[Ya1[x], Ya2[x], Ya3[x], Ya4[x], Yb1[x], Yb2[x], Yb3[x], Yb4[x], pa, pb] -
  (HM[Abs[Ya1[x]] + es, Abs[Ya2[x]] + es, Abs[Ya3[x]] + es, Abs[Ya4[x]] + es,
    Abs[Yb1[x]] + es, Abs[Yb2[x]] + es, Abs[Yb3[x]] + es, Abs[Yb4[x]] + es,
    pa, pb] - HM[Abs[Ya1[x]], Abs[Ya2[x]], Abs[Ya3[x]], Abs[Ya4[x]],
    Abs[Yb1[x]], Abs[Yb2[x]], Abs[Yb3[x]], Abs[Yb4[x]], pa, pb]);
Ya2[x] - es
Ya4[x] - es
HepsilonExpanded = Expand[Hepsilon[x]];
xvals = Range[tauL, tauR, du];
minValue = Min[HepsilonExpanded /. x -> xvals];
minValue > 0
```

```
Out[170]= 1 368 358 052 173 157 750 784 980 624 005 615 999 838 604 825 302 259 045 863 775 739
+
5 842 835 553 631 319 870 482 351 516 979 088 666 926 168 339 925 000 000 000 000 000
163 410 003 294 769 333 102 434 735 655 859 004 318 360 771 458 304 708 840 433 800 211 x
+
5 613 776 768 774 393 197 208 183 097 352 347 101 448 709 013 598 600 000 000 000 000
(3 985 331 314 462 767 651 047 384 677 836 349 008 462 896 331 667 229 785 864 787 038 273 921
x^2) /
2 630 048 574 439 891 329 569 563 014 649 440 384 209 291 212 615 924 815 060 000 000 000 :
000 000
```

```
Out[171]=

$$\frac{12\ 580\ 749\ 316\ 047\ 438\ 413\ 369\ 276\ 419\ 853\ 848\ 666\ 648\ 834\ 170\ 655\ 715\ 923\ 401\ 671 + 233\ 713\ 422\ 145\ 252\ 794\ 819\ 294\ 060\ 679\ 163\ 546\ 677\ 046\ 733\ 597\ 000\ 000\ 000\ 000\ 000 + 11\ 305\ 754\ 343\ 616\ 303\ 782\ 502\ 875\ 718\ 739\ 463\ 369\ 907\ 852\ 699\ 271\ 822\ 451\ 768\ 667\ 733\ x + 112\ 275\ 535\ 375\ 487\ 863\ 944\ 163\ 661\ 947\ 046\ 942\ 028\ 974\ 180\ 271\ 972\ 000\ 000\ 000\ 000\ 000 + (180\ 484\ 421\ 136\ 231\ 189\ 769\ 838\ 350\ 553\ 022\ 421\ 215\ 491\ 022\ 865\ 640\ 143\ 777\ 829\ 928\ 834\ 293 x^2) / 21\ 917\ 071\ 453\ 665\ 761\ 079\ 746\ 358\ 455\ 412\ 003\ 201\ 744\ 093\ 438\ 466\ 040\ 125\ 500\ 000\ 000\ 000 : 000$$

```

Out[175]= True

(C.3)

```
In[176]= Max[Abs[Ya1'[tauL]], Abs[Ya1'[tauR]], Abs[Ya2'[tauL]], Abs[Ya2'[tauR]], Abs[Ya3'[tauL]], Abs[Ya3'[tauR]], Abs[Ya4'[tauL]], Abs[Ya4'[tauR]], Abs[Yb1'[tauL]], Abs[Yb1'[tauR]], Abs[Yb2'[tauL]], Abs[Yb2'[tauR]], Abs[Yb3'[tauL]], Abs[Yb3'[tauR]], Abs[Yb4'[tauL]], Abs[Yb4'[tauR]]] < 2/5
```

Out[176]= True

(C.4)

```
In[177]= Clear[tauL, tauR, dx, T, Za, Zb, x, Ra, Rb, A1, A2, B1, B2];
tauL = 12 / 10;
tauR = 21 / 10;
T = 11 / 10;
dx = 10^-2;
A1[x_, t_] := Ua2d[x, t] - (1 - x - 2 Qanew[t]) Uaapp[x, t] - Qanew[t] x Vaapp[x, t];
A2[x_, t_] := Va2d[x, t] - (1 + x - 2 Qanew[t]) Vaapp[x, t] - Qanew[t] x Uaapp[x, t];
B1[x_, t_] := Ub2d[x, t] - (1 - x - 2 Qanew[t]) Ubapp[x, t] - Qanew[t] x Vbapp[x, t];
B2[x_, t_] := Vb2d[x, t] - (1 + x - 2 Qanew[t]) Vbapp[x, t] - Qanew[t] x Ubapp[x, t];
Za = Integrate[ExpandAll[A1[x, t]^2 + A2[x, t]^2], {t, 0, T}];
Zb = Integrate[ExpandAll[B1[x, t]^2 + B2[x, t]^2], {t, 0, T}];
Ra = polyMax[Za, tauL, tauR, dx];
Rb = polyMax[Zb, tauL, tauR, dx];
k = Sqrt[tauR + 1];
1/2 Sqrt[2 k T + Sinh[2 k T]] / k Sqrt[Max[FU[Ra], FU[Rb]]] + 3 T Sinh[k T] / (2 k 10^5) + 1/5 dx < es
```

Out[187]= True

Case l=1: Section 7

(7.15), please check toy example 1

(7.18)

```
In[188]:= polynomialMin[2 +  $\frac{867}{100} x^2 + 3 x^3 P2[x]$ ,  $\frac{1}{2}$ , 1,  $\frac{1}{10^5}$ ] > 0
```

```
Out[188]= True
```

(7.21)

```
In[189]:= polynomialMin[2 + 3 x^3 P2[x] + 3 x^2, 1,  $\frac{17}{10}$ ,  $\frac{5}{10^5}$ ] > 0
```

```
Out[189]= True
```

(7.34)

```
In[190]:= Clear[y1, y];
```

```
y1[t_] := - $\frac{3 t^8}{341} + \frac{46 t^7}{525} - \frac{487 t^6}{1404} + \frac{1801 t^5}{2640} - \frac{2669 t^4}{4312} + \frac{81 t^2}{185}$ ;
```

```
y[t_] := t^2 y1[t];
```

```
polynomialMin[
```

```
ExpandAll[(y'[x] - 2 y1[x] - x^2 (W1[x] + 75 / 10^6)) / x^2], 0, 17 / 10, 1 / 10^3] > 0
```

```
Out[191]= True
```

(7.35)

```
In[192]:= polynomialMin[
```

```
ExpandAll[(2 / x^2 + 3 x P2[x]) y1[x] + W1[x] - 75 / 10^6], 0, 17 / 10, 2 / 10^4] > 0
```

```
Out[192]= True
```

l=1: Section 8

Check $l_-(t) > 0, t \in [0, 1.8]$

```
In[193]= gtilde[t_] := 29 / 5 + (5324 t^2) / 11721 - (5242 t^3) / 6305 - (453401 t^4) / 8690 -
(1372577 t^5) / 5985 + (11561100 t^6) / 5101 - (104060861 t^7) / 8909 +
(269766735 t^8) / 5476 - (1847962994 t^9) / 11735 + (3127972553 t^10) / 8468 -
(117464741599 t^11) / 181890 + (272712958583 t^12) / 316809 -
(188840245805 t^13) / 212143 + (59891505057 t^14) / 83009 -
(82501815384 t^15) / 179359 + (88378047540 t^16) / 384383 -
(13934342915 t^17) / 156279 + (22540646527 t^18) / 857154 -
(4889665718 t^19) / 857587 + (1193343423 t^20) / 1392527 -
(61031641 t^21) / 765161 + (2655007 t^22) / 766661;
Iminus[t_] := t^2 (gtilde[t] - t / 1600);
Iplus[t_] := t^2 (gtilde[t] + t / 1600);
Itilde[t_] := t^2 gtilde[t];
polynomialMin[gtilde[x] - x / 1600, 0, 18 / 10, 6 * 10^(-4)] > 0
```

Out[197]= True

(8.3)

```
In[198]= Clear[p, x];
P2plus[t_] := (W1[t] - 75 * 10^(-6)) (W1'[t] + 42 * 10^(-5));
p = Simplify[ExpandAll[(Iplus'[x] * x^2 * Iminus[x] - (2 + 3 x^3 P2plus[x]) *
Iplus[x] * Iminus[x] - x^4 (W1[x] + 75 / 10^6) * Iplus[x]) / x^5]];
polynomialMin[p, 0, 12 / 10, 6 * 10^(-5)] > 0
Clear[p, x];
p = Simplify[ExpandAll[(-Iminus'[x] * x^2 * Iplus[x] + (2 + 3 x^3 P2[x]) *
Iplus[x] * Iminus[x] + x^4 (W1[x] - 75 / 10^6) * Iminus[x]) / x^5]];
polynomialMin[p, 0, 12 / 10, 6 * 10^(-5)] > 0
```

Out[199]= True

Out[201]= True

(8.4)

```
In[202]= Max[Abs[Itilde'[6/5] - Iplus'[6/5] / Iplus[6/5] * Iminus[6/5]],
Abs[Itilde'[6/5] - Iminus'[6/5] / Iminus[6/5] * Iplus[6/5]]] < 62 / 10^4
```

Out[202]= True

(8.5)

```

In[203]:= (* Check 1st inequality in (8.5) *)
Fminus[t_] :=
  Itilde'[t] - 2 gtilde[t] - 3 t * P2plus[t] * Itilde[t] - t^2 (W1[t] + 75 / 10^6)
Fplus[t_] :=
  Itilde'[t] - 2 gtilde[t] - 3 t * P2[t] * Itilde[t] - t^2 (W1[t] - 75 / 10^6);
polynomialMin[Fminus[x], 12 / 10, 18 / 10, 6 * 10^(-5)] > 0
{polyMax[Fminus[x], 12 / 10, 18 / 10, 15 * 10^(-5)] < 39 / 10^4,
 polyMax[Fplus[x], 12 / 10, 18 / 10, 5 * 10^(-5)] < 39 / 10^4}

Out[205]= True
Out[206]= {True, True}

In[207]:= (* Check 2nd inequality in (8.5) *)
Clear[p];
p = 7 / 5 * x^2 - 2 - 3 x^3 P2plus[x];
polynomialMin[p, 12 / 10, 18 / 10, 10^(-4)] > 0
Clear[p];
p = 7 / 5 * x^2 + 2 + 3 x^3 P2[x]; polynomialMin[p, 12 / 10, 18 / 10, 4 * 10^(-5)] > 0

Out[207]= True
Out[208]= True

```

(8.7)

```

In[209]:= Clear[t, y];
DSolve[{y'[t] == 7 / 5 y[t] + 39 / 10^4,
 y[6 / 5] == (12 / 10)^3 / 1600, y'[6 / 5] == 62 / 10^4}, y[t], t]

Out[209]= {{y[t] ->  $\frac{1}{700000} e^{-\frac{6\sqrt{\frac{7}{5}}}{5}t - \sqrt{\frac{7}{5}}t}$ 
 $\left( 1353 e^{\frac{12\sqrt{\frac{7}{5}}}{5}} - 310\sqrt{35} e^{\frac{12\sqrt{\frac{7}{5}}}{5}} + 1353 e^{2\sqrt{\frac{7}{5}}t} + 310\sqrt{35} e^{2\sqrt{\frac{7}{5}}t} - 1950 e^{\frac{6\sqrt{\frac{7}{5}}}{5} + \sqrt{\frac{7}{5}}t} \right)}}$ 

In[210]:= Clear[yz];
yz[t_] :=  $\frac{1}{700000} e^{-\frac{6\sqrt{\frac{7}{5}}}{5}t - \sqrt{\frac{7}{5}}t}$ 
 $\left( 1353 e^{\frac{12\sqrt{\frac{7}{5}}}{5}} - 310\sqrt{35} e^{\frac{12\sqrt{\frac{7}{5}}}{5}} + 1353 e^{2\sqrt{\frac{7}{5}}t} + 310\sqrt{35} e^{2\sqrt{\frac{7}{5}}t} - 1950 e^{\frac{6\sqrt{\frac{7}{5}}}{5} + \sqrt{\frac{7}{5}}t} \right);$ 
{yz[9 / 5] < 62 / 10^4, yz'[9 / 5] < 114 / 10^4}

Out[211]= {True, True}

```

(8.8)

```
In[212]:= polynomialMin[-2 - 3 x^3 P2plus[x], 12 / 10, 16 / 10, 10^(-4)] > 0
Integrate[-2 / t^2 - 3 t P2[t], {t, 12 / 10, 16 / 10}] < 244 / 10^3
polynomialMin[3 / 20 x^2 + 2 + 3 x^3 P2[x], 16 / 10, 18 / 10, 5 * 10^(-5)] > 0
polynomialMin[3 / 20 x^2 - 2 - 3 x^3 P2plus[x], 16 / 10, 18 / 10, 8 * 10^(-5)] > 0
Integrate[Fplus[t], {t, 12 / 10, 18 / 10}] < 165 / 10^5
62 / 10^4 + (244 / 10^3 + 2 / 10 * 15 / 100) * 62 / 10^4 + 165 / 10^5 < 96 / 10^4
```

Out[212]= True

Out[213]= True

Out[214]= True

Out[215]= True

Out[216]= True

Out[217]= True

(8.9)

```
In[218]:= Integrate[t^2 (Itilde[t] - 62 / 10^4) (W1[t] - 75 / 10^6), {t, 0, 18 / 10}] > 107 / 100
Out[218]= True
```

$$J_+\left(\frac{9}{5}\right) \geq I\left(\frac{9}{5}\right) \geq J_-\left(\frac{9}{5}\right), \quad J'_+\left(\frac{9}{5}\right) > I'\left(\frac{9}{5}\right) \geq J'_-\left(\frac{9}{5}\right).$$

```
In[219]:= Clear[Jp, Jm];
```

$$Jp[x_] := -\frac{46355}{4182} + \frac{162617x}{2916} - \frac{215987x^2}{2017} + \frac{428192x^3}{3543} - \frac{38323x^4}{412} + \frac{104356x^5}{2011} - \frac{37198x^6}{1725} + \frac{20181x^7}{2999} - \frac{5962x^8}{3809} + \frac{923x^9}{3491} - \frac{17x^{10}}{556} + \frac{11x^{11}}{5279} - \frac{x^{12}}{27017} - \frac{x^{13}}{189311} + \frac{x^{14}}{3302419};$$

$$Jm[x_] := -\frac{12413}{1105} + \frac{171747x}{3038} - \frac{174823x^2}{1612} + \frac{47397503x^3}{388000} - \frac{307187x^4}{3277} + \frac{540785x^5}{10386} - \frac{28489x^6}{1325} + \frac{20281x^7}{3051} - \frac{5023x^8}{3296} + \frac{1218x^9}{4849} - \frac{187x^{10}}{6760} + \frac{4x^{11}}{2441} + \frac{x^{12}}{121108} - \frac{x^{13}}{124156} + \frac{x^{14}}{2629395};$$

$$\left\{ Jp\left[\frac{9}{5}\right] > Iplus\left[\frac{9}{5}\right] > Iminus\left[\frac{9}{5}\right] > Jm\left[\frac{9}{5}\right], \right.$$

$$\left. Jp'\left[\frac{9}{5}\right] > Iplus'\left[\frac{9}{5}\right] > Iminus'\left[\frac{9}{5}\right] > Jm'\left[\frac{9}{5}\right] \right\}$$

Out[221]= {True, True}

(8.11) and note that $J_m(t) < J_p(t)$, then $-J_p(t) < -J_m(t)$

$$\text{In[222]= } \left\{ \frac{83}{100} < -J_p\left[\frac{35}{10}\right] < -J_m\left[\frac{35}{10}\right] < \frac{1051}{1000}, \frac{16}{100} < -J_p'\left[\frac{35}{10}\right] < -J_m'\left[\frac{35}{10}\right] < \frac{57}{100} \right\}$$

Out[222]= {True, True}

(8.13)

```
In[223]= Clear[x, Bt1, Bt2, pt1, pt2, pt3, pt4];
Bt1 = (W1'[x] + 5/10^5) (W1[x] - 6/10^6);
Bt2 = (W1'[x] - 5/10^5) (W1[x] + 6/10^6);
pt1 = x^2 Jp''[x] - 2 Jp[x] - 3 x^3 Bt1 * Jp[x] - x^4 (W1[x] + 6/10^6);
pt2 = x^2 Jp''[x] - 2 Jp[x] - 3 x^3 Bt2 * Jp[x] - x^4 (W1[x] + 6/10^6);
pt3 = x^2 Jm''[x] - 2 Jm[x] - 3 x^3 Bt1 * Jm[x] - x^4 (W1[x] - 6/10^6);
pt4 = x^2 Jm''[x] - 2 Jm[x] - 3 x^3 Bt2 * Jm[x] - x^4 (W1[x] - 6/10^6);
{polynomialMin[pt1, 18/10, 25/10, 7 * 10^(-5)] > 0,
 polynomialMin[pt2, 18/10, 25/10, 7 * 10^(-5)] > 0,
 polynomialMin[-pt3, 18/10, 25/10, 10^(-4)] > 0,
 polynomialMin[-pt4, 18/10, 25/10, 10^(-4)] > 0}
```

Out[228]= {True, True, True, True}

Check $J_{\pm} < 0$ on [2.5, 3.5]

```
In[229]= {polynomialMin[-Jp[x], 25/10, 35/10, 7 * 10^(-3)] > 0,
 polynomialMin[-Jm[x], 25/10, 35/10, 7 * 10^(-3)] > 0}
```

Out[229]= {True, True}

(8.14) and (8.15)

```
In[230]= Clear[x, Ct1, Ct2, pt1, pt2];
Ct1 = (-W2'[x] + 5/10^5) (W2[x] + 6/10^6);
Ct2 = (-W2'[x] - 5/10^5) (W2[x] - 6/10^6);
pt1 = x^2 Jp''[x] - 2 Jp[x] + 3 x^3 Ct1 * Jp[x] - x^4 (W2[x] + 6/10^6);
pt2 = x^2 Jm''[x] - 2 Jm[x] + 3 x^3 Ct2 * Jm[x] - x^4 (W2[x] - 6/10^6);
{polynomialMin[pt1, 25/10, 35/10, 2 * 10^(-4)] > 0,
 polynomialMin[-pt2, 25/10, 35/10, 2 * 10^(-4)] > 0}
```

Out[234]= {True, True}

(8.16)

In[235]= `polyMax[Jm[x], 9 / 5, 7 / 2, 10 ^ (-4)] < $\frac{106}{100}$`

$$\text{Integrate}\left[Jm[t] t^2 W1[t], \left\{t, \frac{9}{5}, \frac{5}{2}\right\}\right] + \text{Integrate}\left[Jm[t] t^2 W2[t], \left\{t, \frac{5}{2}, \frac{7}{2}\right\}\right] - \frac{6}{10^6} * \frac{106}{100} \text{Integrate}\left[t^2, \left\{t, \frac{9}{5}, \frac{7}{2}\right\}\right] > - \frac{401}{1000}$$

Out[235]= True

Out[236]= True

 $H_-(t)$ in Lemma 8.3

In[237]= `Clear[t];`

`DSolve[{Hm'[t] == 18525 / 10^4 * 1 / t^2 Hm[t] - 28 / 10 t Exp[-t],
Hm[7 / 2] == 83 / 100, Hm'[7 / 2] == 16 / 100}, Hm[t], t]`

Out[237]= $\left\{ \left\{ Hm[t] \rightarrow \frac{1}{568400 t^{19/20}} \left(103733 \times 2^{1/20} \times 7^{19/20} + 10788 \times 2^{19/20} \times 7^{1/20} t^{29/10} - 548800 t^{29/10} \text{Gamma}\left[\frac{21}{20}, \frac{7}{2}\right] + 548800 t^{29/10} \text{Gamma}\left[\frac{21}{20}, t\right] + 548800 \text{Gamma}\left[\frac{79}{20}, \frac{7}{2}\right] - 548800 \text{Gamma}\left[\frac{79}{20}, t\right] \right) \right\} \right\}$

(8.19)

In[238]= `$10788 \times 2^{19/20} \times 7^{1/20} - 548800 \text{Gamma}\left[\frac{21}{20}, \frac{7}{2}\right] > 0$`

Out[238]= True

 $H_+(t)$ in Lemma 8.3

In[239]= `Clear[t];`

`DSolve[{Hp'[t] == 2 / t^2 Hp[t] - 27 / 10 t Exp[-t],
Hp[7 / 2] == 1051 / 1000, Hp'[7 / 2] == 57 / 100}, Hp[t], t]`

Out[239]= $\left\{ \left\{ Hp[t] \rightarrow \frac{1}{294000 t} e^{-\frac{7}{2}-t} \left(-1587600 e^{7/2} + 28212975 e^t + 36701 e^{\frac{7}{2}+t} - 1587600 e^{7/2} t - 793800 e^{7/2} t^2 - 264600 e^t t^3 + 24368 e^{\frac{7}{2}+t} t^3 \right) \right\} \right\}$

```
In[240]:= Integrate[ExpandAll[ $\frac{1}{294000 t}$ 
 $e^{-\frac{7}{2}-t} \left( -1587600 e^{7/2} + 28212975 e^t + 36701 e^{\frac{7}{2}+t} - 1587600 e^{7/2} t - 793800 e^{7/2} t^2 - \right.$ 
 $\left. 264600 e^t t^3 + 24368 e^{\frac{7}{2}+t} t^3 \right) t^{19/7} \text{Exp}[-t]], \{t, 7/2, \text{Infinity}\}] < \frac{65}{100}$ 
```

Out[240]= True

Proposition 8.4

```
In[241]:=  $\frac{107}{100} - \frac{401}{1000} + \frac{65}{100} > 0$ 
```

Out[241]= True